An Explicit Numerical Methods Involving Nine- Points Formula for Solving Transport Equation

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Abstract :-

The main aim of this paper is to compare five methods of finite difference schemes involving nine points formulas such as truncations errors and averages , maximum and minimum error. We have found that two of these methods are more accurate. Moreover, these schemes gives better results than the other three schemes .

1- Introduction :-

One of the most basic partial differential equation we can examine is the one dimensional convection – diffusion equation [1]

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}=\alpha\,\frac{\partial^2 T}{\partial x^2} \qquad \cdots (1)$$

Where T(x , t) is a scalar variable , x is the distance , t is the time and u and α are positive constants.

Equation (1) seems like a simple partial differential equation at first glance, but when trying to approximate the solution of equation (1) numerically, we find that are large errors introduced into our approximate solution. Initial condition is

$$T\,(\,x\,,0\,)=\exp\,(\,-\frac{(\,x-x_o\,)^2}{\alpha}\,) \qquad .\ 0\le x\le X \qquad \cdots (2)$$

With boundary conditions are

$$T\,(\,0\,,t\,)=\frac{1}{\sqrt{4t+1}}\exp\,(\,-\frac{(\,x_o+u\,t\,)^2}{\alpha\,(\,4t+1)}\,)\ ,$$

$$T\,(\,X\,,t\,)=\frac{1}{\sqrt{4t+1}}\exp\,(\,-\frac{(\,X-x_o-u\,t\,)^2}{\alpha\,(\,4t+1)}\,) \qquad \cdots (3)$$

The exact solution is

$$T\,(\,x\,,t\,)=\frac{1}{\sqrt{4t+1}}\exp\,(\,-\frac{(\,x-x_o-u\,t\,)^2}{\alpha\,(\,4t+1)}\,)\ ,\quad t\ge 0 \quad \cdots (4)$$

We will compare the numerical solution with both of the exact solutions to determine the errors produced by the finite difference approximation of equation (1) and thus find the “ best ” scheme to solve this problem .

2- Numerical Solution:-

A uniform rectangular grid is superimposed on the domain as shown in figure (1). The mesh lengths in the x and t direction are denoted by Δx and Δt respectively. We use the method of finite differences to find an approximation T_j^n to the solution T at the (j , n) grid point , that is T_j^n is an approximation to $T\,(\,x_j\,,t_n\,)$.

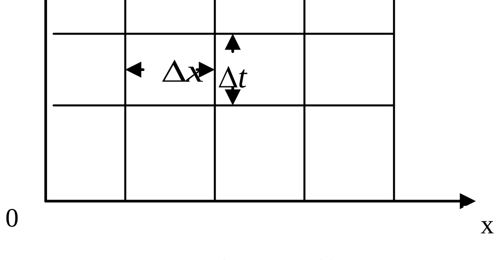

Figure (1): The finite difference grid

In the method of finite difference the derivatives approximating the partial differential equation are approximated by " combinations " (often differences) of value at grid points.

$x_j = j\,\Delta x$, $j=0(1)M$ and $t_n = n\,\Delta t$, $n=0(1)N$

It can be shown that the first and second derivative of T at the point $(\,x_j\,,t_n)$ of eight- order [7] satisfies

$$\frac{\partial T}{\partial x}\Big|_j^n=\frac{1}{10080(\Delta x)}[T_{j-4}^n+36T_{j-3}^n+686T_{j-2}^n-6524T_{j-1}^n+6524T_{j+1}^n- T_{j+4}^n-36T_{j+3}^n-686T_{j+2}^n]+O\{(\Delta x)^8\} \qquad \cdots (5)$$

$$\frac{\partial^2 T}{\partial x^2}\Big|_j^n=\frac{1}{20160\,(\Delta x)^2}[-T_{j-4}^n-48T_{j-3}^n- 1372T_{j-2}^n+26096T_{j-1}^n+26096T_{j+1}^n- 49350T_j^n-T_{j+4}^n-48T_{j+3}^n-1372T_{j+2}^n]+O\{(\Delta x)^8\} \quad ....(6)$$

since $T_j^n = O\{(\Delta x)^8\}$ the approximation is said to be eight-order accurate in Δx. Replacing both derivatives in the convection-diffusion equation gives a finite difference equation which is solved for the approximations T_j^n .

3 - Upwind Differences with Nine-point scheme (US&9):-

Using a forward difference approximation for the time derivative and a backward difference approximation for the spatial derivative and central difference approximation of eight - order equation (6) for the second derivative, to approximate equation (1) , we obtain a new formula for this method with nine-point (US&9) , as

$$T_j^{n+1}=-\frac{s}{20160}T_{j-4}^n-\frac{s}{420}T_{j-3}^n-\frac{343s}{5040}T_{j-2}^n+ (c+\frac{1631}{1260}s)T_{j-1}^n+(1-c-\frac{4935}{2016}s)T_j^n+\frac{1631}{1260}\,s\,T_{j+1}^n -\frac{343}{5040}T_{j+2}^n-\frac{s}{420}T_{j+3}^n-\frac{s}{20160}T_{j+4}^n \cdots (7)$$

where $c=\frac{u\,\Delta t}{\Delta x}$ is courant number and the diffusion number $s=\frac{\alpha\ \Delta t}{(\Delta x)^2}$, this is a fully explicit method as the “new” value T_j^{n+1} is defined explicitly in terms of known values at the time level $t = t_n$. to determine wether this scheme is stable, the Fourier mode [3 , 5 , 6] $T_j^n = (G)^n e^{ikj\Delta x}$, where G is the amplification factor and $i = \sqrt{-1}$ and k=0,1,…,N , is substituted into equation (7) , after some algebra we get

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 \qquad \cdots (8)$$

where $f(x) = |G|^2 - 1$, $x = \cos k$ and the coefficients

$$b_0 = (\frac{971}{420} s)^2 + \frac{971}{210} s(c-1) + 2c(c-1) \ ,$$

$$b_1 = \frac{1640}{315} s + 2c(1-c) - \frac{6193}{630} cs - \frac{79622}{6615} s^2$$

$$b_2 = -\frac{171}{315} s + \frac{1811}{315} cs + \frac{3187723}{(630)^2} s^2 \ , \ b_3 = -\frac{4}{105} s - \frac{53}{105} cs - \frac{14609}{(105)^2} s^2 \ , \ b_4 = -\frac{11}{504} s^2 - \frac{23c+1}{630} s \ \ b_5 = \frac{1232}{198450} s^2 - \frac{cs}{630} \ , \ b_6 = \frac{s^2}{1260} \ , \ b_7 = \frac{s^2}{33075} \ , \ b_8 = \frac{s^2}{(1260)^2}$$

equation (8) is stable if $f(x) \le 0$ for all $x \in [-1, 1]$, if x=0 then $f(0) \le 0$ for all $0 < s \le \frac{420}{971}[1 - c + \sqrt{1-c^2}]$ for value 0 < c < 1 If x = -1 then $f(-1) \le 0$ this scheme is stable for

$$0 < s \le \frac{1}{2A}[-(Ec + I) + \sqrt{(Ec + I)^2 - 4A(Bc + Dc^2)}\ ]$$

where A=26.71085 ,
B= - 4 , D=4 , E=20.67302 , F=1.024455E-07, H=1.227018E-07 and I=-10.33651For all c > 0 and this scheme is unconditionally stable if x = 1 for all s , c > 0 .

4 – The Lax– Wendroff with Nine-point scheme (L-WS&9) :-

The Lax–Wendroff scheme is another fully explicit scheme [6]. This scheme was developed by P.D. Lax and B.Wendroff, by using Taylor's expansion

$$T_j^{n+1} = T(x_j, t_{n+\Delta t}) = T_j^n + \Delta t (\frac{\partial T}{\partial t})_j^n + \frac{1}{2}(\Delta t)^2 (\frac{\partial^2 T}{\partial t^2})_j^n + \cdots$$

where

$x_j = j\Delta x$ and $t_n = n\Delta t$, $j = 0, \pm 1, \pm 2, \cdots$, $n = 0, 1, 2, \cdots$

The differential equation can be used to eliminate the t-derivatives because it gives that $\frac{\partial}{\partial t} = \alpha \frac{\partial^2}{\partial x^2} - u \frac{\partial}{\partial x}$ so

$$T_j^{n+1} = T_j^n + \alpha\, \Delta t (\frac{\partial^2 T}{\partial x^2})_j^n - u\, \Delta t (\frac{\partial T}{\partial x})_j^n + \frac{(\alpha\, \Delta t)^2}{2} (\frac{\partial^4 T}{\partial x^4})_j^n - \alpha u (\Delta t)^2 (\frac{\partial^3 T}{\partial x^3})_j^n + \frac{(u\Delta t)^2}{2} (\frac{\partial^2 T}{\partial x^2})_j^n \quad \cdots (9)$$

Where ,

$$(\frac{\partial^3 T}{\partial x^3}) = \frac{1}{2(\Delta x)^3}[T_{j+2}^n - 2T_{j+1}^n + 2T_{j-1}^n - T_{j-2}^n] \ ,$$

$$(\frac{\partial^4 T}{\partial x^4}) = \frac{1}{(\Delta x)^4}[T_{j+2}^n - 4T_{j+1}^n + 6T_j^n - 4T_{j-1}^n + T_{j-2}^n] \ ,$$

$$T_j^{n+1} = T_j^n (1 + 3s^2 - \frac{4935}{2016}(s + \frac{c^2}{2})) + T_{j-1}^n (\frac{26096}{20160}(s + \frac{c^2}{2}) + \frac{6524c}{10080} - 2s^2 - sc) + T_{j+1}^n ((\frac{26096}{20160}(s + \frac{c^2}{2}) - \frac{6524c}{10080} - 2s^2 + sc) + T_{j-2}^n (\frac{s^2}{2} + \frac{sc}{2} - \frac{686c}{10080} - \frac{1372}{20160}(s + \frac{c^2}{2})) + T_{j+2}^n (\frac{s^2}{2} - \frac{sc}{2} + \frac{686c}{10080} - \frac{1372}{20160}(s + \frac{c^2}{2})) - (\frac{c}{10080} + \frac{s + \frac{c^2}{2}}{20160}) T_{j-4}^n + (\frac{c}{10080} - \frac{s + \frac{c^2}{2}}{20160}) T_{j+4}^n \ \frac{48}{20160}(s + \frac{c^2}{2}) + \frac{36c}{10080}) T_{j-3}^n + (-\frac{48}{20160}(s + \frac{c^2}{2}) + \frac{36c}{10080}) T_{j+3}^n \qquad \cdots (10)$$

and $(\frac{\partial T}{\partial x})_j^n$ from equation (5). Then equation (9) has the form

where once again $c = \frac{u\Delta t}{\Delta x}$ is the courant number and $s = \alpha \frac{\Delta t}{(\Delta x)^2}$ is the diffusion number. The amplification factor of the general Fourier mode is

$$G = -\frac{1}{10080}(s + \frac{c^2}{2})(8\cos^4 k\Delta x + 192\cos^3 k\Delta x + 2736\cos^2 k\Delta x - 26240\cos k\Delta x - 1371) + 2s^2\cos^2 k\Delta x - 4s^2\cos k\Delta x + 1 - \frac{4935}{2016}(s + \frac{c^2}{2}) + 2s^2 + 2ic\sin k\Delta x\ ((1 - \cos k\Delta x) + \frac{1}{5040}(4\cos^3 k\Delta x + 18\cos^2 k\Delta x + 684\cos k\Delta x - 3280))$$

after some algebra we get equation (8) with coefficients

$$b_o = \frac{33929737}{5644800}(s + \frac{c^2}{2})^2 + 4s^4 + 4s^2 - \frac{1645}{336}(s + \frac{c^2}{2}) - \frac{1645}{168}s^2(s + \frac{c^2}{2}) - 4(cs)^2 - (\frac{82}{63}c)^2$$

$$b_1 = \frac{18737}{26460}(s + \frac{c^2}{2})^2 - 8s^2 + \frac{1645}{84}s^2(s + \frac{c^2}{2}) - 16s^4 + 8(cs)^2 + \frac{1558}{2205}c^2$$

$$b_2 = \frac{5675299}{8467200}(s + \frac{c^2}{2})^2 + 24s^4 + 4s^2 - \frac{235}{24}s^2(s + \frac{c^2}{2})^2 + \frac{650539}{(630)^2}c^2$$

$$b_3 = -16s^4 - 8(cs)^2 + \frac{160327}{226800}c^2 - \frac{2553}{1800}(s + \frac{c^2}{2})^2$$

$$b_4 = 4s^4 + 4(cs)^2 + \frac{13}{240}c^2 - \frac{311}{12096}(s + \frac{c^2}{2})^2 \ ,$$

$$b_5 = \frac{630784}{(10080)^2}(s + \frac{c^4}{2}) - \frac{7040}{(5040)^2}c^2$$

$$b_6 = \frac{1}{1260}(s + \frac{c^2}{2}) + \frac{23120}{(5040)^2}c^2 \ , \ b_7 = \frac{1}{33075}(s + \frac{c^2}{2})^2 + \frac{c^2}{(210)^2} \ , \ b_8 = \frac{1}{(1260)^2}(s + \frac{c^2}{2}) + \frac{c^2}{(630)^2}$$

equation (8) is stable if $f(x) \le 0$ for all $x \in [-1, 1]$ this scheme is stable for all s , c > 0

5– The Leap-Frog with Nine-point scheme (L-FS&9) :-

The leap-frog scheme adopts it's name from the fact that it behaves in a similar manner to that of frog . It "leaps" from one time interval to another in order to get a central time difference, and then spreads it's "legs" to determine the space differences at the time level in between . This can be observed in figure (2).

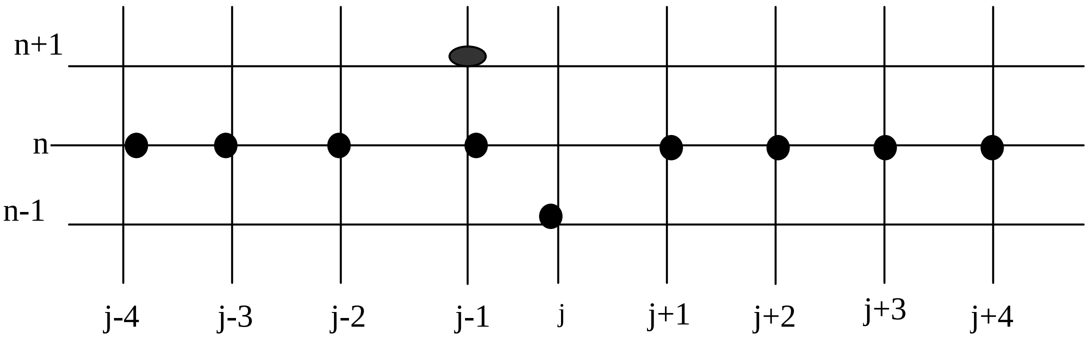

Figure (2): Calculation points for the leap-frog scheme with nine-points

The leap-frog scheme was developed using central difference approximations for both the spatial and time derivatives and eight–order centered difference approximating for $\frac{\partial^2 T}{\partial x^2}$ equation (6). The derivation of this scheme is below, substituting the central difference approximations

$$\frac{\partial T}{\partial t}\Big|_j^n \cong \frac{T_j^{n+1}-T_j^{n-1}}{2\,\Delta t} \quad , \quad \frac{\partial T}{\partial x}\Big|_j^n \cong \frac{T_{j+1}^{n}-T_{j-1}^{n}}{2\,\Delta x}$$

into equation (1) and rearranging gives the scheme

$$T_j^{n+1} = T_j^{n-1} + \frac{s}{10080}[-T_{j-4}^n - 48T_{j-3}^n - 1372T_{j-2}^n - 49350T_j^n - 1372T_{j+2}^n$$
$$-48T_{j+3}^n - T_{j+4}^n] + (\frac{26096}{10080}s + c)\,T_{j-1}^n + (\frac{26096}{10080}s - c)\,T_{j+1}^n \qquad \cdots (11)$$

where $c = \frac{u\Delta t}{\Delta x}$ and $s = \alpha \frac{\Delta t}{(\Delta x)^2}$.

As can be seen by this equation , the leap-frog scheme is an explicit scheme in that it has only one term involving the new time level n+1. It dose however need a spatial technique to begin evaluating the approximate to the solution , as equation (11) involves the time levels n-1, n and n+1. Initial condition as given in equation (2), is used to determine the values of T_j^0 ,

however the problem is to obtain values for T_j^{n-1}. This can be done by using any one-step scheme [2] .

However the more accurate the scheme the better the final approximation to the solution will be. Taking this into consideration we used the Lax-Wendroff scheme which we found earlier to more accurate then of upwind scheme. Using Fourier analysis , the amplification factor satisfied the quadratic

$$G^2 + G\ [\,s\,(\frac{1}{630}\cos^4 k\Delta x + \frac{4}{105}\cos^3 k\Delta x + \frac{19}{35}\cos^2 k\Delta x$$
$$-\frac{1}{35}\cos k\Delta x - \frac{457}{1680}) + \frac{235}{48} + 2\,i\,c\,\sin k\Delta x\ ] - 1 = 0$$

solving for both roots we obtain

$$G = \frac{1}{2}[-2ic\sin k\Delta x - s(\frac{1}{630}\cos^4 k\Delta x + \frac{4}{105}\cos^3 k\Delta x +$$
$$\frac{19}{35}\cos^2 k\Delta x - \frac{1}{35}\cos k\Delta x - \frac{457}{1680}) - \frac{235}{48} \pm \sqrt{f(x)}]$$

$$b_8 = \frac{1}{(630)^2}s^2,\ b_7 = \frac{4}{33075}s^2,\ b_6 = \frac{1}{315}s^2,\ b_5 = \frac{13}{315}s^2$$

$$b_4 = \frac{22049}{75600}s^2 + \frac{47}{3024}s,\ b_3 = \frac{47}{126}s - \frac{163}{315}s^2,\ b_2 = 4c^2 + \frac{893}{168}s - \frac{1237}{4200}s^2$$

$$b_1 = \frac{457}{29400}s^2 - \frac{47}{168}s,\ b_o = (\frac{457}{1680}s)^2 - \frac{21479}{8064}s + \frac{64441}{(48)^2} - 4c^2$$

where $f(x)$ as equation (8) with coefficients which can only occur if $x \in [-1, 1]$ we get three cases the first if x=0 then $f(0) \le 0$. For all $0 < s \le \frac{3316320}{403} + 3360\sqrt{c^2-1}$ for all c > 1 . The second and third case if x=-1 and x = 1 then $f(-1) \le 0$ and f (1) < 0 for all s , c > 0 .

6– The second order explicit upwind with Nine-point scheme :-

As the name would suggest , the second order explicit upwind scheme is developed using upwind approximations of both the spatial and time derivatives, after substituting these into equation (1) as

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} + u\,[c\,(\frac{T_j^n - T_{j-1}^n}{\Delta x}) + (1 - c)\,(\frac{3T_j^n - 4T_{j-1}^n + T_{j-2}^n}{2\,\Delta x})]$$
$$= \frac{\alpha}{20160\,(\Delta x)^2}[-T_{j-4}^n - T_{j+4}^n - 48\,T_{j-3}^n - 48\,T_{j+3}^n - 1372\,T_{j-2}^n -$$
$$1372\,T_{j+2}^n + 26096\,T_{j-1}^n + 26096\,T_{j+1}^n - 49350\,T_j^n\,]$$

rearrangement gives the second-order explicit upwind equation as

$$T_j^{n+1}=\frac{s}{20160}\,[-T_{j-4}^{n}-48\,T_{j-3}^{n}+26096\,T_{j+1}^{n}-1372\,T_{j+2}^{n}$$
$$-T_{j+4}^{n}-48\,T_{j+3}^{n}\,]+\,(\frac{c\,(c-1)}{2}-\frac{1372}{20160}s)\,T_{j-2}^{n}+(c\,(2-c)+$$
$$\frac{26096}{20160}s)T_{j-1}^{n}+(1-\frac{3}{2}c+2\,c^2-\frac{4935}{2016}s)\,T_j^{n}\quad\cdots(12)$$

where $c=\frac{u\Delta t}{\Delta x}$ and $s=\alpha\frac{\Delta t}{(\Delta x)^2}$. As it uses only the known time level n to calculate the approximation at time n+1 . Fourier analysis was used to examine the stability and errors introduced by the second order explicit upwind scheme. We again see that $f(x)$ as equation (8) with coefficients

$$b_o=\frac{5}{4}c^4+c^3-3\,c^2-2\,c+(\frac{971}{420})^2s^2-\frac{971}{210}s\,(1-c)-\frac{971}{140}s\,c^2$$
$$b_1=-2c^4+5c^3-4c^2+4c+\frac{373}{30}s\,c^2-\frac{4553}{315}c\,s$$
$$b_2=4c^4-11c^3-9c^2-2c+\frac{1636}{105}c\,s-\frac{2683}{252}s\,c^2+\frac{637859}{2\,(630)^2}s^2$$
$$b_3=-3c^4+9c^3-6c^2-\frac{23}{630}s+\frac{1639}{3150}s\,c^2-\frac{3943}{630}c\,s-\frac{1058041}{793800}s^2$$
$$b_4=2c^4-4c^3+2c^2-\frac{s}{630}+\frac{523}{1260}c\,s-\frac{9371}{105840}s^2-\frac{643}{1260}s\,c^2$$
$$b_5=-\frac{11}{315}s\,c^2+\frac{c\,s}{30}+\frac{1241}{88200}s^2\quad,\quad b_6=\frac{c\,s}{630}-\frac{s\,c^2}{630}+\frac{3}{3920}s^2$$
$$b_7=\frac{23}{2\,(1260)^2}s^2\quad,\quad b_8=\frac{s^2}{(1260)^2}$$

equation (8) is stable if $f(x)\le 0$ for all $x\in[-1,1]$ this scheme is stable for all s , c > 0

7– The (1,2,9) two – point explicit with Nine-point scheme :-

The (1,2,9) two-point explicit method was obtained by Noye [4] using centered - differencing for the space derivative at $(j-\frac{1}{2}\,,\,n)$ for the time derivative and eight- order centered difference to approximate $\partial^2 T/\partial x^2$ at (j , n-1). This leads to the (1,2,9) two-point explicit method

$$T_j^{n+1}=T_{j-1}^{n-1}+(\frac{c}{2}-1+\frac{26096}{20160}s)\,T_{j-1}^{n}+(1-\frac{c}{2}-\frac{4935}{2016}s)\,T_j^{n}$$
$$+\frac{s}{20160}[-T_{j-4}^{n-1}-T_{j+4}^{n-1}-48\,T_{j-3}^{n-1}-48\,T_{j+3}^{n-1}-1372\,T_{j-2}^{n-1}-$$
$$1372\,T_{j+2}^{n-1}+26096\,T_{j+1}^{n-1}\,]\qquad\cdots(13)$$

where $c=\frac{u\Delta t}{\Delta x}$ and $s=\alpha\frac{\Delta t}{(\Delta x)^2}$. As can be deduced from the name , the (1,2,9) two-point method only involves one point on the new time level , implying it is in fact an explicit method that uses three time levels. This can be observed in figure (3) bellow.

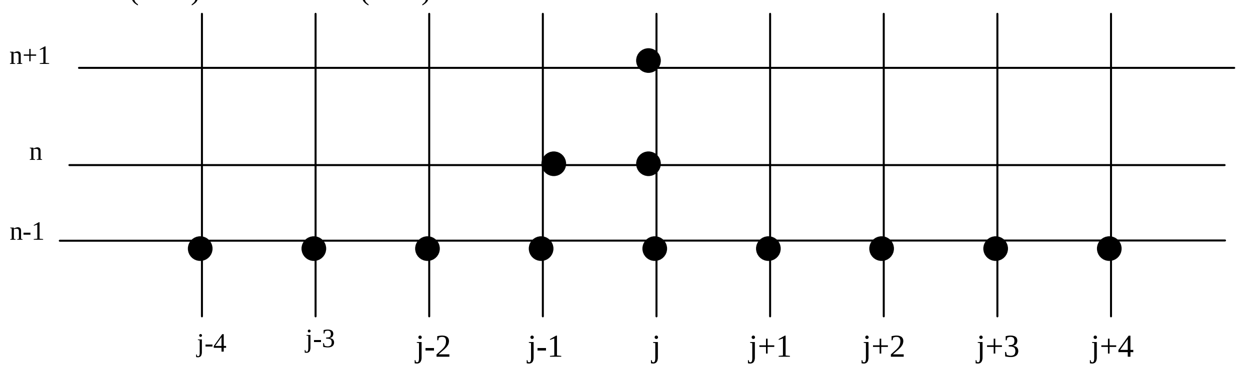

Figure (3):Calculation points for (1,2,9) two-point explicit method

After completing the Fourier analysis as for the previous methods, it is found that (1,2,9) two-point explicit method has an amplification factor of

$$G=\frac{1}{2}[\,(\frac{c}{2}-1+\frac{233}{180}s)\,(\cos k\Delta x-i\sin k\Delta x)+1-\frac{c}{2}-\frac{4935}{2016}s\pm\sqrt{f(x)}\,]$$

where f(x) as equation (8) with coefficients

$$b_8=b_7=b_6=b_5=0\,,\ b_4=-\frac{s}{315}\ ,\ b_3=-\frac{23}{315}s$$
$$b_2=\frac{c^2}{2}+2+(\frac{233}{45})^2\,s^2-2\,c+\frac{233}{90}c\,s-\frac{94}{15}s$$
$$b_1=2c-2-\frac{c^2}{2}+\frac{229423}{60480}s-\frac{5389}{1440}c\,s$$
$$b_o=\frac{1661}{160}c\,s-c-\frac{104641}{45360}s+\frac{3552706001}{(90720)^2}s^2$$

equation (8) is stable if $f(x)\le 0$ for all $x\in[-1,1]$ this scheme is stable for all s , c > 0.

8– The Treatment of Adjacent Points of Boundary :-

If we use the finite difference schemes involving nine-point difference formulas to solve equation (1) and if j=1,2,3 and m-1,m-2,m-3 then equations (7), (10), (11), (12) and (13) will contain points out of the nodal network. suggest in this case the treatment such that the value of the function at the points on the boundary of network is equal to the average of the nearest two points . That is

$$T_0 = \frac{T_{-1}+T_1}{2} \rightarrow T_{-1} = 2T_0 - T_1 \quad ,$$

$$T_{-1} = \frac{T_{-2}+T_0}{2} \rightarrow T_{-2} = 3T_0 - 2T_1$$

$$T_{-2} = \frac{T_{-3}+T_{-1}}{2} \rightarrow T_{-3} = 4T_0 - 3T_1 \quad ,$$

$$T_m = \frac{T_{m-1}+T_{m+1}}{2} \rightarrow T_{m+1} = 2T_m - T_{m-1}$$

$$T_{m+1} = \frac{T_m+T_{m+2}}{2} \rightarrow T_{m+2} = 3T_m - 2T_{m-1} \quad ,$$

$$T_{m+2} = \frac{T_{m+1}+T_{m+3}}{2} \rightarrow T_{m+3} = 4T_m - 3T_{m-1}$$

9– Numerical Test :-

In this section we will discussed the numerical results which get from test the schemes using to solve transport equation (1), as accurate order to these schemes and average absolute error (Average $|error|$), maximum absolute error (Max. $|error|$) and minimum of T (Min. $|error|$). The effects of numerical diffusion are seen in the following numerical test applied to the methods just described . The one-dimensional transport equation (1) was solved on $0 \le x \le 1$ with initial condition (2) and the boundary conditions (3) using $\alpha = 0.006$, 0.0006 , $u = 0.6$, $c = 0.1$, $\Delta x = 0.01$ and s=0.1, 0.01 respectively with $x_0 = 0.2$. When the numerical solution was compared with the exact solution (4) at t=1 the results from these numerical test are summarized in tables (1) and (2) and if equation (1) solved on $0 \le x \le 9$ using $\alpha = 0.005, 0.0005$ and 0.002, $u = 0.8$, $c = 0.4$, $\Delta x = 0.025$ and $s = 0.1$, 0.01 and 0.04 respectively with $x_0 = 1$ and t=5 such that the results from these numerical test are summarized in tables (3) to (5) , we discussed in table (6) the truncation error of these methods. Compute the results on a Pentium III .

table (1) :- Numerical results for $s = 0.1$, $c = 0.4$, $\alpha = 0.005$

Method	Average $\|erroe\|$	Max. $\|erroe\|$	Min. $\|erroe\|$
UD&9	5.011106E-03	6.925240E-02	1.401298E-45
L-W&9	1.130966E-04	1.633644E-03	-1.251412E-14
L-F&9	1.369216E-02	1.177202E-01	-2.004039E-01
Second-order	1.108074E-03	1.660281E-02	-7.973640E-04
(1,2,9)	9.558598E-04	1.152104E-02	-8.247018E-17

table (2) :- Numerical results for $s = 0.01$, $c = 0.4$, $\alpha = 0.0005$

Method	Average $\|erroe\|$	Max. $\|erroe\|$	Min. $\|erroe\|$
UD&9	4.700575E-03	1.563067E-01	-1.121039E-44
L-W&9	6.996484E-04	2.557443E-02	-7.333943E-03
L-F&9	9.421433E-04	3.466838E-02	-7.286977E-12
Second-order	3.988415E-03	1.009401E-01	-5.49270E-02
(1,2,9)	9.558616E-04	3.540678E-02	-7.049166E-04

table (3) :- Numerical results for $s = 0.04$, $c = 0.4$, $\alpha = 0.002$

Method	Average $\|erroe\|$	Max. $\|erroe\|$	Min. $\|erroe\|$
UD&9	5.471174E-03	1.071656E-01	-7.006492E-45
L-W&9	2.594909E-04	5.834609E-03	-1.605737E-05
L-F&9	9.521303E-04	1.800804E-02	-5.530100E-19
Second-order	2.039599E-03	4.408810E-02	-1.755445E-02
(1,2,9)	9.558600E-04	1.816010E-02	-3.193723E-08

table (4) :- Numerical results for $s = 0.1$, $c = 0.1$, $\alpha = 0.006$

Method	Average $\lvert erroe \rvert$	Max. $\lvert erroe \rvert$	Min. $\lvert erroe \rvert$
UD&9	1.578436E-02	6.386578E-02	8.052027E-09
L-W&9	7.676445E-06	3.284216E-05	4.181190E-10
L-F&9	UNSTABLE	UNSTABLE	-17.355400
Second-order	2.231248E-03	7.847250E-03	7.391824E-10
(1,2,9)	7.163336E-03	2.101240E-02	5.858156E-10

table (5) :- Numerical results for $s = 0.01$, $c = 0.1$, $\alpha = 0.0006$

Method	Average $\lvert erroe \rvert$	Max. $\lvert erroe \rvert$	Min. $\lvert erroe \rvert$
UD&9	2.990734E-02	2.388063E-01	-2.633555E-24
L-W&9	5.685376E-04	5.552202E-03	-7.513975E-05
L-F&9	1.750695E-03	1.367477E-02	-1.821688E-44
Second-order	1.514816E-02	1.221100E-01	-5.551219E-02
(1,2,9)	8.409956E-03	6.548887E-02	-3.664358E-04

table (6) :- Truncation error

Method	Truncation error
UD&9	$-\frac{1}{2}(1-c)\Delta x\frac{\partial^2 T}{\partial x^2}+\cdots$
L-W&9	$-\frac{1}{2}c\,\Delta x\frac{\partial^2 T}{\partial x^2}+\cdots$
L-F&9	$-\frac{(\Delta x)^2}{6}(1-c^2)\frac{\partial^3 T}{\partial x^3}+\cdots$
Second-order	$-\frac{(\Delta x)^2}{6}(1-c\quad)(2-c)\frac{\partial^3 T}{\partial x^3}+\cdots$
(1,2,9)	$-\frac{(\Delta x)^2}{12}(1-c\quad)(1-2c)\frac{\partial^3 T}{\partial x^3}+\cdots$

10 – Conclusion

Looking at table (6) we see that the first two schemes are first order accurate. As discussed in the text the Lax-Wendroff scheme gives a more accurate approximate solution than does the Upwind scheme. Looking at the next three second order accurate schemes. The only way to determine the most accurate scheme is therefore via the truncation errors . Comparing the (1 , 2 , 9) two – point explicit scheme has approximately half the truncation error of the second order explicit upwind scheme, implying the (1, 2, 9) two – point explicit scheme is more accurate. So from the five methods discussed throughout this project, the two most accurate are the Lax – Wendroff method and the (1 , 2 , 9) two – point explicit method .

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الملخص

الهدف الأساسي من هذا البحث هو المقارنة بين خمس أساليب من أساليب الفروقات المحددة و المتضمنة صيغ النقاط التسع وذلك من ناحية خطأ القطع والقيمة المطلقة لمعدل الخطأ Average |error| وأكبر قيمة مطلقة للخطأ Maximum |error| وأصغر تقريب عددي لدرجة الحرارة T (MinimumT) عند الزمن النهائي وقد وجدنا بأن أثنان من هذه الأساليب كانت أكثر دقة و أعطت نتائج جيدة مقارنة مع الأساليب الثلاث الأخرى .