ω-Perfect Mappings

Y. Y. Yousif
Department of Mathematics, College of Education, Ibn Al-haitham ,University of Baghdad

Abstract
In this paper, we shall introduce a new kind of Perfect (or proper) Mappings, namely ω-Perfect Mappings, which are strictly weaker than perfect mappings. And the following are the main result: (a) Let f :X→Y be ω-perfect mapping of a space X onto a space Y, then X is compact (Lindeloff), if Y is so. (b) Let f : X→Y be ω-perfect mapping of a regular space X onto a space Y. then X is paracompact (strongly paracompact) if Y is so paracompact (strongly paracompact). (c) Let X be a compact space and Y be a p^*-space then the projection p:X×Y→Y is a ω-perfect mapping. Hence, X×Y is compact (paracompact, strongly paracompact) if and only if Y is so.

Introduction
The notion of perfect mapping was introduced in Bourbaki (1) and he stated and proved several theorems concerning mappings and Engliking (2) introduced equivalence definition of perfect mapping. Also the notion of ω-open set, ω-closed set and ω-closed mappings introduced in Hdeib (3). In this paper we introduce a new concept, which is the concept of ω-Perfect Mapping and prove new results.

Notations:

For any set X, |X| denotes the cardinal number of X.

ω, denotes the cardinal number of integers.

For other notions or notations not defined here we follow closely Engelking (2).

Basic Definitions

Definition 2.1 (2)
A point x of a space X is called a condensation point of the set $A \subseteq X$ if every arbitrary nbd of the point x contains an uncountable subset of this set.

Definition 2.2 (3)
A subset of a space X is called ω-closed if it contains all its condensation points. The complement of a ω-closed set is called ω-open set.

Observe that A is ω-open if and only if for every $x \in A$ there is an open nbd U of x with $|U-A| \leq \omega$.

Definition 2.3 (3)
A mapping $f : X \rightarrow Y$ is called ω-closed mapping if it is maps closed sets onto ω-closed sets.

Definition 2.4 (3)
A mapping $f : X \rightarrow Y$ is called compact mapping if for each compact closed subset K of Y , $f^{-1}(K)$ is compact.

Definition 2.5 (3)
A space X is called p^*-space if the intersection of countable many open sets is a ω- open set.

Some Basic Results

Now we are ready to introduce a new concept, which is the concept of ω-perfect mapping and definition here:

Definition 3.1
A mapping $f : X \rightarrow Y$ is called a ω-perfect mapping if it is continuous, ω-closed and for each $y \in Y$, $f^{-1}(y)$ is compact.

Remark: 3.2
Every perfect mapping is ω-perfect mapping. But the converse is not true for example.

Example: 3.3
Let (X,τ) be a topological space, we define a new topology τ' on X as follows: $G \in \tau'$ if and only if G is an ω-open set in τ, then τ' will be an expansion of τ and therefore will be Functionally Hausdorff and Urysohn if τ is so, now the identity mapping $f : (X,\tau) \rightarrow (X,\tau')$ will be

a ω-perfect mapping wich is not perfect mapping (since it is not closed).

Theorem: 3.4

(a) A ω-closed subset of compact space is compact.

(b) If $f : X \to Y$ is a continuous mapping from Hausdorff space X onto Y , then f is ω-perfect mapping if and only if for each $y \in Y$ and any open set U such that $f^{-1}(y) \subset U$, there exists an ω-open set O_y such that $y \in O_y$ and $f^{-1}(O_y) \subset U$.

(c) Every compact, ω-open subset A of a space X is of the form G\B, where G is open and B is a finite set, in particular A is G_δ-set.

Proof:

(a) Let X be a compact space, A be ω-closed subset of X and $\underset{\sim}{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of A , since A is ω-closed then X-A is ω-open so, for each $x \in X-A$, there is open set V_x such that $x \in V_x$ and $|V_x \cap A| \leq \omega$, also $\{U_\alpha : \alpha \in \Lambda\} \cup \{V_x : x \in X-A\}$ is an open cover of compact space X, so it has a finite subcover say $\{U_{\alpha_i} : i = 1,2,...,n\} \cup \{V_{x_i} : i = 1,2,...,n\}$, but $\bigcup_{i=1}^{n}\{V_{x_i} \cap A\}$ is finite, so for each $x_m \in \bigcup_{i=1}^{n}\{V_{x_i} \cap A\}$ choose $U_{x_m} \in \underset{\sim}{U}$ such that $x_m \in U_{\alpha_{x_m}}$, then $\{U_{\alpha_i} : i = 1,2,...,n\} \cup \{U_{\alpha_{x_m}} : x_m \in \bigcup_{i=1}^{n}[V_{x_i} \cap A]\}$ a finite subcover.

(b) (⇒) suppose that $f : X \to Y$ is ω-perfect mapping and $y \in Y$, let U be any open set such that $f^{-1}(y) \subseteq U$, then X-U is closed in X so f(X-U) is ω-closed in Y, let $O_y = Y - f(X-U)$, then O_y ω-open and $y \in O_y$ such that $f^{-1}(O_y) = X - f^{-1}[f(X-U)] \subseteq U$.

(⇐) suppose that the assumption hold, so it's eight to proof that the mapping $f : X \to Y$ is ω-closed, let A be any closed sub set of X and $y \in Y - f(A)$. By assumption there exist a ω-open set O_y such that $y \in O_y$ and $f^{-1}(O_y) \subset U$. It is easy to show that $O_y \subset Y - f(A)$. Hence Y-f(A) is ω-open then f(A) is ω-closed.

(c) Since A is ω-open, then for each $x \in A$ there is a nbd U_x of x such that $|U_x \cap (X-A)| \leq \omega$, now $\{U_x : x \in A\}$ is an open cover of A so it has a finite subcover $U_1, U_2, \ldots, U_n$; $A \subset \bigcup_{i=1}^{n} U_i$ where $|U_i \cap (X-A)| \leq \omega$ for each $i=1,2,\ldots,n$. Now $U_i \cap (X-A) = \bigcup_{m=1}^{n} \{x_{i,m}\}$, therefore $A = \cup [U_i \setminus \bigcup_{m=1}^{n} \{x_{i,m}\}] = \bigcup_{i=1}^{n} (U_i \setminus B)$ where $B \subset \bigcup_{m=1}^{n} \{x_{i,m}\}$.

Corollary3.5

Let X is hereditary compact space, then every ω-open subset of X is a G_δ-set. In particular every ω-open subset of the real line is a G_δ-set.

Proof:

Let A be an ω-open subset of X, since X is hereditary compact, then A is compact, by theorem 3.4 (c), we have A is G_δ-set.

Remark3.6

The converse of theorem 3.4 (c) is not true. For example take usual topology (IR, τ_u) and $A=[a,b] \subset IR$, then A is compact, G_δ-set, but A is not ω-open .

The following theorem is a generalization of the well known theorem that the compact (Lindloff) property is preserved under taking counter image by perfect mapping.

Theorem 3.7

Let $f : X \rightarrow Y$ be ω-perfect mapping of a space X onto a space Y, then X is compact, if Y is so.

Proof:

Let $\underset{\sim}{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X. since $f^{-1}(y)$ is compact, $f^{-1}(y) \subset \bigcup_{i=1}^{n} U_{\alpha_i}$. Denote $O_y = Y - f(X - \bigcup_{i=1}^{n} U_{\alpha_i})$. Since f is ω-closed, O_y is ω-open for each $y \in Y$, so there exists an open nbd O_y' of y such that $|O_y' \cap (X - O_y)| \leq \omega$, now $O_y' = [O_y \cap O_y'] \cup [O_y' \cap (X - O_y)]$ therefore, $f^{-1}(y)$ is contained in a union of finite member of $\underset{\sim}{U}$, since $\{O_y' : y \in Y\}$ is an open cover of Y and Y is compact, $\{O_y' : y \in Y\}$ has a finite subcover, therefore X is the union of finite many member of $\{f^{-1}(O_y') : y \in Y\}$, since each $f^{-1}(y)$ is contained in the union of finite many member of $\underset{\sim}{U}$, consequently X is the union of finite many member of $\underset{\sim}{U}$. Hence X is compact.

Corollary 3.8
Let f : X→Y be ω-perfect mapping of a space X onto a space Y, then X is lindeloff if Y is so.

Theorem 3.9
Let f: X→Y be ω-perfect mapping of a regular space X onto a space Y.

(a) If Y is paracompact, then X is paracompact.

(b) If Y is strongly paracompact, then X is strongly paracompact.

Proof:

(a) Let $\underset{\sim}{U}$ be an open cover of X. It suffices to show that $\underset{\sim}{U}$ has a σ-locally finite refinement. Since $f^{-1}(y)$ is compact then it is paracompact relative to X. for each y in Y, $\underset{\sim}{U}$ has an open locally finite refinement in X which cover $f^{-1}(y)$, say $\underset{\sim}{A}_y = \{A_\alpha : \alpha \in \Lambda_y\}$. Denote $O_y = Y - f(X - \bigcup_{\alpha \in \Lambda_y} A_\alpha)$. Since f is ω-closed, O_y is ω-open for each y in Y. hence there exists an open nbd O_y' of y such that $|O_y' \cap (X - O_y)| \leq \omega$. Put $O_y \cap O_y' = G_y$, $O_y' \cap (X - O_y) = H_y$. then $f^{-1}(H_y)$ is contained in the union of a σ-locally finite refinement of $\underset{\sim}{U}$ whose member are open in X. Also $f^{-1}(G_y) \subset \bigcup_{\alpha \in \Lambda_y} A_\alpha$. Therefore $f^{-1}(O_y')$ is covered by a σ-locally finite refinement B_y of $\underset{\sim}{U}$ whose members are open in X. since Y is paracompact, $\{ O_y' : y \in Y\}$ has an open locally finite refinement $\underset{\sim}{V}$ which covers Y. let $S = \{f^{-1}(V) \cap B : f^{-1}(V) \subset f^{-1}(O_y'), B \in B_y, V \in \underset{\sim}{V} \}$.

It is easy to see that S is an open σ-locally finite refinement of $\underset{\sim}{U}$.

The proof of (b) can be obtained by a similar method (one uses the characterization of strongly paracompactness in terms of star countable refinements.

Corollary 3.10

Let $f : X \rightarrow Y$ be a perfect mapping of a regular space X onto a strongly paracompact space Y. then X is strongly paracompact.

Corollary 3.11

Let $f : X \rightarrow Y$ be a perfect mapping of a regular space X onto a paracompact space Y, then X is paracompact.

Theorem 3.12

Let $f : X \rightarrow Y$ be continuous mapping from X onto Y, where Y is locally compact huasdorff p^*-space then f is ω-perfect mapping if and only if f is compact mapping.

Proof:

(⇒) follows from theorem (3.7)

(⇐) let $f : X \rightarrow Y$ be a continuous compact mapping, where Y is a locally compact, hausdorff, p^*-space. It suffices to show that f is ω-closed. Let F be a closed subset of X. assume that f (F) is not ω-closed so there exists a point $y_0 \in Y - f(F)$ such that for every nbd V of y_0, $|V \cap f(F)| > \omega$. Since Y is locally compact, there is an open nbd G of y_0 such that cl(G) is compact. Observe now $f(F) \cap cl(G)$ is not compact. Indeed, if it is, then it is easy to see that it is ω-closed, so there exists a nbd M of y_0 such that $|M \cap f(F)| \leq \omega$, which is impossible. Now cl(G) is compact so $f^{-1}[cl(G)]$ is compact and $F \cap f^{-1}[cl(G)]$ is compact subset of X, therefore $f[F \cap f^{-1}[cl(G)]] = f(F) \cap cl(G)$ is compact, which is a contradiction. Hence f(F) is ω-closed.

Theorem 3.13

Let X be a compact space and Y be a p^*-space then the projection $p : X \times Y \rightarrow Y$ is a ω-perfect mapping.

Proof:

We have $p : X \times Y \rightarrow Y$ is continuous and for each $y \in Y$, $p^{-1}(y) = X \times \{y\}$ is compact, since X is compact and {y} is compact. Now, let $y \in Y$ and U be an open set in $X \times Y$ such that $p^{-1}(y) = X \times \{y\} \subset U$. For each $(x,y) \in X \times \{y\}$, let O_x, $O_{y(x)}$ be open nbds of x and y such that $(x,y) \in O_x \times O_{y(x)} \subset U$. now $\{O_x : x \in X\}$ is an open cover of X, therefore it has a finite subcover $\{O_{x_i}\}_{i=1}^{n}$. Hence $X \times \{y\} \subset \bigcup_{i=1}^{n} O_{x_i} \times O_{y(x_i)} \subset U$. Let $O_y = \bigcap_{i=1}^{n} O_{y(x_i)}$ then

$X\times\{y\}\subset\bigcup_{i=1}^{n}O_{x_i}\times O_y\subset U$ and O_y is an ω-open set since Y is a p^*-space. Thus, for each y in Y, there is an ω-open set O_y , such that y $\in O_y$ and $p^{-1}(O_y)\subset U$. Therefore by theorem 3.4, p is a ω-closed mapping.

Theorem 3.14

Let X be a topological space and Y be a topological space such that there exists an F_δ-set which is not ω-closed. If the projection $p:X\times Y\rightarrow Y$ is ω-perfect mapping then X is countable compact.

Proof:

Let $\bigcup_{i=1}^{\infty}A_i$ be a F_δ-subset of Y which is not ω-close. Let X be not countable compact, then there exists a decreasing sequence $\{B_i\}_{i=1}^{\infty}$ of closed subset of X such that $\bigcap_{i=1}^{\infty}B_i=\phi$. Let $F=\bigcup_{i=1}^{\infty}(B_i\times A_i)$, then we can see that F is a closed subset of X×Y. now, for every point (x,y) in X×Y, p(x,y)=y, $P(F)=\bigcup_{i=1}^{\infty}A_i$ is not ω-close. Therefore the projection is not ω-closed. Which is a contradiction. Hence the result.

Theorem 3.15

A space Y is p^*-space if and only if for any compact space X the projection $p:X\times Y\rightarrow Y$ is a ω-perfect mapping.

Proof:

($\Rightarrow$) follows from theorem (3.13)

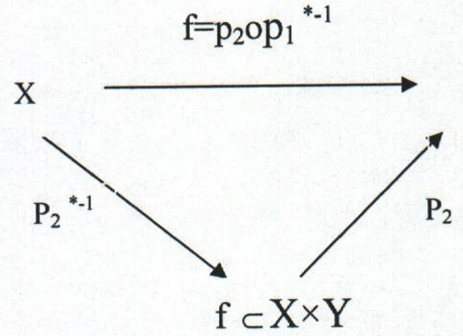
($\Leftarrow$) assume that Y is not p^*-space and for any compact space X the projection $p:X\times Y\rightarrow Y$ is ω-perfect. Let X=IR, the set of real numbers with the usual topology, then by theorem (3.14), X is countable compact which is a contradiction.

Theorem 3.16

Let X and Y be two space each with the property that every compact subset is ω-closed, if $f:X\rightarrow Y$ and f (considered as a subspace of X×Y) is compact, then f is weakly continuous (i.e., for every open set $U\subset Y$, $f^{-1}(U)$ is ω-open).

Proof:

Let p_1: X×Y→X and p_2: X×Y→Y. be the projections, then X and rang f are compact sets, as image of compact set under p_1 and p_2, let $p_1^{*}=p_1|_f$ .observe that p_1^{*} is ω-perfect mapping. Indeed, if $A\subseteq f$ is closed, then A is compact, so $p_1^{*}(A)$ is compact, hence it is ω-perfect mapping. Since f is a function defined on X, p_1^{*} is a bijection onto X. This, together with the fact p_1^{*} is ω-perfect mapping, implies that for every open set $V\subset f$, $p_1^{*}(V)$ is ω-open in X. now $f=p_2 o p_1^{*-1}$, hence f has the required property.



Theorem 3.17

Let X be a topological space and Y be a Hausdorff, p^{*}-space, then the foloowing holds:

(a) X×Y is compact if and only if Y is so.

(b) X×Y is paracompact if and only if Y is so.

(c) X×Y is strongly paracompact if and only if Y is so.

References

1. Bourbaki, N. (1975) General Topology, Addison-Wesley publishing company, Vol.1.
2. Englking, R. (1989) Outline of General Topology, Amsterdam.
3. Hdeib, H. Z. (1982) ω-closed mapping, Revista Colombian a de Mathematics, Vol. XVI, p(65-78).

التطبيقات التامة من النمط-ω

يوسف يعكوب يوسف

قسم الرياضيات، كليه التربية/ ابن الهيثم، جامعة بغداد

الخلاصة

في هذا البحث سنقدم نوعا" جديدا" من التطبيقات التامة (أو المناسبة) سميناها التطبيقات التامة من النمط-ω ، التي تعد وعلى سبيل الحصر أضعف من التطبيقات التامة، واهم النتائج التي توصلنا إليها هي : (ا) ليكن $f : X\to Y$ تطبيقا" شاملا" وتام من النمط-ω من الفضاء X إلى الفضاء Y. فان X يكون فضاء متراص (ليندلوف) أذا كان Y فضاء متراص (ليندلوف). (ب) ليكن $f : X\to Y$ تطبيقا" شاملا" وتام من النمط-ω من الفضاء المنتظم X إلى الفضاء Y، فان X يكون شبه متراص (شبه متراص قوي) أذا كان Y شبه متراص (شبه متراص قوي). (ج) ليكن X فضاء متراصا و Y فضاء p^*- فان الإسقاط $p:X\times Y\to Y$ يكون تطبيقا" تاما" من النمط-ω . لذلك $X\times Y$ يكون متراص (شبه متراص، شبه متراص قوي) أذا كان Y كذلك.