Rings with acc (dcc) on semi-essential ideals and semi – uniform Rings

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Abstract

Let R be a commutative ring with unity. In this paper we study rings satisfying ascending (descending) chain condition on semi-essential ideals, where a nonzero ideal I of a ring R is semi-essential, if $I \cap P \neq (0)$, for any nonzero prime ideal P of R. Moreover, we study semi-uniform rings, where a ring R is called semi-uniform, if every nonzero ideal of R is semi-essential.

Introduction

In this paper, all rings are assumed to be commutative and unital unless otherwise stated.

Various kinds of chain conditions were studies by several authors, especially chain condition on essential ideals was studied in (1), (2), (3). In section 1 of this paper, we study ascending (descending) chain condition on semi-essential ideals, where a nonzero ideal I of a ring R is said be semi- essential if $I \cap P \neq (0)$ for each nonzero prime ideal P of R, (4). In section 2, we study semi – uniform rings, where a ring said to be semi – uniform, if every nonzero ideal of R is semi-essential (4).

S.1. Rings with acc (dcc) on semi-essential ideals

We give the basic properties of rings that satisfy acc (dcc) on semi – essential ideals. For convince, we give the following definitions:

Definition1.1 A nonzero ideal I of a ring R is called an essential ideal, if $I \cap J \neq (0)$, for each nonzero ideal J of R, (5).

Definition1.2 A nonzero ideal I of a ring R is called semi-essential if $I \cap P \neq (0)$, for each nonzero prime ideal P of R, (4).

Definition13 A ring R is said to satisfy acc (dcc) on semi-essential ideals if every ascending (descending) chain of semi-essential ideals terminates.

Remarks and examples 1.4

1.It is clear that every essential ideal of a ring R is semi-essential, however the converse is not true , as the following example shows :

In the ring $(Z_{12}, +_{12}, \cdot_{12})$, the ideal $<\bar{3}>$ is semi-essential since the only prime ideals in Z_{12} are $<\bar{2}>$, $<\bar{3}>$, and $<\bar{3}> \cap <\bar{2}> = <\bar{6}> \neq <\bar{0}>$, but $<\bar{3}> \cap <\bar{4}> = <\bar{0}>$.

2.If R satisfies acc (dcc) on semi-essential ideals, then R satisfies acc (dcc) on essential ideals.

Proof. By (1.4(1)), every essential ideal is semi-essential, hence every ascending (descending) chain of essential ideals is ascending (descending) chain of semi-essential ideals and so it must terminates.

3.acc on semi-essential ideals, does not imply dcc on semi-essential ideals as the following example shows:

In the ring of integers Z, every nonzero ideal is essential, so semi-essential. But Z is Noetherian and not Artinian. Thus Z satisfies acc on semi-essential ideals, but it does not satisfy dcc on semi-essential ideals.

4. If I is a semi-essential ideal of R, J is an ideal of R such that $J \supseteq I$, then J is semi-essential.

Proof. It is clear.

5.If $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ is an ascending chain of semi-essential ideals of a ring R, then $I = \bigcup_{i=1}^{\infty} I_i$ is semi-essential.

Proof. It is easy, so is omitted.

The following proposition gives a characterization for rings with acc (dcc) on semi-essential ideals:

Proposition 1.5 Let R be a ring. The following are equivalent:

1-R satisfies acc (dcc) semi-essential ideals.

2-Any collection $\{I_i\}_{i \in \Lambda}$ of semi-essential ideals has a maximal (minimal) element.

Proof. (1) $\Rightarrow$ (2). If R satisfies acc on semi-essential ideals. Let ´ = $\{I_i\}_{i\in\Lambda}$ be a collection of semi-essential ideals of R. Suppose ´ has no maximal element. Let $I_1 \in$ ´ . Then I_1 is not a maximal element of ´ , so $\exists\ I_2 \in$ ´ such that $I_1 \subset I_2$. But I_2 is not a maximal element of ´ , so $\exists\ I_3 \in$ ´ such that $I_1 \subset I_2 \subset I_3$.Continuing this process, we get $I_1 \subset I_2 \subset I_3 \subset \ldots$an infinite ascending chain of semi-essential ideals, which contradict (1). Thus ´ has a maximal element.

Similarly, R satisfies dcc on semi-essential ideals, implies ´ has a minimal element.

(2) $\Rightarrow$ (1). Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ be an ascending chain of semi-essential ideals of R. Let ´ $= \{I_i : i = 1,2,..\}$. Then ´ has a maximal element, say I_n. But $\forall$ i >n, $I_i \supseteq I_n$, hence $I_i = I_n$, $\forall$ i > n. It follows that $I_n = I_{n+1} = \ldots$, that is the chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ terminates and R satisfies acc on semi-essential ideals.

Similarly, if any collection of semi-essential ideal has a minimal element, then R satisfies dcc on semi-essential ideals.

The following result is a characterization of Noetherian rings.

Theorem 1.6 Let R be a ring. Then R is Notherian iff every semi-essential ideal is finitely generated.

Proof. If R is Notherian, then every ideal is finitely generated, so every semi-essential ideal is finitely generated.

Conversely, if every semi-essential ideal of R is finitely generated, then every essential ideal is finitely generated, since every essential ideal is semi-essential ideal, by Rem. 1.4 (1). Thus R is Notherian, by (6, Exc.7, p.20).

Proposition 1.7 Let R be a ring which satisfies acc (dcc) on semi-essential ideals. If I is a semi-essential ideal of R, then R/I is Notherian (resp. Artinian) ring.

Proof. Suppose R satisfies acc on semi-essential ideals. To prove R/I is Notherian. Assume $J_1/I \subseteq J_2/I \subseteq \ldots$ is an ascending chain of ideals in R/I. Then $J_1 \subseteq J_2 \subseteq \ldots$ is an ascending chain of ideals in R, but for each i=1, 2,..., $J_i \supseteq I$, hence J_i is a semi-essential ideal, by Rem. 1.4 (4). Thus $J_1 \subseteq J_2 \subseteq \ldots$ is an ascending chain of semi-essential ideals of R, so $\exists$ n $\in Z_+$ such that $J_n = J_{n+1} = \ldots$ It follows that $J_n/I = J_{n+1}/I = \ldots$ and R/I is Notherian.

Similarly, R/I is Artinian, if R satisfies dcc on semi-essential ideals.

Remark 1.8 The converse of prop. 1.7 may not be true in general, for example:
The ring Z does not satisfy dcc on semi-essential ideals (see Rem. 1.4 (3)). But, for any ideal I of Z, $Z/I \cong Z_n$, for some $n \in Z_+$. However, Z_n is an Artinian ring.
Notice that, we have no example of a ring R does not satisfy acc on semi-essential ideals and R/I is a Notherian ring for some semi-essential ideal I of R.

Proposition 1.9 Let R be a ring which satisfies acc (dcc) on semi-essential ideals. If I is an ideal of R such that $<0> \neq I \subseteq L(R)$. Then R/I satisfies acc (dcc) on semi-essential ideals. Where L(R) = intersection of all prime ideals of R.

Proof. Let R satisfies acc on semi-essential ideals. Assume $J_1/I \subseteq J_2/I \subseteq \dots$ is an ascending chain of semi-essential ideals in R/I. Then $J_1 \subseteq J_2 \subseteq \dots$On the other hand, $I \subseteq L(R)$, so $I \subseteq P$ for any prime ideal P of R. It follows that P/I is a prime ideal of R/I for each prime ideal P of R. Moreover, we can see that J_i is a semi - essential ideal of R, $\forall i = 1,2, \dots$ as follows: suppose that J_i is not a semi-essential ideal for some i, so that there exists a nonzero prime ideal P of R, such that $J_i \cap P = <0>$, which implies that $(J_i/I) \cap (P/I) = < o_{R/I} >$. Thus J_i/I is not semi-essential ideal, which is a contradiction. Therefore $J_1 \subseteq J_2 \subseteq \dots$ is an ascending chain of semi-essential ideals in R, hence $\exists n \in Z_+$ such that $J_n = J_{n+1} = \dots$ Thus $J_n /I = J_{n+1} /I = \dots$ and R/I satisfy acc on semi-essential ideals.
Similarly, R/I satisfies dcc on semi-essential ideals. If R satisfies dcc on semi-essential ideals & $<0> \neq I \subseteq L(R)$.
Recall that, if R is a ring, then socle of R (denoted by Soc(R)) is the intersection of all essential ideals (5). We introduce the following:

Definition 1.10 Let R be a ring. The intersection of all semi-essential ideals of R is called a semi socle of R and denoted by S.Soc(R).
It is clear that $S.Soc(R) \subseteq Soc(R)$, and any semi-essential ideal I of R, $I \supseteq S.Soc(R)$.
In (3), the following had been proved:
(1)If R satisfies dcc on essential ideals, then Soc® is essential.
(2)If R satisfies dcc on essential ideals, then R satisfies acc on essential ideals.
We prove the following:

Proposition 1.11 If R satisfies dcc on semi-essential ideals and S.Soc(R) is semi-essential, then R satisfies acc on semi-essential ideals.

Proof. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ be an ascending chain of semi-essential ideals of R. But, $I_i \supseteq$ S.Soc(R), $\forall i = 1,2,\ldots$ and since S.Soc(R) is semi-essential, I_i is semi-essential. Then I_1 / S.Soc(R) $\subseteq I_2$ /S.Soc(R) $\subseteq \ldots$ is an ascending chain of ideals in R/S.Soc(R), but R/S.Soc(R) is Artinian,by Prop.1.7. It follows that R/S.Soc(R),is Noetherian, hence $\exists n \in Z_+$ such that I_n/S.Soc(R) = I_{n+1} /S.Soc(R) =... Thus $I_n = I_{n+1} = \ldots$

Proposition 1.12 Let R be a ring such that S.Soc(R) is a semi-essential ideal. Then R satisfies acc (dcc) on semi-essential ideals $\Leftrightarrow$ R/S.Soc(R) is Noetherian (Artinian).

Proof. ($\Rightarrow$) It follows, by Prop. 1.7 .

($\Leftarrow$) If R/S.Soc(R) is Noetherian. Assume $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ is an ascending chain of semi-essential ideals of R. But, $I_i \supseteq$ S.Soc(R), $\forall i = 1,2,\ldots$, hence I_1 / S.Soc(R) $\subseteq I_2$ /S.Soc(R) $\subseteq \ldots$ is an ascending chain of ideals in R/S.Soc(R). Hence, it must terminate and so the chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ terminates.

A similar proof, R satisfies dcc on semi-essential ideals, if R/S.Soc(R) is an Artinian ring.

Now, we trun out attention on the direct sum of rings, which satisfy acc (dcc), but first, we prove the following results:

Proposition 1.13 Let I, J be ideals of rings R_1 and R_2 respectively, let $R = R_1 \oplus R_2$. If I is semi-essential or J is semi-essential, then $I \oplus J$ is a semi-essential ideal in R.

Proof. If I is a semi-essential ideal in R_1. To prove $I \oplus J$ is a semi-essential ideal in R. Let P be a prime ideal of R. Then, either $P = P_1 \oplus R_2$ or $P = R_1 \oplus P_2$, where P_1 is a prime ideal of R_1, P_2 is a prime ideal of R_2, by (7, Exc.4, p. 53). Thus, either

(1)$(I \oplus J) \cap P = (I \oplus J) \cap (P_1 \oplus R_2) = (I \cap P_1) \oplus (J \cap R_2) = (I \cap P_1) \oplus J$, and since I is semi-essential, $I \cap P_1 \neq \langle 0 \rangle$. Thus $(I \cap P_1) \oplus J \neq \langle 0 \rangle$, or

(2) $(I \oplus J) \cap P = (I \oplus J) \cap (R_1 \oplus P_2) = (I \cap R_1) \oplus (J \cap P_2) = I \oplus (J \cap P_2)$, and since J is semi-essential, $J \cap P_2 \neq \langle 0 \rangle$. Thus $I \oplus (J \cap P_2) \neq \langle 0 \rangle$.

Therefore $I \oplus J$ is a semi-essential ideal in R.

Similarly, $I \oplus J$ is a semi-essential ideal in R, if J is a semi-essential ideal in R_2 .

Corollary 1.14 Let R_1, R_2 be rings, let $R = R_1 \oplus R_2$, if I is a semi-essential ideal in R_1 and J is a semi-essential ideal in R_2, then $I\oplus \langle 0\rangle$, $\langle 0\rangle \oplus J$ are semi-essential ideals in R.

Proof. It follows directly, by Prop. 1.13

Remark 1.15 The converse of Prop. 1.13 is not true, as the following example shows:

Let $R = Z_6 \oplus Z_{12}$.

Let $I = \langle \bar{2}\rangle \oplus \langle \bar{4}\rangle$. Let P be a prime ideal of R. Then, either $P = Z_6 \oplus P_2$ or $P = P_1 \oplus Z_{12}$, where P_1, P_2 are a prime ideals in Z_6, Z_{12} respectively. Hence, if $P = Z_6 \oplus P_2$, then $I \cap P = (\langle \bar{2}\rangle \oplus \langle \bar{4}\rangle) \cap (Z_6 \oplus P_2) = \langle \bar{2}\rangle \oplus (\langle \bar{4}\rangle \cap P_2) \neq \langle 0\rangle$.

If $P = P_1 \oplus Z_{12}$, then $I \cap P = (\langle \bar{2}\rangle \oplus \langle \bar{4}\rangle) \cap (P_1 \oplus Z_{12}) = (\langle \bar{2}\rangle \cap P_1) \oplus \langle \bar{4}\rangle \neq \langle 0\rangle$. It follows that I is a semi-essential ideal in R, but $\langle \bar{2}\rangle$ is not a semi-essential ideal in Z_6, $\langle \bar{4}\rangle$ is not a semi-essential ideal in Z_{12}.

Next we have the following:

Proposition 1.16 Let R_1, R_2 be rings, let $R = R_1 \oplus R_2$. If R satisfies acc (dcc) on semi-essential ideals, then R_1 and R_2 satisfy acc (dcc) on semi-essential ideals.

Proof. Assume R satisfy acc on semi-essential ideals. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ be an ascending chain of semi-essential ideals in R_1. Then $I_1 \oplus \langle 0\rangle \subseteq I_2 \oplus \langle 0\rangle \subseteq \ldots$. But $I_i \oplus \langle 0\rangle$ is semi-essential ideal in R, $\forall i = 1,2, \ldots$(by Cor.1.14), hence the chain $I_1 \oplus \langle 0\rangle \subseteq I_2 \oplus \langle 0\rangle \subseteq \ldots$ is an ascending chain of semi-essential ideals in R, so it must terminate. It follows that the chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ terminates. Thus R_1 satisfies acc on semi-essential ideals.

Similarly R_1 satisfies dcc on semi-essential ideals, R_2 satisfies acc (dcc) on semi-essential ideals.

S.2 Semi – Uniform Rings.

In this section, we study semi – uniform rings and give one of their basic properties.

Definition 2.1 A ring R is called semi – uniform ring, if every nonzero ideal of R is semi-essential, (4).

Remarks 2.2

(1)Every uniform ring is semi – uniform, but the converse is not true, as the following example shows:

The ring Z_{36} is semi – uniform, but it is not uniform.

(2)Every chained ring is uniform, so it is semi- uniform.

(3)The homomorphic image of a semi- uniform ring need not be semi- uniform ring, for example:

Let $f : (Z,+,.) \rightarrow (Z_{12}, +_{12}, \cdot_{12})$ defined by $f(x) = \bar{x}$, $\forall x \in Z$.

It is clear that f is a ring homomorphism and onto.

However, Z is uniform and hence semi- uniform, but Z_{12} is not semi-uniform, since the ideal $<\bar{4}>$ of Z_{12} is not semi-essential.

Theorem 2.3 Let R_1, R_2 be rings, let $R = R_1 \oplus R_2$. R is a semi-uniform ring $\Leftrightarrow$ R_1 and R_2 are semi- uniform rings.

Proof. ($\Rightarrow$) If R is semi- uniform ring. To prove R_1 is semi- uniform. Let I be a non zero ideal of R_1, then $I\oplus <0>$ is an ideal of R, so it is semi- essential. Now, if P is any prime ideal of R_1, then by (7, Exc.4, p. 53), $P \oplus R_2$ is a prime ideal of R. It follows that $(I\oplus <0>) \cap (P \oplus R_2) \neq <0>$, which implies that $(I \cap P) \oplus <0> \neq <0>$. Thus $I \cap P \neq <0>$, and so I is a semi- essential ideal in R_1. Therefore R_1 is a semi-uniform ring.

Similarly $R_2$is semi- uniform.

($\Leftarrow$) If R_1 and R_2 are semi - uniform rings. To prove R is semi-uniform. Let I be any non zero ideal of R, then $I = J\oplus K$, for some ideals J, K of R_1, R_2 respectively. Since $I \neq <0>$, either $J \neq <0>$ or $K\neq <0>$. Assume $J \neq <0>$. Let P be any prime ideal of R. By (7 , Exc.4 ,p. 53), either $P = P_1\oplus R_2$ or $P = R_1\oplus P_2$ for some prime ideals P_1, P_2 of R_1, R_2 respectively. So that, if $P = P_1\oplus R_2$, then $I\cap P = (J\oplus K) \cap (P_1 \oplus R_2) = (J\cap P_1)\oplus (K\cap R_2) = (J\cap P_1)\oplus K$. But J is a nonzero ideal of R_1 and R_1 is semi- uniform, so J is semi-essential and hence $J\cap P_1 \neq <0>$. It follows that $I\cap P \neq <0>$. If $P = R_1\oplus P_2$, then $I\cap P = (J\oplus K) \cap (R_1 \oplus P_2) = J\oplus(K \cap P_2)$, which is non zero, since $J \neq <0>$. Thus $I\cap P \neq <0>$.

Similarly, if $K \neq <0>$, then $I\cap P \neq <0>$. Therefore I is semi-essential. Thus R is semi uniform.

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مجلة ابن الهيثم للعلوم الصرفة والتطبيقية المجلد20 (2) 2007

خاصيتي السلسلة الصاعدة(النازلة) على المثاليات شبه الواسعة والحلقات شبه المنتظمة

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الخلاصة

لتكن R حلقة ابدالية ذات محايد. في هذا البحث درسنا الحلقات التي تحقق خاصيتي السلسلة على المثاليات شبه الواسعة كتعميم للحلقات التي تحقق خاصيتي السلسلة على المثاليات الواسعة ، إذ إنه إذا كان I مثالي غير صفري في R، فان I يسمى مثاليا شبه واسع إذا كان $I \cap P \neq (0)$ لكل مثالي غير صفري أولي P في R . فضلا عن ذلك درسنا الحلقات شبه المنتظمة، إذ إن الحلقة R تسمى شبه منتظمة ، إذا كان كل مثالي غير صفري فيها شبه واسع .