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# Approximate solution of Lane-Emden problem via modified Hermite operation matrix method

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**Abstract**

Lane-Emden equations are singular initial value problems and they are important in mathematical physics and astrophysics. The aim of this present paper is presenting a new numerical method for finding approximate solution to Lane-Emden type equations arising in astrophysics based on modified Hermite operational matrix of integration. The proposed technique is based on taking the truncated modified Hermite series of the solution function in the Lane-Emden equation and then transferred into a matrix equation together with the given conditions. The obtained result is system of linear algebraic equation using collection points. The suggested algorithm is applied on some relevant physical problems as Lane-Emden type equations.

## Introduction:

Lane-Emden equations are initial value problems with singularity at the origin. They have the attention of many mathematician and physics because of their importance in many field such as a astrophysics and mathematical physics [1-5]. Lane-Emden problem has been studied by many researchers and solved using different technique. For example, authors in [6] solved singular Lane-Emden equation using Legender wavelets while in [7]. Bernstein operational matrix was used and applied Chebyshev operational matrix technique to find approximate solution for such problem [8]. Some other approximate solutions are used such as variation iteration method [9], Bifurcation technique [10] to compute numerical solution of linear and nonlinear Lane-Emden equation. For more studies, concerning Lane-Emden type equations are can see [11-13]. The collocation method has been a very important and useful technique to find the approximate solution of Lane-Emden equation method based on hybrid of the rational Bernoulli functions used in [14], collocation scheme with third degree B-spline suggested by [15], In [16-17], collocation method was utilized to solve system of Lane-Emden equation depending on Chebyshev polynomials. Moreover, orthogonal polynomial and wavelets establish a relation with fast approximate techniques and they are successfully used in many fields such as optimal control problem [18-20].

The main important of this work is considering modified Hermite polynomials together with perational matrix of integration for solving linear singular Lane-Emden equation. Implementation of the suggested method reduces the original problem to a system of linear algebraic equations, which can easily solve with computer.

The rest sections of this work are arranged as follows: Sections 2 and 3 present some important properties of modified Hermite polynomials with basic formulation of them. Section 4, devotes to the operational matrix of integration for modified Hermite polynomial functions with useful theorem. Section 5, summarizes the utilized of obtained operation matrix to find an approximate solution of linear singular Lane-Emden equation. Illustrate examples are given in section 6 to demonstrate the application of operation matrix of integration for modified Hermite and some concluding remarks are listed in section 7.

## 1. Modified Hermite Polynomials: Definition and Properties:

For $n \in R$, modified Hermite polynomial $lH_m^a(t)$, can be defined by

$$\sum_{m=0}^{\infty} \frac{H_m^a(t)}{m!} = a^{2t\tau - \tau^2}, \quad |\tau| < \infty, \quad a > 0 \tag{1}$$

Expand $\sum_{m=0}^{\infty} \frac{H_m^a(t)}{m!}$ as a Maclurin series in $\tau$ and equate coefficients, then $H_m^a(t)$can be found as:

$$H_m^a(t) = m! \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{-1^k (2t)^{m-2k} (\log a)^{m-k}}{k!(m-2k)!} \tag{2}$$

In particular, for $a = 2$, one can obtain the first few modified Hermite polynomials

$H_0^2(t) = 1$

$H_1^2(t) = 2At$

$H_2^2(t) = 4A^2t^2 - 2A$

$H_3^2(t) = 8A^3t^3 - 12A^2t$

$H_4^2(t) = 16A^4t^4 - 48A^3t^2 + 12A^2$

where $A = \log 2$. The five first modified Hermite polynomials are plotted in figure 1.

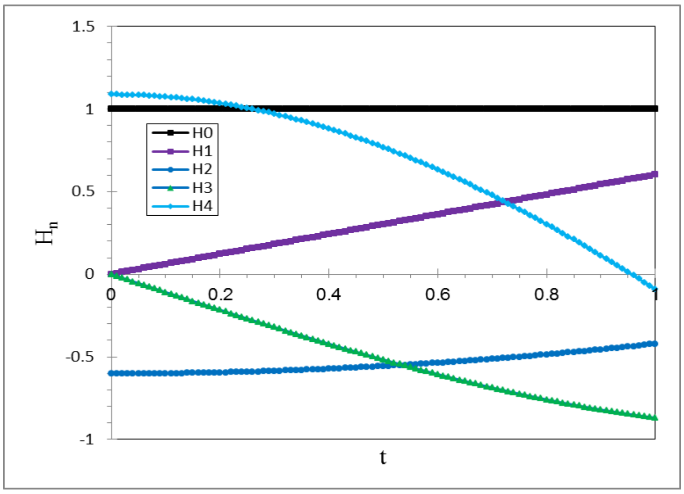


**Fig.1:** Graph of modified Hermite polynomials of degrees 0, 1, 2, 3, 4 and 5

Also, the modified Hermite polynomials satisfy recurrence relation given by

$$H^2_{m+1} = A(2t\ H^2_m(t) - 2mH^2_{m-1}(t)) \qquad (3)$$

where $m = 1,2, \dots, H^2_0(t) = 1, H^2_1(t) = 2At$

The next recurrence relation connects the modified Hermite polynomials with their first derivative.

$$\dot{H}^2_m(t) = 2m\ A\ H^2_{m-1}(t) \qquad (4)$$

Combining the two recurrence relations 3 and 4, yields another relation.

$$\ddot{H}^2_m(t) = 2t\ A\ \dot{H}^2_{m-1}(t) + 2mA\ H^2_m(t) \qquad (5)$$

The polynomials $H^2_m(t)$ have the orthogonally property, that is

$$\int_{-\infty}^{\infty} H^2_m(t)H^2_n(t)e^{-t^2}\ dt = \begin{cases} 2^m\ m!\ A^{m^2}(\pi/A)^{1/2} & m = n \\ 0 & m \neq n \end{cases}$$

The modified Hermite polynomials $H^2_m(t)$ form an orthogonal basis of the Hilbert space function satisfying.

$$\int_{-\infty}^{\infty} |f(t)|^2\ w(t)dt < \infty$$

## 2. Function Approximation

Suppose that $\{H^2_0(t)\ H^2_1(t)\ H^2_2(t) \dots H^2_m(t)\} \subset L^2_w[-\infty, \infty]$ is the set of modified Hermite polynomials, $\gamma = apan\{H^2_0(t)\ H^2_1(t)\ H^2_2(t) \dots H^2_m(t)\}$. Suppose that $u(t)$ is arbitrary function of $L^2_w[-\infty, \infty]$, then.

$$\|u(t) - u^*(t)\|_2 \leq \|u(t) - u_m(t)\|_2, \quad \forall u_m(t) \in \gamma.$$

where $u^*(t)$ is the unique best approximation.

**Theorem 1**

A function $u(t)$ defined on $[-\infty, \infty]$ can be expanded as a finite sum of modified Hermite polynomials, then $u_m(t) = \sum_{i=0}^{m} c_i\, H^2_i(t)$ converge uniformily to $u(t)$, where $c_i =< u(t), H^2_i(t) >$.

## 3. Modified Hermite Operational Matrix of Integration

The integration of vector $H^2_m(t)$ can be expressed as.

$$\int_0^t H^2_m(x)dx = I_H\ H^2_m(t) \qquad (6)$$

where the matrix $I_H$ is the $(m + 1) \times (m + 1)$ operational matrix of integration for modified Hermite polynomials.

The integration of $H_m^2 f(t)$ of order m can be obtained as follows.

$$\int_0^t H_m^2(x)\,dx = \frac{1}{2A(m+1)}\{H_{m+1}^2(t) - H_{m+1}^2(0)\,H_0^2\} \tag{7}$$

where

$$H_{m+1}^2(0) = \begin{cases} (-1)^{\frac{m+1}{2}} A^{\frac{m+1}{2}} \frac{(m+1)!}{(\frac{m+1}{2})!} & \text{m odd} \\ 0 & \text{m even} \end{cases} \tag{8}$$

The matrix $I_H$ can be written as

$$I_A = \begin{bmatrix} 0 & \frac{1}{2A} & \cdots & 0 & 0 & 0 & 0 \\ \frac{-1}{2A} & 0 & \frac{1}{4A} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{\frac{m+1}{2}} A^{\frac{m+1}{2}} \frac{(m+1)!}{\left(\frac{m+1}{2}\right)!} & \vdots & \vdots & \vdots & \vdots & \frac{1}{2A(m+1)} & 0 \end{bmatrix}$$

For even $m$ the last row will be.

$$\begin{bmatrix} 0 & \frac{1}{2A} & 0 & \frac{1}{4A} & 0 & \cdots & \frac{1}{2A(m+1)} & 0 & 0 \end{bmatrix}$$

## 4. Formulation of New Trial Solution via Operational Matrix of Integration:

In the present section, a computational technique is presented to treat Lane-Emden equation with collocation method together with modified Hermite operational matrix of integration.

Consider Lane-Emden type equation.

$$\ddot{u}(t) + \frac{\alpha}{t}\,\dot{u}(t) + f(t,u) = g(t) \tag{9}$$

ogether with,

$$u(0) = \lambda_0\,,\ \dot{u}(0) = \lambda_1 \tag{10}$$

By expanding $\ddot{u}(t)$ in modified Herimte polynomials we can get.

$$\ddot{u}(t) = C^T H^2(t) \tag{11}$$

where the vectors c and $H^2(t)$ are $[c_0\ \ c_1\ \ \dots\ \ c_m]^T$ and $[H_0^2\ \ H_1^2\ \dots\ H_m^2]^T$ recpectively.

Integrating Eq. 11 from 0 to t, yields.

$$\dot{u}(t) - \dot{u}(0) = \int_0^t c^T H^2(\tau)\,d\tau$$

$$\dot{u}(t) = \dot{u}(0) + c^T I_H\,H^2(t) = \lambda_1 + c^T I_H H^2(t) \tag{12}$$

where $I_H$ is the operational matrix integration for modified Hermite polynomial. Then integrating Eq. 12, to obtain.

$$u(t) = \lambda_0 + \lambda_1 t + c^T I_H \int_0^t H^2(\tau)\, d\tau \tag{13}$$

Eq. 13 represents the formulation of the solution for Eq. 9.

After substituted Eqns. 11, 12 and 13 into Eq. 9, and then applied collocation points, a linear set of algebraic equations is obtained with $c_0, c_1, \ldots, c_m$.

## 1. Numerical Examples:

Three numerical examples are solved for showing the activity of the proposed method.

**Example 1**

The first standard linear Lane-Emden type is

$$\ddot{u}(t) - \frac{2}{t}\,\dot{u}(t) + 1 = 0 \tag{14}$$

subject to $u(0) = 1$, $\dot{u}(0) = 0$

Modified Hermite polynomials is used with n =0, to approximate the second derivative of the solution, that is

$$\ddot{u}(t) = c_0 H_0^2(t) \tag{15}$$

Integrate Eq. 15 twice from 0 to t with boundary condition, yields

$$\dot{u}(t) = \dot{u}(0) + c_0 I_H\, H^2(t) \tag{16}$$

where $I_H = \begin{bmatrix} 0 & \frac{1}{2A} & 0 \end{bmatrix}$ and $H^2(t) = [H_0^2 \quad H_1^2 \quad H_2^2]^T$

Then the solution $u(t)$ can be obtained by integrating Eq. 16 ,

$$u(t) = u(0) + \dot{u}(0)t + c_0 \begin{bmatrix} \frac{1}{4A} & 0 & \frac{1}{8A^2} \end{bmatrix} \begin{bmatrix} H_0^2(t) \\ H_1^2(t) \\ H_2^2(t) \end{bmatrix} \tag{17}$$

Now substitute the approximation of $\ddot{u}$ (t), $\dot{u}$(t) into Eq. 14 and discretize it using collocation point thus one equation with one unknown coefficient can be obtained. therefore the value $c_0 = -.3333$ is obtained.

$$u(t) = 1 - \frac{1}{3}\left(\frac{1}{4A} H_0^2(t) + \frac{1}{8A^2} H_1^2(t)\right) = 1 - \frac{t^2}{6} \tag{18}$$

Figure 2 shows an identical between the obtained numerical solutions against the exact one.

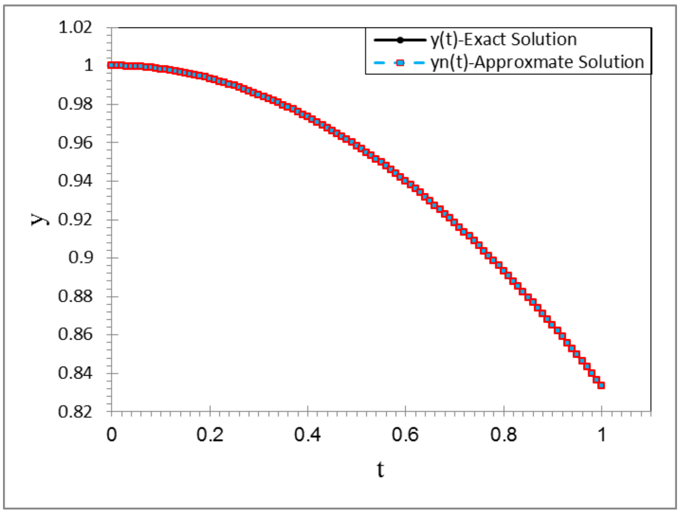


**Fig. 2:** The obtained approximate solution with $n = 0$ and the exact $u(t) = 1 - \frac{t^2}{6}$ for example 1.

**Example 2**

The second slandered linear singular initial Lane-Emden equation is given by

$$\ddot{u}(t) - \frac{2}{t}\,\dot{u}(t) + u(t) = 0$$

Subject to $u(0) = 1,\ \dot{u}(0) = 1$

exact solution $y(t) = \frac{\sin(t)}{t}$.

here we assume $\ddot{u}(t) = \sum_{n=0}^{8} c_n\, H_n^2(t)$ ,

By applying the same method and solve the system of equations we approximate the solution. Table 1 and figure 3 clarify the comparison between the numerical and exact solutions.

**Table 1**: The obtained numerical solution with $n = 8$ and the exact one for example 2.

| $t$ | Present method (n=8) | Exact solution |
|---|---|---|
| 0 | 1 | 1 |
| 0.1 | 0.017453284 | 0.017453284 |
| 0.2 | 0.017453257 | 0.017453257 |
| 0.3 | 0.017453213 | 0.017453213 |
| 0.4 | 0.017453151 | 0.017453151 |
| 0.5 | 0.017453071 | 0.017453071 |
| 0.6 | 0.017452974 | 0.017452974 |
| 0.7 | 0.017452858 | 0.017452858 |
| 0.8 | 0.017452725 | 0.017452725 |
| 0.9 | 0.017452575 | 0.017452575 |
| 1 | 0.017452406 | 0.017452406 |

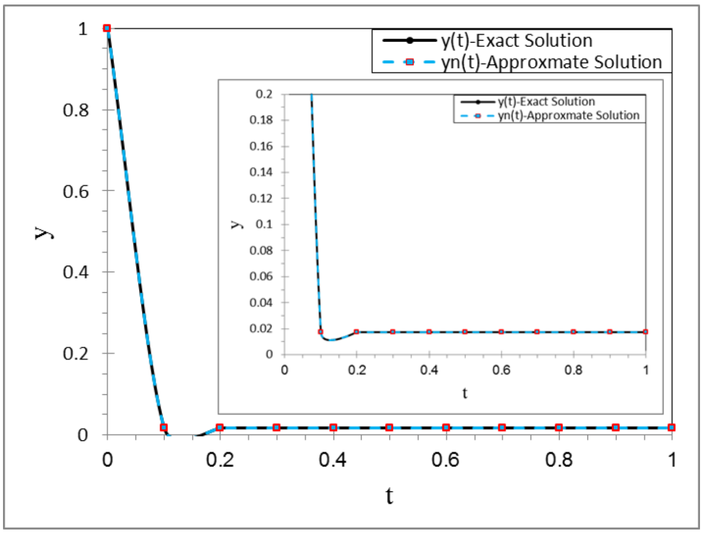


**Fig. 3**: The obtained approximate solution with $n = 8$ and the exact one for example 2.

**Example 3**

The third standard singular Lane-Emden equation is [1]

$$\ddot{u}(t) + \frac{8}{t}\,\dot{u}(t) + tu = t^5 - t^4 + 44t^2 - 30t$$

subject to $u(0) = 0\,,\ \ \dot{u}(0) = 0.$

where the analytic solution is $u(t) = t^4 - t^3$.

Let $\ddot{u}(t) = \sum_{n=0}^{2} c_n\, H_n^2(t)$

By applying the presented method and solve the system of equations, one can get the following approximate

$$u(t) = \frac{3}{4A^2}H_0^2(t) - \frac{3}{4A^2}H_1^2(t) + \frac{3}{4A^3}H_2^2(t) - \frac{1}{8A^3}H_3^2(t) + \frac{1}{16A^4}H_4^2(t) \tag{19}$$

The result of Eq. 19 is $u(t) = t^4 - t^3$.

The test solved example shows to clear the applicability and validity of the presented operation matrix method with a few terms of modified Hermite basis functions comparing with other method. The obtained results are listed in both Figures and Tables.

Table 2 and figure 4 show the comparison between numerical solution with n =2 and the analytic one.

**Table 2:** The obtained approximate solution with $n = 2$ and the exact one for example 3.

| $t$ | Present method n=2 | Result in [1] n=4 | Exact solution |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 0.1 | 0.017453284 | 0.017453284 | 0.017453284 |
| 0.2 | 0.017453257 | 0.017453257 | 0.017453257 |
| 0.3 | 0.017453213 | 0.017453213 | 0.017453213 |
| 0.4 | 0.017453151 | 0.017453151 | 0.017453151 |
| 0.5 | 0.017453071 | 0.017453071 | 0.017453071 |
| 0.6 | 0.017452974 | 0.017452974 | 0.017452974 |
| 0.7 | 0.017452858 | 0.017452858 | 0.017452858 |
| 0.8 | 0.017452725 | 0.017452725 | 0.017452725 |
| 0.9 | 0.017452575 | 0.017452575 | 0.017452575 |
| 1 | 0.017452406 | 0.017452406 | 0.017452406 |

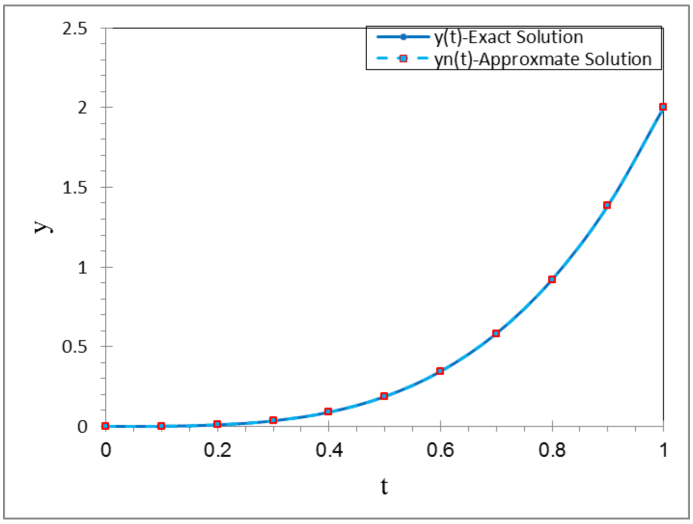


**Fig. 4:** The obtained approximate solution with n =2 and the exact one for example 3.

**Conclusion:**

An important certain linear initial value problem with singularity called Lane-Emden problem is treated in this work. The proposed technique is based on the application of modified Hermite operation matrix of integration. The idea of the proposed method essentially based on transforming the original Lane-Emden problem together with some conditions to a set of linear algebraic equations. The test-solved examples showed to clear the applicability and validity of the presented operation matrix method with a few terms of basis functions comparing with oher method. The obtained results are listed in both figures and tables.

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# الحل التقريبي لمشكلة لين-إمدن عبر طريقة مصفوفة عمليات هيرميت المعدلة

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البحث مستل من رسالة ماجستير الباحث الاول

**معلومات البحث:**

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**الكلمات المفتاحية:**

*طريقة التجميع، معادلة لين إمدن، وظيفة هيرميت المعدلة، مصفوفة التشغيل للتكامل، مشكلة القيمة الأولية المفردة.*

**الخلاصة:**

معادلات لين-إمدن هي مشاكل ذات قيمة مبدئية مفردة وهي مهمة في الفيزياء الرياضية والفيزياء الفلكية. الهدف من هذه البحث هو تقديم طريقة عددية جديدة لإيجاد حل تقريبي لمعادلات نوع لين-إمدن الناشئة في الفيزياء الفلكية على أساس مصفوفة هيرميت المعدلة للتكامل. تعتمد التقنية المقترحة على أخذ سلسلة هيرمت المعدلة من أقتطاع الحل في معادلة لين-إمدن، ثم نقلها إلى معادلة مصفوفة مع الشروط المحددة. النتيجة التي تم الحصول عليها هي نظام المعادلة الجبرية الخطية باستخدام نقاط الجمع. يتم تطبيق الخوارزمية المقترحة على بعض المشكلات المادية ذات الصلة مثل معادلات نوع لين-إمدن.