The Kernel and Range of Idempotent Matrices

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Abstract
In this paper we give some properties that's command with the Kernel and Range of Idempotent Matrices , and non – singular idempotent Matrices .

1- Introduction :
In [3] J.J. Koliha and V.Rakocevic present new results on the non-singularity of A+B, where A and B are idempotent matrices also show that it's equivalent to non-singularity of any linear combination cA+dB where c,d $\neq 0$ and c+d$\neq$0 ,and They studied the Nullity and Rank of cA+dB .
Throughout , this paper all Matrices considered are square and commutative unless other wise stated. In this paper we present some basic properties of an idempotent Matrices and relations between the Range and the Kernel space and give some results on the non – singularity of the difference and sum of idempotent Matrices , we recall that :

1- Amatrix A is said to be idempotent if $A^2 = A$ [**2**]
2- Akernel space of Matrix A is the set of all solutions to the equation $A\overline{x} = 0$ and denoted by N (A)

$N(A) = \{ \overline{x} \in R^n : A\overline{x} = 0\}$. [4]

3- Arange space of amatrix A is the set of all solutions to the equation $A\overline{x} = \overline{y}$ and we denoted by R(A) . [4]

Idempotent Matrices :
In this section we give some properties of the Kernel and Range of idempotent Matrices .

Lemma 2.1 : [1]
Let $A \in C^{nxn}$ be idempotent Matrix then .
1- $A\overline{x} = \overline{x}$ if and ony if $\overline{x} \in R(A)$.
2- N(A) = R (I-A) .

Proposition 2-2 :
If A and B are idempotent commutative Matrices and $\overline{y} \in R(B)$, then R [B(I-A)] = N (A) .

Proof :
Let $\overline{\chi} \in R[B(I-A)], then\, \overline{\chi} = B(I-A)\overline{\chi}$
Then $A\overline{\chi} = AB(I-A)\overline{\chi} = 0 and\, \overline{\chi} \in N(A)$
There fore $R[B(I-A)] \subseteq N(A)$ …… (1)
Now let
$\overline{y} \in N(A), then A\overline{y} = 0 and BA\overline{y} = 0$
Implies that $BA\overline{y} - B\overline{y} = -B\overline{y}$
and $B(I-A)\,\overline{y} = B\overline{y}$ but $B\overline{y} = \overline{y}$ [Since $\overline{y} \in R(B)$]
Implies that $B(I-A)\overline{y} = \overline{y}$ and $\overline{y} \in$ [B(I– A)]
Therefore $N(A) \subseteq R\,[B(I-A)]$ ……….. (2)
From (1) and (2) we get
R[B(I – A)] = N (A) ■

Proposition 2.3 :
Let A and B be idempotent Matrices ,If N (A) = N(B), then $N(A) \cap R(B) = \{0\}$.

Proof :
Let $\overline{x} \in N(A) \cap R(B)$, then $\overline{x} \in N(A) = N(B)$
And $\overline{x} \in R(B)$, then $\overline{x} = B\overline{x}$ and $B\overline{x} = 0$
Then $\overline{x} = 0$ and $N(A) \cap R(B) = \{0\}$
Similarly we get $N(B) \cap R(A) = \{0\}$ ■

Proposition 2.4 :
If A and B are idempotent Matrice and $\overline{y} = AB\,\overline{y}$, then $R(A) \cap R(B) = N(A-B)$.

Proof :
Let $\overline{x} \in R(A) \cap R(B)$ then
$A\overline{x} = \overline{x}$ and $B\overline{x} = \chi$ implies $A\overline{x} = B\overline{x}$
And $(A-B)\overline{x} = 0$ and $\overline{x} \in N(A-B)$
Therefore $R(A) \cap R(B) \subseteq N(A-B)$ ………. (1)
Now let $y \in N(A-B)$, then $(A-B)\overline{y} = 0$ implies
$A\overline{y} = B\overline{y}$
and hence $\overline{y} \in R(A)$ also $B\overline{y} = BA\overline{y} = \overline{y}$
Then $\overline{y} \in R(B)$ and $\overline{y} \in R(A) \cap R(B)$
Therefore $N(A-B) \subseteq R(A) \cap R(B)$ …….. (2)
From (1) and (2) we get
$R(A) \cap R(B) = N(A-B)$ ■

Proposition 2.5 :

If A and B are commutative idempotent Matrices and $R(A) \cap R(B) = \{0\}$

Then $N(A+B) = N(A) \cap N(B)$

Proof :

Let $\bar{x} \in N(A+B)$, then $A\bar{x} = -B\bar{x}$

Multiplication from left by A we get

$A\bar{x} = -AB\bar{x} \in R(A)$ also

$A\bar{x} = -BA\bar{x} \in R(B)$

Since $R(A) \cap R(B) = \{0\}$ and A and B are Commutative then $A\bar{x} \in R(A) \cap R(B) = \{0\}$

Implies that $A\bar{x} = 0$ and $\bar{x} \in N(A)$

Since $A\bar{x} = -B\bar{x}$ then $B\bar{x} = 0$ and $\bar{x} \in N(B)$

and hence $\bar{x} \in N(A) \cap N(B)$, therefore

$N(A+B) \subseteq N(A) \cap N(B)$ ………………. (1)

Now . let $\bar{y} \in N(A) \cap N(B)$, then

$A\bar{y} = 0$ and $B\bar{y} = 0$

Then $(A+B)\bar{y} = 0$, hence $\bar{y} \in N(A+B)$

Therefore $N(A) \cap N(B) \subseteq N(A+B)$ ………… (2)

From (1) and (2) we get

$N(A+B) = N(A) \cap N(B)$ ■

3- Non singular idempotent Matrices .

in this section we give some properties of non singular idempotent Matrices .

Proposition 3.1 :

Let A and B are commutative idempotent Matrices, and let $R(A) \cap R(B) = \{0\}$ and $N(A) \cap N(B) = \{0\}$, then $(A+B)$ and $(I - AB)$ are Non singular .

Proof :

To show that (A+B) and (I – AB) are non singular we shall proof that

$N(A+B) = \{0\}$ and $N(I - AB) = \{0\}$ Now, let

$(A+B)\bar{x} = 0$, then $A\bar{x} = -B\bar{x} = -AB\bar{x} \in R(A) \cap R(B) = \{0\}$

and $\bar{x} \in N(A) \cap N(B) = \{0\}$ implies that

$\bar{x} = 0$, So $N(A+B) = \{0\}$, then (A+B) is non singular

Now , let $(I - AB)\bar{x} = 0$, implies that

$\bar{x} = AB\bar{x} \in R(A) \cap R(B) = \{0\}$

and $\bar{x} \in N(A) \cap N(B) = \{0\}$, hence $\bar{x} = 0$

hence $N(I - AB) = \{0\}$ and (I– AB) is non singular. ■

Proposition 3.2 :

Let A and B are idempotent Mattrices , if (A+B) and (A – B) are Non singular Matrices , then the Product is also Non Singular .

Proof :

Let (A+B) and (A – B) are non singular Matrices, then

$(A+B)\bar{x} = 0$. implies $A\bar{x} = -B\bar{x} = -AB\bar{x}$

$(A - B)\bar{x} = 0$. implies $A\bar{x} = B\bar{x} = BA\bar{x}$

Now $[(A+B)(A - B)]\bar{x} = A\bar{x} - AB\bar{x} + BA\bar{x} - B\bar{x} = 0$

Then $N[(A+B)(A - B)] = \{0\}$

hence [(A+B) (A – B)] is non singular ■

Proposition 3.3 :

Let A and B are idempotent Matrices , then if $R(A) \cap R(B) = \{0\}$ and $N(A) \cap N(B) = \{0\}$ Then (A – B) is non singular .

Proof :

Let $(A - B)\bar{x} = 0$, then $A\bar{x} = B\bar{x} \in R(A) \cap R(B) = \{0\}$

and $\bar{x} \in N(A) \cap N(B) = \{0\}$,thus $N(A - B) = \{0\}$

hence (A – B) is non singular ■

Proposition 3.4 :

If A and B are a non singular idempotent Matrices , then $R(A) \cap R(B) = N(A - B)$.

Proof :

Let $\bar{x} \in R(A) \cap R(B)$, then $\bar{x} = A\bar{x}$ and $\bar{x} = B\bar{x}$

Implies $A\bar{x} = B\bar{x}$ and $(A - B)\bar{x} = 0$ hence

$\chi \in N(A - B)$, there fore

$R(A) \cap R(B) \subseteq N(A - B)$. ……… (1)

Now , let $\bar{y} \in N(A - B)$, then $(A - B)\bar{y} = 0$

So $A\bar{y} = B\bar{y}$

Multiplication from left by (I – A) we get (I–A) Ay = (I–A) B y, So

$(I - A) B\ \overline{y} = 0$, implies that $B\ \overline{y} - AB\overline{y} = 0$

but $B\overline{y} = A\ \overline{y}$ So $A\overline{y} - AB\overline{y} = 0$

Implies $A (I - B)\ \overline{y} = 0$, So $(I - B)\ \overline{y} \in N(A) = 0\}$

[Since A is non sinular]

Therefore $(I - B)\ \overline{y} = 0$ and $\overline{y} = B\ \overline{y}$, So $\overline{y} \in R(B)$

Similarly $\overline{y} \in R(A)$, hence $\overline{y} \in R(A) \cap R(B)$

Therefore $N(A - B) \subseteq R(A) \cap R(B)$ ……… (2)

From (1) and (2) we get $R(A) \cap R(B) = N(A - B)$ ■

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الملخص :

في هذا البحث قدمنا بعض الخواص التي تتعلق بنواة ومدى المصفوفات المتحايدة والمصفوفات المتحايدة الغير المنفردة .