A Numerical solution of Transport Problems

Taghreed Abdul Kareem

College of Education, University of misan

Abstract: We apply the variational iteration method for solving the transport problems. The suggested algorithm is quite efficient and is practically well suited for use in these problems. The proposed iterative scheme finds the solution without any discritization, linearization, or restrictive assumptions. Several examples are given to verify the reliability and efficiency of the method. The composition is presented between variational iteration method and Adomian Decomposition Method(ADM) and we found that the proposed technique solves this type of problems without using Adomian's polynomials can be considered as a clear advantage of this algorithm over the decomposition method and the result accuatte than ADM .several examples are given to verify the reliability and efficiency of the method.

Key Words; variational iteration method, transport problems.

1. Introduction

In this paper, our work stems mainly from the variational iteration method[4,7-11,13]. The basic motivation of this paper is to propose mathematical technique without imposing perturbation, restrictive assumptions or linearization. It is used to solve effectively, easily, and accurately a large class of non-linear problems with approximations converging rapidly to accurate solutions, where the approximate solution of the VIM in the main is readily obtained upon using the obtained Lagrange multiplier and on the selective initial approximate. The variational iteration method changes the differential equation to a recurrence sequence of functions, where the limit of that sequence is considered as the solution of the partial differential equations. The main advantage of the method is that it can be applied directly to all types of nonlinear differential and integral equations, homogeneous or inhomogeneous, with constant or variable coefficients [1, 14-16]. Moreover, the proposed method is capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution.

The transport problems has also been widely employed as a model for chemical reaction processes and this usually entails the inclusion of lower-order

terms that describe convection and reaction [12,5]. the considered convection-diffusion problem involves many tubes inside which convection occurs transport equations have been sought in terms of cross-section averaged fields [2, 3, 20]. In these cases it is crucial to understand how the micro scale flow may be approximated by averaged models. Many research areas such as chemical engineering, biomechanics, and porous media are interested by variants of such a simple generic convection-diffusion problem.
In this paper, we investigate the model of unsteady convection-diffusion by using VIM. This study shows that, in this particular context, an averaged description can capture only large scale features of the exact solution, the convergence of which can be made as precise as necessary. Numerical methods have provided solutions to problems satisfying a fairly wide range of conditions. Among them are restrictive the explicit predictor method [17], the alternating direction implicit (ADI) method [18], and Taylor's approximation [6].
Consider the following transport equation:

$$\frac{\partial U}{\partial t} + A_1(x,y)\frac{\partial U}{\partial x} + A_2(y)\frac{\partial U}{\partial y} + (B_1\frac{\partial^2 U}{\partial x^2} + B_2\frac{\partial^2 U}{\partial y^2}) = 0 \quad ,in\ \Omega\times J$$

$$U(x,y,t) = H_1(x,t) \quad ,on\ \partial\Omega\times J$$

$$U(x,y,0) = H_2(x) \quad ,in\ \Omega\ , \qquad \ldots(1)$$

where $\Omega = (0, 1) \times (0, 1)$, $J = (0, T)$, A1(x, y), A2 (y) are smooth functions and H1, H2 are positive constants.

The paper is organized as follows. The second section presents a generalization of the VIM. The third section the sufficient conditions are presented to guarantee the convergence of the method. The fourth section given the examples and Some numerical results to illustrate the effectiveness and the useful of the variational iteration method. In section 5, we presented discussion of our work. at the last section the Conclusions are presented.

2. Variational Iteration Method:

To illustrate the basic concept of the technique, we consider the following general differential Equation

$$L(u) + N(u) + R(u) = g(x,t), \qquad \ldots (2a)$$

with specified initial condition: $u_0 = u(x,0)$
…(2b)

where L is a linear operator , N is a nonlinear operator and R is a linear operator, and g(x) is an inhomogeneous term. According to the VIM [3,10,13], we can construct a correction functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)[L(u_n(x,\tau)) + N(\tilde{u}_n(x,\tau) + R(\tilde{u}_n(x,\tau) - \tilde{g}(x,\tau)]d\tau \quad \ldots(3)$$

where $\lambda(\tau)$ is called general Lagrange multiplier [1,15,16], which can be identified optimally via the variational theory and integration by parts. The iterates u_n denote the nth order approximate solutions, where n refers to the number of iterates. $\tilde{u}_n$ is considered as restricted variations so that their variations are zero, $\delta \tilde{u}_n = 0$ [8]. The successive approximation u_{n+1}, $n \geq 0$ of the solution $u(x,t)$ will be obtained by using the determined Lagrange multiplier and any selective function u_0.

To find the optimal value of $\lambda(\tau)$, we applied the restricted variations of correction functional (3) and integrating by part, noticing that $\delta u(0,x) = 0$, in the following form:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\tau)[(u_n(x,\tau))_\tau + R(\tilde{u}_n(x,\tau)) + N(\tilde{u}_n(x,\tau)) - \tilde{g}(x,t)]d\tau$$

$$= \delta u_n(t) + \lambda \delta u_n(\tau)\Big|_{\tau=t} - \int_0^t \lambda' \delta u_n(\tau) d\tau = 0$$

yields the following stationary conditions:

$$\delta(u_n) : \lambda' = 0$$

$$\delta(u_n) : 1 + \lambda\Big|_{\tau=t} = 0$$

So, the Lagrange multiplier in this case can be identified as follows: $\lambda = -1$

Consequently, we can write the equation (3) as a successive approximation as follows:

$$u_1(x,t) = u_0(x,t) + \int_0^t (-1)[L(u_0(x,\tau)) + R(u_0(x,\tau)) + N(u_0(x,\tau)) - g(x,t)]d\tau$$

$$u_2(x,t) = u_1(x,t) + \int_0^t (-1)[L(u_1(x,\tau)) + R(u_1(x,\tau)) + N(u_1(x,\tau)) - g(x,t)]d\tau$$

$$u_3(x,t) = u_2(x,t) + \int_0^t (-1)[L(u_2(x,\tau)) + R(u_2(x,\tau)) + N(u_2(x,\tau)) - g(x,t)]d\tau$$

So on, where by finding the nth order approximation. Finally summing up iterates to yields,

$$U_M = \sum_{n=0}^{M} u_n, \quad M \geq 1$$

The general solution obtained by the VIM can be written as: $u(x,t) = \lim_{M \to \infty} U_M$

3. Convergence Analysis of the VIM

In this section, we will study the convergence analysis as the same manner in [16] of the variational iteration method to the linear equations. we consider linear partial differential equations of the form:

$$L(u) + R(u) = g(x,t), \quad 0 \leq x \leq \beta \ , \ 0 \leq t \leq T \qquad \dots (4)$$

where, $L(u)$ and $R(u)$, are linear time and space derivative operators, respectively. Then we construct the following correction functional for $u(x,t)$

$$u_{m+1}(x,t) = u_m(x,t) + \int_0^t \lambda(\tau)[L(u_m) + R(u_m) - g(x,\tau)]d\tau \ , m = 0,1,2,3,... \qquad \dots (5)$$

Now, we show that the sequence $\{u_m(x,t)\}_{m=0}^{\infty}$ given by (5) with $u_0(x,t) = u_0$ converges to the exact solution of (4). To do this we state the following theorem.

Theorem 2 [16]: Let $u(x,t) \in (C^2(\Re))^n, (x,t) \in \Re = [0,\beta] \times [0,T]$ be the exact solution of (4) and $u_m(x,t) \in (C^2(\Re))^n$ be the solution of the sequence

$$u_{m+1}(x,t) = u_m(x,t) + \int_0^t \lambda(\tau)[L(u_m) + R(u_m) - g(x,\tau)]d\tau$$

, with $u_0(x,t) = u_0$.

If $E_{m+1}(x,t) = u_m(x,t) - u(x,t)$ and $\|R(E_m(x,t))\|_2 \leq \|E_m(x,t)\|_2$, then the functional sequence $\{u_m(x,t)\}_{m=1}^{\infty}$ defined by (5) converge to $u(x,t)$.

4. Numerical Examples

To show the efficiency of our method described in the previous part, we present some examples of 2D transport problem.

Example1 We consider the following homogeneous transport problem[19]

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0 \quad ,(x,y,t)\in\Omega\times J$$

subject to the initial condition

$$u(x,y,0)=\exp[-(x-0.05)^2-(y-0.05)^2]$$

The exact solution is given by

$$u(x,y,t)=\frac{1}{4t+1}\exp[\frac{-(x-t-0.05)^2}{4t+1}-\frac{(y-t-0.05)^2}{4t+1}] \quad \ldots(6)$$

To solve the problem (6) by using VIM, we consider a correction functional (3) as:

$$u_{n+1}(x,y,t)=u_n(x,y,t)+\int_0^t \lambda(\tau)[\frac{\partial u_n(x,y,\tau)}{\partial t}+\frac{\partial \tilde{u}_n(x,y,\tau)}{\partial x}+\frac{\partial \tilde{u}_n(x,y,\tau)}{\partial y}-\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial x^2}-\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial y^2}]d\tau \quad \ldots (7)$$

where, λ is a general Lagrange multiplier. The value of λ can be found by considering $\frac{\partial \tilde{u}_n(x,y,\tau)}{\partial x},\frac{\partial \tilde{u}_n(x,y,\tau)}{\partial y},\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial x^2}$ *and* $\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial y^2}$ as restricted variations (i.e. $\delta(\frac{\partial \tilde{u}_n(x,y,\tau)}{\partial x})=\delta(\frac{\partial \tilde{u}_n(x,y,\tau)}{\partial y})=\delta(\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial x^2})=\delta(\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial y^2})=0$ in equation(7), then integrating the result by part to obtain $\lambda=-1$. Then the correction functional (7) becomes in the following formula:

$$u_{n+1}(x,y,t)=u_n(x,y,t)-\int_0^t[\frac{\partial u_n(x,y,\tau)}{\partial t}+\frac{\partial u_n(x,y,\tau)}{\partial x}+\frac{\partial u_n(x,y,\tau)}{\partial y}-\frac{\partial^2 u_n(x,y,\tau)}{\partial x^2}-\frac{\partial^2 u_n(x,y,\tau)}{\partial y^2}]d\tau \quad \ldots (8)$$

Using the above iteration formulas (8) and the initial approximations, we can obtain the following approximations:

u1(x,y,t) = $\exp[-(x-0.05)^2-(y-0.05)^2$ -(-2x+0.1)
$\exp[-(x-0.05)^2-(y-0.05)^2$ t-(-2y+0.1)

$\exp[-(x-0.05)^2-(y-0.05)^2$ t-4 $\exp[-(x-0.05)^2-(y-0.05)^2$ t+(-2x+0.1)2 $\exp[-(x-0.05)^2-(y-0.05)^2$ t+(-2y+0.1)2 $\exp[-(x-0.05)^2-(y-0.05)^2$ t

= $\exp[-(x-0.05)^2-(y-0.05)^2$ [1-(-2x+0.1) t -(-2y+0.1) t -4t+(-2x+0.1)2 t -(-2y+0.1)2 t].

So on.

we observed that the variational iteration solution has a good convergence to the exact solution, where from the theorem 1 we have the proof of convergence as the following form:

The error function can be written as the following

$$E_{m+1}(x,y,t)=E_m(x,y,t)-\int_0^t[\frac{\partial E_m(x,y,\tau)}{\partial \tau}+\frac{\partial E_m(x,y,\tau)}{\partial x}+\frac{\partial E_m(x,y,\tau)}{\partial y}-\frac{\partial^2 E_m(x,y,\tau)}{\partial x^2}-\frac{\partial^2 E_m(x,y,\tau)}{\partial y^2}]d\tau$$

By using integration by parts we conclude that

$$E_{m+1}(x,y,t)=E_m(x,y,t)-E_m(x,y,\tau)\Big|_0^t-\int_0^x[\frac{\partial E_m(x,y,\tau)}{\partial x}+\frac{\partial E_m(x,y,\tau)}{\partial y}-\frac{\partial^2 E_m(x,y,\tau)}{\partial x^2}-\frac{\partial^2 E_m(x,y,\tau)}{\partial y^2}]d\tau$$

We know that $E_{m+1}(x,y,0)=0,\ m=0,1,2,\ldots,$ and therefore:

$$E_{m+1}(x,y,\tau)=-\int_0^x[\frac{\partial E_m(x,y,\tau)}{\partial x}+\frac{\partial E_m(x,y,\tau)}{\partial y}+(-\frac{\partial^2 E_m(x,y,\tau)}{\partial x^2})+(-\frac{\partial^2 E_m(x,y,\tau)}{\partial y^2})]d\tau$$

Therefore

$$\|E_{m+1}(x,y,t)\|=\left\|\int_0^x[\frac{\partial E_m(x,y,\tau)}{\partial x}+\frac{\partial E_m(x,y,\tau)}{\partial y}+(-\frac{\partial^2 E_m(x,y,\tau)}{\partial x^2})+(-\frac{\partial^2 E_m(x,y,\tau)}{\partial y^2})]d\tau\right\|$$

$$\|E_{m+1}(x,y,t)\|_2\le\int_0^x\left\|\frac{\partial E_m(x,y,\tau)}{\partial x}\right\|+\left\|\frac{\partial E_m(x)}{\partial y}\right\|x,y,\tau+\left\|\frac{\partial^2 E_m(x,y,\tau)}{\partial x^2}\right\|+\left\|\frac{\partial^2 E_m(x,y,\tau)}{\partial y^2}\right\|ds$$

Since $\left\|\frac{\partial E_m(x,y,\tau)}{\partial x}\right\|_2\le\|E_m(x,y,\tau)\|_2,\ \left\|\frac{\partial^2 E_m(x,y,\tau)}{\partial x^2}\right\|_2\le\|E_m(x,y,\tau)\|_2$

we have $\|E_{m+1}(x,y,\tau)\|_2\le\int_0^x M\|E_m(x,y,\tau)\|_2\,ds$

Then for $0\le t\le L$ we obtain

$$\|E_1(x,y,t)\|_2\le M\int_0^x\|E_0(s,y,\tau)\|_2\,ds\le M\max_{\tau\in[0,T],\,x,y\in[0,L]}\|E_0(x,y,\tau)\|_2\int_0^t d\tau$$

$$=M_2\max_{\tau\in[0,T],\,x,y\in[0,L]}\|E_0(x,y,t)\|_2\,t,$$

$$\left\|E_2(x,y,t)\right\|_2 \le M \int_0^t \left\|E_1(x,y,\tau)\right\|_2 \, ds \le M^2 \max_{\tau\in[0,T],\, x,y\in[0,L]} \left\|E_0(x,y,\tau)\right\|_2 \int_0^x \tau \, d\tau$$

$$= M^2 \max_{\tau\in[0,T],\, x,y\in[0,L]} \left\|E_0(x,y,t)\right\|_2 \frac{t^2}{2!},$$

$$\left\|E_3(x,y,t)\right\|_2 \le M \int_0^t \left\|E_2(x,y,\tau)\right\|_2 \, d\tau \le M^3 \max_{\tau\in[0,T],\, x,y\in[0,L]} \left\|E_0(x,y,\tau)\right\|_2 \int_0^t \frac{\tau^2}{2!} \, d\tau$$

$$= M^3 \max_{\tau\in[0,T],\, x,y\in[0,L]} \left\|E_0(s,y)\right\|_2 \frac{t^3}{3!},$$

$\vdots$

$$\left\|E_m(x,y,t)\right\|_2 \le M \int_0^t \left\|E_{m-1}(x,y,\tau)\right\|_2 \, d\tau \le M^m \int_0^t \max_{\tau\in[0,T],\, x,y\in[0,L]} \left\|E_0(x,y,\tau)\right\|_2 \frac{\tau^{m-1}}{(m-1)!} \, d\tau$$

$$= \max_{\tau\in[0,T],\, x,y\in[0,L]} \left\|E_0(x,y,t)\right\|_2 \frac{(M_2\ t)^m}{m!},$$

We have

$$\max_{\tau,\, y\in[0,L]} \left\|E_0(x,y,\tau)\right\|_2 \frac{(M_2\ t)^m}{m!} \le \max_{\tau\in[0,T],\, x,y\in[0,L]} \left\|E_0(x,y,\tau)\right\|_2 \frac{(M_2\ L)^m}{m!} = \max_{\tau\in[0,T],\, x,y\in[0,L]} \left\|E_0(x,y,\tau)\right\|_2 \frac{(4L)^m}{m!}$$

Where, M=4 is a constant . Therefore $\left\|E_m(x,y,t)\right\|_2 \to 0$ as $m \to \infty$ and hence the functional sequence $\{u_m(x,y,t)\}_{m=1}^{\infty}$ converges to $u(x,y,t)$.

Therefore, the proof is completed. □

We explained the Comparison between the variational iteration method and Adomian Decomposition Method [19] for examples 1 and 2 in tables 1 and 2

Table 1: Comparison between the VIM and the ADM solutions with exact solution at t=0.1 .

$x=y$	u_e	$u_{6(VIM)}$	$u_{8(VIM)}$	$\lvert u_e - u_{6(VIM)}\rvert$	$\lvert u_e - u_{8(VIM)}\rvert$	$\lvert u_e - u_{6(ADM)}\rvert$
0.2	0.6360083002	0.6361891213	0.6360056232	1.8082E-4	2.6768 E-6	1.07142E-3
0.4	0.6638583223	0.6633757839	0.6637679406	4.8253E-4	9.0381 E-5	9.38058E-4
0.6	0.6180937749	0.6176824587	0.6180724912	4.1131E-4	2.1283 E-5	1.65995E-4
0.8	0.5133335863	0.5135497714	0.513396939	2.1618E-4	6.3348 E-5	5.81031E-4
1.0	0.3802868740	0.3806033402	0.3802965943	3.1646E-4	9.7201 E-6	4.67080E-5

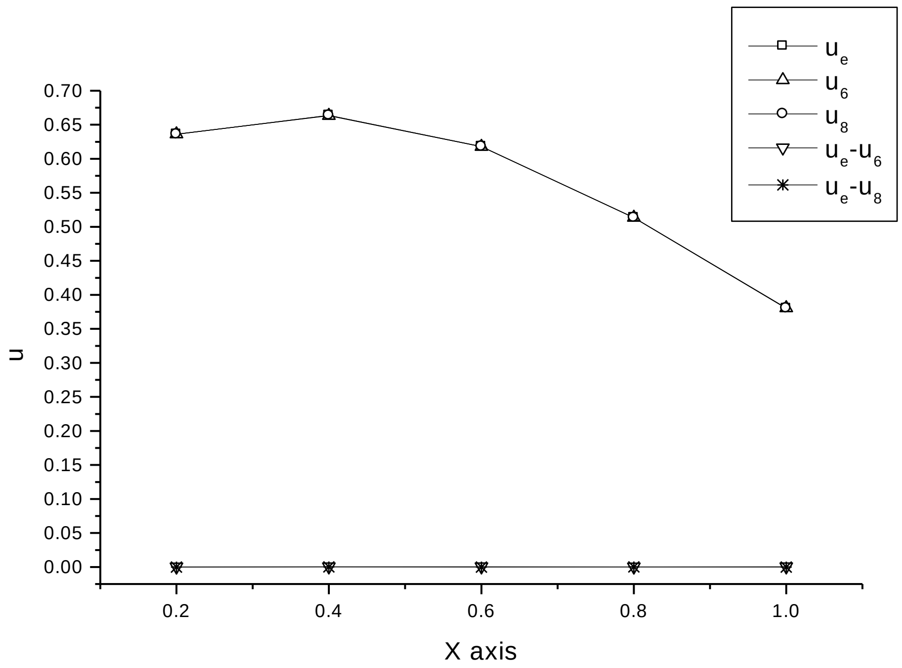

Figure (1) Comparison between exact solution and VIM solutions for u(x,y,t) at $t = 0.1$

Example 2 : We next consider the following transport problem[19]

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \quad ,(x, y, t) \in \Omega \times J \qquad \ldots(9)$$

subject to the initial condition $u(x, y, 0) = \sin(\pi x)\sin(\pi y)$

To solve the problem (9) by using VIM, we consider a correction functional (3) as:

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^t \lambda(\tau)[\frac{\partial u_n(x,y,\tau)}{\partial t} - \frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial x^2} - \frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial y^2}]\, d\tau$$

… (10)

The value of λ can be found by considering $\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial x^2}$ *and* $\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial y^2}$ as restricted variations (i.e. $\delta(\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial x^2}) = \delta(\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial y^2}) = 0$ in equation(10), then integrating the result by part to obtain $\lambda = -1$. Then the correction functional (10) becomes in the following formula:

$$u_{n+1}(x,y,t) = u_n(x,y,t) - \int_0^t [\frac{\partial u_n(x,y,\tau)}{\partial t} - \frac{\partial^2 u_n(x,y,\tau)}{\partial x^2} - \frac{\partial^2 u_n(x,y,\tau)}{\partial y^2}]\, d\tau$$

… (11)

Using the above iteration formulas (11) and the initial approximations, we can obtain the following approximations:

$$u_1 := \sin(\pi x)\sin(\pi y) - 2\sin(\pi x)\,\pi^2 \sin(\pi y)\, t$$

$$u_2 := \sin(\pi x)\sin(\pi y) - 2\sin(\pi x)\,\pi^2\sin(\pi y)\, t + 2\sin(\pi x)\,\pi^4\sin(\pi y)\, t^2$$

$$u_3 := \sin(\pi x)\sin(\pi y) - 2\sin(\pi x)\,\pi^2\sin(\pi y)\, t + 2\sin(\pi x)\,\pi^4\sin(\pi y)\, t^2 - \frac{4}{3}\sin(\pi x)\,\pi^6\sin(\pi y)\, t^3$$

and so on;

The solution for the convection-diffusion equation (9) in a series form is given by

$$u(x,y,t) = \sin(\pi x)\sin(\pi y)(1 - 2\pi^2 t + \frac{(2\pi^2 t)^2}{2} - \frac{(2\pi^2 t)^3}{6} + \ldots)$$

It can be easily observed that the above series is equivalent to the exact solution[19]

$$u(x,y,t) = e^{-2t\pi^2}\sin(\pi x)\sin(\pi y).$$

Now , when apply theorem 1 to this problem the condition of convergence is achieved, and has the following form:

$$\left\|E_m(x,y,t)\right\|_2 \le M\int_0^t \left\|E_{m-1}(x,y,\tau)\right\|_2 d\tau \le \max_{\tau\in[0,T],\, x,y\in[0,L]} \left\|E_0(x,y,\tau)\right\|_2 \frac{(M\ t)^m}{m!},$$

$$\le \max_{\tau\in[0,T],\, x,y\in[0,L]} \left\|E_0(x,y,\tau)\right\|_2 \frac{(M_2 T)^m}{m!}$$

where, $M = 2$ is a constant. Therefore $\left\|E_m(x,y,t)\right\|_2 \to 0$ and as $m \to \infty$ hence the functional sequence $\{u_m(x,y,t)\}_{m=1}^{\infty}$ converges to $u(x,y,t)$.

Table 2: Comparison between the VIM and the exact solution ,and the Absolute errors for several iterations of the VIM solution at t=0.1 .

$x = y$	u_e	$u_{8(VIM)}$	$u_{10(VIM)}$	$\lvert u_e - u_{8(VIM)} \rvert$	$\lvert u_e - u_{10(VIM)} \rvert$
0.1	0.01325407743	0.01335459157	0.01325774805	1.0051 E-4	3.6705 E-6
0.3	0.09082416342	0.09151294104	0.09084931644	6.8877 E-4	2.5153 E-5
0.5	0.1386904783	0.1397422568	0.1387288876	1.0517 E-3	3.8409 E-5
0.7	0.09065734670	0.09134485969	0.09068245396	6.8751 E-4	2.5107 E-5
0.9	0.01315113952	0.01325087288	0.01315478149	9.9733 E-5	3.6420 E-6

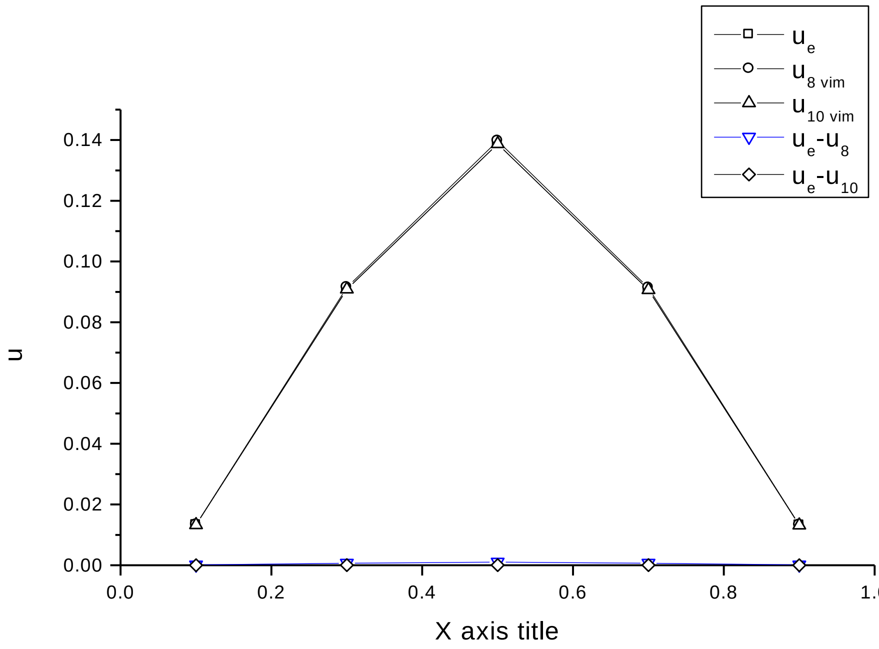

Figure (2) Comparison between exact solution and VIM solutions for u(x,y,t) at $t = 0.1$.

Example 3: We next consider the following non homogeneous transport problem[19]

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=3x^2-6x+2t+1 \quad ,(x,y,t)\in \Omega\times J$$

subject to the initial condition $u(x,y,0)=x^3+y$

The exact solution is given by $u(x,y,t)=x^3+y+t^2$ …(12)

To solve the problem (12) by using VIM, we consider a correction functional (3) as:

$$u_{n+1}(x,y,t)=u_n(x,y,t)+\int_0^t \lambda(\tau)[\frac{\partial u_n(x,y,\tau)}{\partial t}+\frac{\partial \tilde{u}_n(x,y,\tau)}{\partial x}+\frac{\partial \tilde{u}_n(x,y,\tau)}{\partial y}-\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial x^2}-\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial y^2}-\tilde{g}(x,y,t)]\,d\tau$$

… (13)

where $g(x,y,t)$ is the non homogeneous term. The value of λ can be found by considering $\frac{\partial \tilde{u}_n(x,y,\tau)}{\partial x},\frac{\partial \tilde{u}_n(x,y,\tau)}{\partial y},\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial x^2},\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial y^2}$ *and* $\tilde{g}(x,y,t)$ as restricted variations (i.e. $\delta(\frac{\partial \tilde{u}_n(x,y,\tau)}{\partial x})=\delta(\frac{\partial \tilde{u}_n(x,y,\tau)}{\partial y})=\delta(\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial x^2})=\delta(\frac{\partial^2 \tilde{u}_n(x,y,\tau)}{\partial y^2})=\delta\,\tilde{g}(x,y,t)=0$ in equation(13), then integrating the result by part to obtain $\lambda=-1$. Then the correction functional (13) becomes in the following formula:

$$u_{n+1}(x,y,t)=u_n(x,y,t)-\int_0^t[\frac{\partial u_n(x,y,\tau)}{\partial t}+\frac{\partial u_n(x,y,\tau)}{\partial x}+\frac{\partial u_n(x,y,\tau)}{\partial y}-\frac{\partial^2 u_n(x,y,\tau)}{\partial x^2}-\frac{\partial^2 u_n(x,y,\tau)}{\partial y^2}-g(x,y,t)]\,d\tau$$

… (14)

Using the above iteration formulas (14) and the initial approximations, we can obtain the following approximations:

$$u_1(x,y,t)=x^3+6xt-t+3tx^2+t^2-6xt+t+y-3tx^2=x^3+t^2+y$$

$$u_2(x,y,t)=0$$

$$u_3(x,y,t)=0,$$

We observe the appearance of noise terms between the components of u1. we obtain the exact solution in the form [19] $u(x,y,t)=x^3+y+t^2$.

5. Discussion

In this paper, we have used the VIM for solving the Unsteady Convection-Diffusion Problems. The initial condition as a function of x and y solution region of this problem is bounded by $0 \le x, y \le 1$, $t \ge 0$. We should be note that only 6-10 iterations were needed to obtain the approximately accurate solutions for the examples 1 and 2, i.e. when $n \ge 6$ the results are converging to the exact solution $u(x, y, t)$. The obtained results by using VIM are compared with the exact solution, which correspond to the various values of x and y . Also, we presented the absolute errors for the solution in several iterations and compared it with absolute errors of Adomian Decomposition Method(ADM) for example 1, and we compared absolute errors for the solution in several iterations for example 2, all these errors are listed in Tables (1,2) and represented graphically in Figures (1,2) at $t = 0.1$. While, in example 3 we need only one iterate for arrive to the exact solution. The results show that the iterate approximation solutions obtained by using first sixth terms of this method are very well converged to the exact solution. From The tables, one can also see that the accuracy of this method increases with increasing the iterations. With other means, the errors are decreasing with increasing the number of iterations. The results we got from the VIM were better than the results obtained by (ADM) in accuracy.

6. Conclusions

In this paper, the variation iteration method has been successfully employed to obtain the approximate analytical solutions of Unsteady Convection-Diffusion Problems. The method has been applied directly without using linearization or any restrictive assumptions. The comparison of the numerical results of VIM with other solutions by using other methods show that the variational iteration method is a powerful mathematical tool to solving linear partial differential equations and faster in convergence to exact solution.

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حل مسالة التوصيل الحراري عدديا

م.م. تغريد عبدالكريم حاتم

كلية التربية/جامعة ميسان

الخلاصة:

في هذا البحث استخدمنا طريقة التغايرالتكرارية في حل مسالة التوصيل الحراري.الطريقة المقترحة كفوءة وملائمة لحل هذه المسالة. تم الحصول على الحل التحليلي باستخدام هذه الطريقة مباشرة من دون استخدام أي فرضيات.كما تم اعطاء الشرط الضروري والكافي للتقارب.المقارنة بين طريقتنا المقترحة وطريقة ادومين تم تقديمها حيث بينت انها لا تحتاج الى متعددات حدود لاكرانج كما في طريقة ادومين حيث يكون الحل بصزرة مباشرة كما ان نتائجها ادق من طريقة ادومين . ايضا تم تقديم امثلة لتوضيح عمل الطريقة وكفاءتها.