The Stability Analysis of Steady State Solution of nonlinear Reaction-Diffusion System

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Abstract

The stability analysis for state solution of nonlinear reaction-diffusion systems with the Lotka-Volterra type interaction where studied which arise in the solution of flow of homogeneous and traveling wave solution, the resulting of this analysis shows that the system is stable when under condition, and we find that the system is stable if $\mu > \frac{A(x)}{k}$, however when **(k)** is increase then the stable region is increase, and when **(A**) is increase then the unstable region is increase.

1-Introduction:

Reaction-diffusion give rise to texture synthesis based on the simulation of a process of local nonlinear interactions, which has been proposed as a model of biological pattern formation.Turing. A. in (1952) was first proposed a chemical mechanism to account for pattern formation in Biological morphogenesis, he postulated that patterning is governed primarily by two concurrently operation processes:

Diffusion of morphogenesis through the tissue and chemical reaction that produce and destroy morphogenesis at a rate that depends, among other things, on their concentrations. Such mechanisms are called Reaction – Diffusion (RD) systems [12].

A system of nature whatsoever that exist in state in stable, in one sense or another , if small disturbance or change in the system do not exists in time-dependent state in which the planets move about the sun in an orderly fashion. It is know that if small additional celestial body is introduced into the system, then the original state is stable to small disturbance. Similar equation of stability arises in every physical problem [4].

Rubinsein in(1948) is able to derive a coupled system in a one –dimensional case includes heat conduction in heterogeneous media. [9]

Masuda in (1983) showed the solution of the reaction – diffusion system exist globally for every $\beta > 1$.[7]

Chow Tanya in (1996) studied the derivation of similarity solutions for one-dimensions coupled reaction – diffusion equations , these solutions are obtained of one-parameter group methods. [1]

Kan-on and fang in (1997) proved the linearized stability of the traveling waves. [3]

Said kouachi (2001) studied construct polynomial functional (according to solution of the coupled reaction–diffusion equations) which give L^P-bounds for solutions. And he deduces the existence of global solutions. [10]

Yuzo Hosono in (2003) studied the traveling wave solution of a two species competition-diffusion systems with the Lotka-Volterra type interaction and find that the rate ε_2 diffusion coefficients of the former species over the latter is small enough. [13]

In this paper , we will study the stability analysis of nonlinear reaction-diffusion systems with the Lotka-Volterra type interaction.

2- Model of equations:

The reaction-diffusion systems with Lotka-Volterra type interaction is given by:

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2} + u[(a(x) - u - v)]$$
$$\frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial x^2} + v[(a(x) - u - v)] \qquad \text{.......................(1)}$$

$u(x,t)$ and $v(x,t)$ represent the densities of two competing species at location (x) and time (t) ,The diffusion rate μ is positive constants, and $a(x)$ denotes the intrinsic growth rate of aspects.

With initial conditions:

$$u(x,0) = u_0$$
$$v(x,0) = v_0$$

Where u_0 and v_0 are constant interior solutions of (1) and the boundary conditions: [2].

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$$

For dimensionless form, we introduce the following dimensionless quantities: [11].

$x = XL \qquad t = L^2T, \qquad u = UL, \qquad v = VL, \qquad , \qquad a = AL$

Substitute these non-dimensional quantities into equation (1) and in the conditions ,we get:

$$\frac{\partial U}{\partial T} = \mu \frac{\partial^2 U}{\partial X^2} + U(X,T)[A(x) - U(X,T) - V(X,T)]$$
$$\frac{\partial V}{\partial T} = \mu \frac{\partial^2 V}{\partial X^2} + V(X,T)[A(x) - U(X,T) - V(X,T)] \qquad \text{.......(2)}$$

With the boundary conditions:

$$\frac{\partial U}{\partial X} = \frac{\partial V}{\partial X} = 0 \text{..........................(3)}$$

Stability analysis has been recently studied by numerous and it is of great interest because of the growing industrial importance.

Assume that the value of the concentrations U(X,T) and V(X,T) has the following form: [4].

$$U(X,T) = U_1(X) + U_2(X,T) \quad ....(4)$$
$$V(X,T) = V_1(X) + V_2(X,T)$$

Where $U_1(X)$ and $V_1(X)$, denote the steady state case, $U_2(X,T)$ and $V_2(X,T)$ denote the disturbance case.

If we substituted (4) in equations, (2) we get the following two systems and take the linear term only since the linear term implies to stability [4] we get:

The steady state system:

$$\mu \frac{d^2U_1}{dX^2} + U_1(X,T)A(x) - (U_1(X,T))^2 - V_1(X,T)U_1(X,T) = 0......(5a)$$

$$\mu \frac{d^2U_1}{dX^2} + V_1(X,T)A(x) - (V_1(X,T))^2 - V_1(X,T)U_1(X,T) = 0.......(5b)$$

With the boundary conditions:

$$\frac{\partial U}{\partial X} = \frac{\partial V}{\partial X} = 0$$

and the second system:

$$\frac{\partial U_2}{\partial T} = \mu \frac{\partial^2 U_2}{\partial X^2} + A(x)U_2(X,T) - V_1(X,T)U_2(X,T) - V_2(X,T)U_1(X,T)..............(6a)$$

$$\frac{\partial V_2}{\partial T} = \mu \frac{\partial^2 V_2}{\partial X^2} + +A(x)V_2(X,T) - V_2(X,T)U_1(X,T) - V_1(X,T)U_2(X,T)..............(6b)$$

with the related boundary:

$$\frac{\partial U}{\partial X} = \frac{\partial V}{\partial X} = 0$$

3-Solution of the steady state case:

The solution of $(5a)$ and $(5b)$ with the boundary conditions:

$$U_1(X) = \mu\Delta\psi + [A(x) - (1+\tau h)\theta]\psi + \lambda\psi......................(7a)$$

$$V_1(X) = \mu\Delta\varphi + [A(x) - (1+\tau g)\theta]\varphi + \lambda\varphi..........................(7b)$$

where (ψ,φ) eigenfunction associated, $(h),(g)$ smooth function, (τ) & (θ) are positive constants, and (λ) is real value $[5]$.

4- Stability analysis (disturbance case):

Assume that the value of $U_2(X,T)$ and $V_2(X,T)$, has the following form [6] and [8]:

$$U_2(X,T) = F_1 e^{ik(X-CT)} \quad ....(8)$$
$$V_2(X,T) = F_2 e^{ik(X-CT)}$$

Hear $(C = C_1 + iC_2)$ is an eigenvalue represent the speed of the wave, the functions F_1 and F_2 are the constant amplitudes, k is the wave number. The flow is stable if the linearized equation corresponds to eigenvalue C with negative part $C_2 < 0$ for presented configurations.

Now, substitute (8) in the equation $(6a),(6b)$ we get respectively:

$$-ik(C_1 + iC_2)F_1 = -\mu k^2 F_1 + A(x)F_1 - F_1[\mu\Delta\psi + [A(x) - (1+\tau h)]\psi + \lambda\psi] - F_2[\mu\Delta\varphi + [A(x) - (1+\tau g)\theta]\varphi + \lambda\varphi]$$

$$-ik(C_1 + iC_2)F_2 = -\mu k^2 F_2 + A(x)F_2 - F_2[\mu\Delta\varphi + [A(x) - (1+\tau g)\theta]\varphi + \lambda\varphi] - F_1[\mu\Delta\psi + [A(x) - (1+\tau h)]\psi + \lambda\psi]$$

by separate the real part and imaginary part, we get:

$$kC_2F_1 = -\mu k^2 F_1 + A(x)F_1 - F_1[\mu\Delta\psi + [A(x) - (1+\tau h)]\psi + \lambda\psi] - F_2[\mu\Delta\varphi + [A(x) - (1+\tau g)\theta]\varphi + \lambda\varphi]............................(9a)$$

$$kC_2F_2 = -\mu k^2 F_2 + A(x)F_2 - F_2[\mu\Delta\varphi + [A(x) - (1+\tau g)\theta]\varphi + \lambda\varphi] - F_1[\mu\Delta\psi + [A(x) - (1+\tau h)]\psi + \lambda\psi]..................................(9b)$$

multiply (9b) by(-1) and add to (9a) we get:

$$kC_2(F_1 - F_2) = -\mu k^2(F_1 - F_2) + A(x)(F_1 - F_2)..........(10)$$
$$\Rightarrow (kC_2 + \mu k^2 - A(x))(F_1 - F_2) = 0..........(11)$$

Such that $(F_1 - F_2) \neq 0$ since $F_2 \neq F_1$ then:

$$kC_2 + \mu k^2 - A(x) = 0..........(12)$$

$$\Rightarrow \quad C_2 = \left(\frac{-\mu k^2 + A(x)}{k}\right)$$

$$\Rightarrow C_2 = \left(\frac{A(x)}{k} - \mu k\right)..................................(13)$$

implies to the system is stable when:

$$\mu k > \frac{A(x)}{k}$$

and unstable when:

$$\mu k < \frac{A(x)}{k}$$

The neutral stability curve, when $(C_2 = 0)$ in the equation (13) is:

$$\mu = \frac{A(x)}{k^2}..................(14)$$

μ	$A = 0.1\ k$	$A = 0.2\ k$
0.1	1.0000-1.0000	1.4142-1.4142
0.2	0.7071-0.7071	1.0000-1.0000
0.3	0.5773-0.5773	0.8164-0.8164
0.4	0.5000-0.5000	0.7071-0.7071
0.5	0.4472-0.4472	0.6324 -0.6324
0.6	0.4082-0.4082	0.5773 -0.5773
0.7	0.3779-0.3779	0.5345 -0.5345
0.8	0.3535-0.3535	0.5000 -0.5000
0.9	0.3333-0.3333	0.4714 -0.4714
1.0	0.3162-0.3162	0.4472 -0.4472

Table (1) . The state solution of stability in Equation (14) when $A = 0.1, \quad A = 0.2$, $\quad 0 \le \mu \le 1$ and show that when (k) is increase then the stable region is increase, also shown in figures (1) and (2).

μ	$k = 0.1$ A	$k = 0.2$ A
0.1	0.0010	0.0040
0.2	0.0020	0.0080
0.3	0.0030	0.0120
0.4	0.0040	0.0160
0.5	0.0050	0.0200
0.6	0.0060	0.0240
0.7	0.0070	0.0280
0.8	0.0080	0.0320
0.9	0.0090	0.0360
1.0	0.0100	0.0400

Table (2) . The state solution of stability in Equation (14) when $k = 0.1, \ k = 0.2$, $\quad 0 \le \mu \le 1$ and show that when (A) is increase then the unstable region is increase, also shown in figures (3)

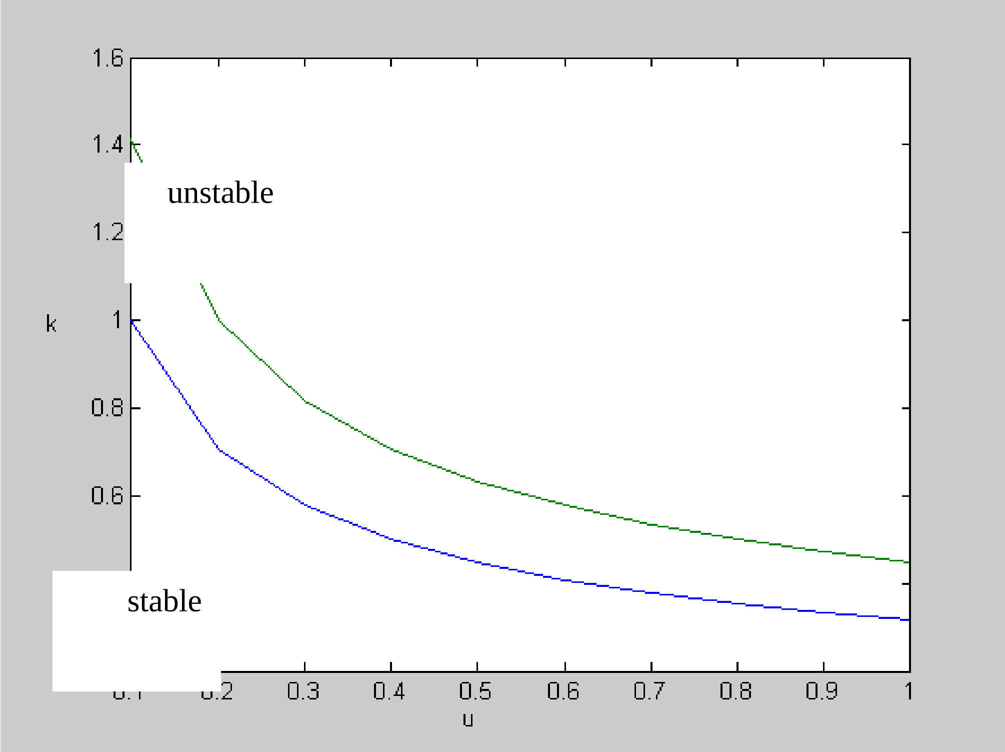

Fig (1). The natural stability curve in (14) when $A = 0.1$, $A = 0.2$, $0 \le \mu \le 1$ **and show that when (k) is increase then the stable region is increase.**

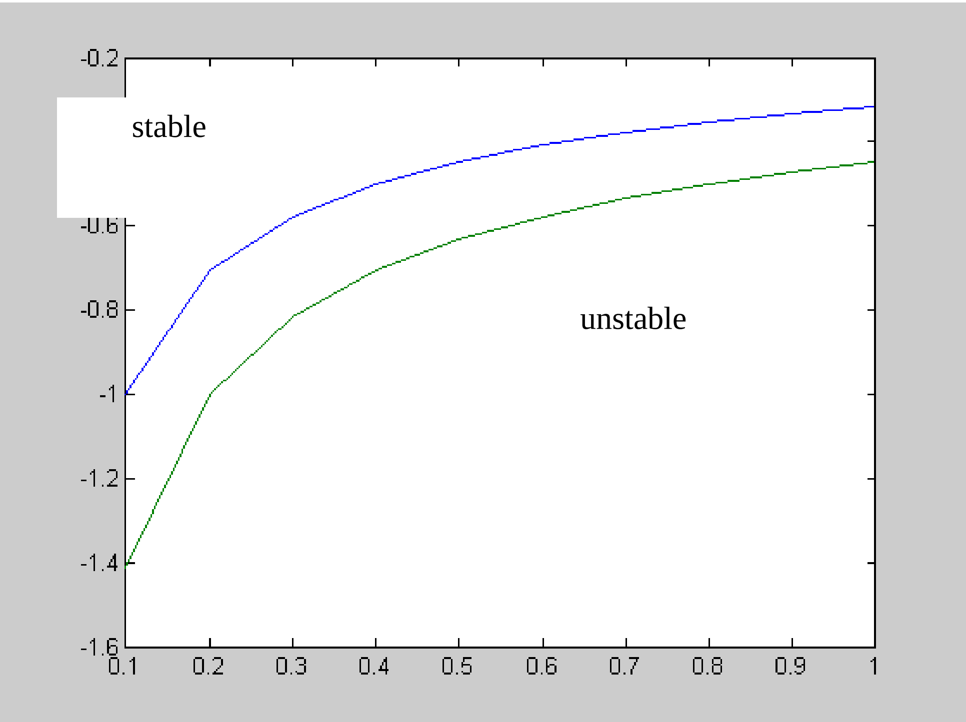

Fig (2). The natural stability curve in (14) when $A = 0.1$, $A = 0.2$, $0 \le \mu \le 1$ **and show that when (k) is increase then the stable region is increase.**

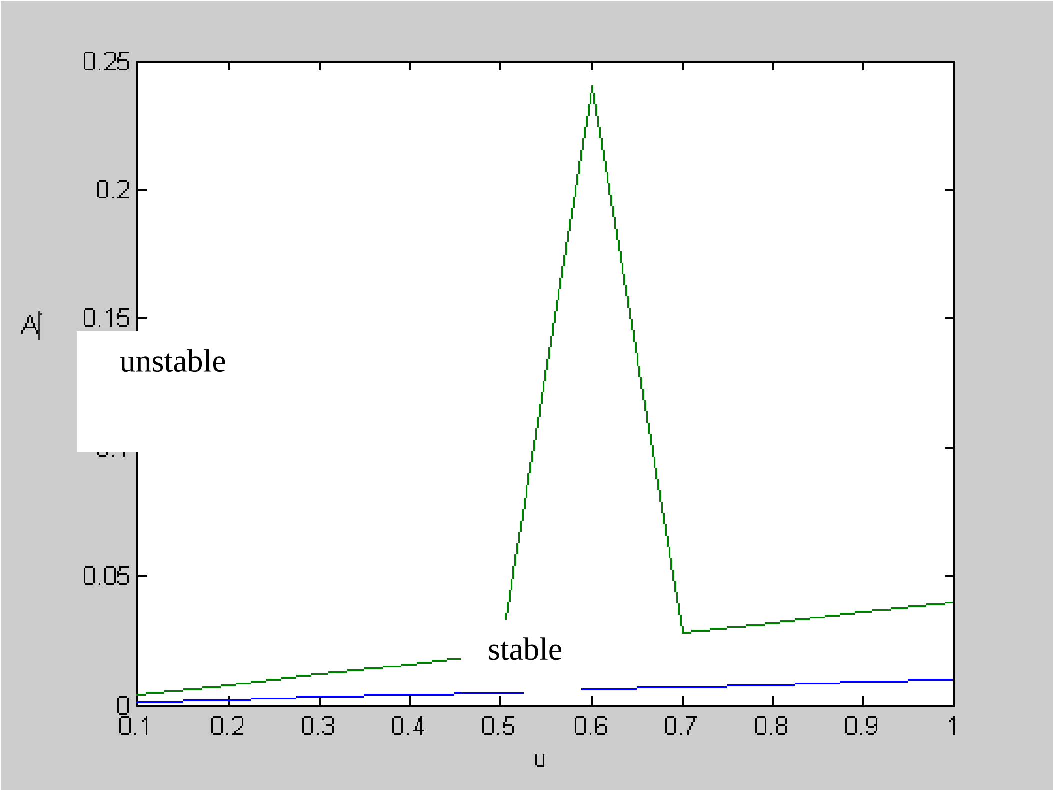

Fig (3). The natural stability curve in (16) when $k = 0.1,\ \ k = 0.2$, $0 \le \mu \le 1$ and show that when (A) is increase then the unstable region is increase

6-Conclusion:

we find form the Stability analysis (disturbance case) that the Stability Analysis of Steady State Solution of nonlinear Reaction-Diffusion

System is stable when $\mu > \frac{A(x)}{k}$, and unstable when $\mu < \frac{A(x)}{k}$ and find the neutral curve, however when **(k)** is increase then the stable region is increase, and when **(A)** is increase then the unstable region is increase, also shown in figures (1) ,(2)and(3) and table (1) and (2)

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تحليل الاستقرارية للحلول اللازمنية لنظام انتشار– تفاعل لاخطي

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الملخص:

تم دراسة استقرارية الحلول اللازمنية لنموذج تفاعل Lotka-Volterra وهو من أنظمة الانتشار – التفاعل اللاخطية (nonlinear reaction-diffusion) والتي تظهر في حل مشاكل جريان الموائع المتجانسة وحلول انتقال الموجة (traveling wave) وقد تبين أن هذا النظام يكون مستقر تحت شروط والشرط هو $\mu > \frac{A(x)}{k}$ والتي فيها يكون النظام مستقر ، كما تبين انه كلما زادت قيمة k زادت مناطق الاستقرارية ، وكلما زادت قيمة A زادت مناطق عدم الاستقرارية وهذا لكون أن قيمة μ تتناسب طرديا" مع قيمة A.