Modern Roman Domination in Graphs

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Abstract

In this paper, "a Modern Roman Domination" is introduced, which is a new model of graph domination. A modern Roman dominating function on a graph $G = (V;E)$ is a labeling function $f: V(G) \rightarrow \{0,1,2,3\}$ such that every vertex with label 0 is adjacent to two vertices, one of them of label 2 and the other of label 3.And every vertex with label 1 is adjacent to a vertex with label 2 or 3. The weight of a modern Roman dominating function f is $w(f) = \sum_{v \in V} f(v)$.The "Modern Roman Domination Number" $\gamma_{mr}(G)$ is the minimum $f(V) = \sum_{v \in V} f(v)$ over all such functions of G. In this paper, some properties of this new model of graph domination are introduced.

Keywords: modern dominating set; modern Roman domination number; Roman dominating function.

Mathematical subject classification: 05C69

1. Introduction

Let $G = (E, V)$ be a finite undirected and connected simple graph with a set $V(G)$ of vertices of order $n$and a set$E(G)$ of edges of size$m$. For a vertex$v \in V(G)$, the degree of a vertex v of any graph G is defined as the number of edges incident on$v$.It is denoted by$deg(v)$. The *minimum* and *maximum degrees* of vertices in $G$are denoted by $\delta(G)$ and$\Delta(G)$, respectively. The *open neighborhood*$N(v)$ of v is the set of vertices adjacent to$v$, and the *closed neighborhood* $N[v] = N(v) \cup \{v\}$. The subgraph of Ginduced by the vertices in $D$is denoted by $G[S]$. For more information refer to [(1)].

In his article published in 1999, Ian Stewart discussed a strategy of Emperor Constantine for defending the Roman Empire [(2)]. Motivated by this article, Cockayne et al.[(3)] defined a Roman dominating function (RDF) on a graph $G = (V, E)$to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function $f$is $w(f) = \sum_{v \in V} f(v)$. The Roman domination number of a graph$G$, denoted by $\gamma_R(G)$,is the minimum weight of all possible Roman dominating functions. For more details on (RDF), see [(4-6)]. Excellent treatment of domination types can be find in [(7-8)].

Here, a new model of graph domination is introduced, based on Roman domination function called “modern Roman domination”(MRDF). This definition will identify the ways of defense in war zones with four weapon types; a light weapon for pedestrians, medium weapons and heavy weapons such as tanks, missiles and the forth weapon is air force. The conditions for the success of this defense strategy are that a light weapon is supported by heavy weapons and air force coverage. The medium weapon is supported with heavy weapon or air force coverage.

The defense strategy of modern Roman domination is based in the fact that every place in which there is established a modern Roman legion (labels 2 and 3 in the modern Roman dominating function) is able to protect themselves under external attacks; and that every unsecured place (labels 0 and 1) such that (labels 0) must have stronger neighbors (label 2 and 3), while (labels 1) must have stronger neighbor (label 2 or 3). In that way, if an unsecured place (a label 0) is attacked, then the stronger neighbors could send the two legions in order to defend the weak neighbor vertex (label 0) from the attack.

2. Modern Roman dominating function

In this section a new model of domination in graphs namely, "*Modern Roman Domination"* is introduced.

Definition 2.1

A modern Roman dominating function (*MRDF*) *on a graph* $G = (V; E)$ *is a labeling function* $f: V(G) \to \{0,1,2,3\}$ *such that every vertex with label* 0 *is adjacent to two vertices one of them is of label* 2 *and the other is of label* 3 *and every vertex with label* 1 *is adjacent to a vertex with label* 2 *or* 3. *The modern Roman domination number* $\gamma_{mR}(G)$ *of* G *is the minimum* $f(V) = \sum_{v\in V} f(v)$ *over all such functions of* G.

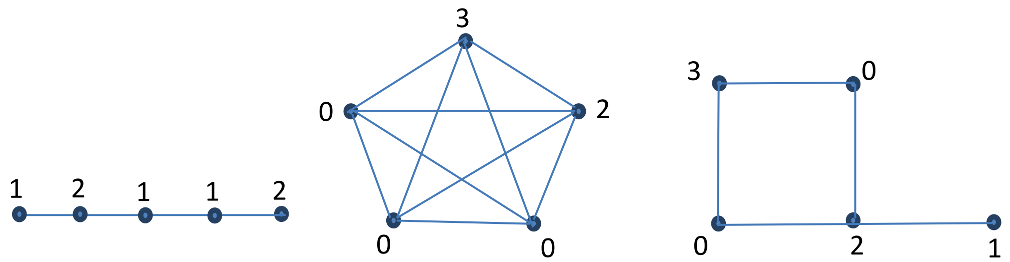

Figure 1: Modern Roman domination for various graphs

Let f be a *modern Roman dominating function* for G and let $V(G) = V_0 \cup V_1 \cup V_2 \cup V_3$ be the sets of vertices of G induced by f, where $V_i = \{v \in V : f(v) = i\}$, for all $i \in \{0, 1, 2,3\}$. It is clear that for any modern Roman dominating function f of a graph G, we have $f(V) = \sum_{v\in V} f(v) = 3|V_3| + 2|V_2| + |V_1|$

A modern Roman dominating function f can be represented by the ordered partition (R_0, R_1, R_2, R_3) of $V(G)$.

Proposition2.2. *If* G *is a graph of order* n, *having modern Roman domination* $\gamma_{mR}(G)$, *then*

1) *If* $n \geq 4$, *then* $5 \leq \gamma_{mR}(G) \leq 2n$.
2) *If there are two vertices that are adjacent to all other vertices in* G, *then* $\gamma_{mR}(G) = 5$.
3) *If* G *is null, then* $2\gamma = \gamma_{mR}(G)$.
4) $V_2 \neq \emptyset$.
5) $V_2 \cup V_3$ *is the dominating set for the induced subgraph* $G[V_0]$.
6) *If* v *is a pendant vertex, then* $f(v) \neq 0$.
7) *If* v *is an isolated vertex, then* $f(v) = 2$.

Proof.

1) There are two cases depending on the graph G whether it is connected or not, to get the lower bound as follows.

Case 1: If G is disconnected of order four, then there are two cases as follows.

i) If one of its components is K_1 and the other component is K_3, then the vertex of K_1 must be labeled by 2.The minimum label can label the vertices of K_3 by 1, 1, 2. Thus, the minimum weight in this case is 6.

ii) If two components are isomorphic to K_2 , then the minimum labels for each component is $1, 2$. Again, the minimum weight in this case is 6

Case 2: If G is connected, then the minimum weight can be got when there are two vertices that are adjacent to all other vertices.

If G is a null graph, then all vertices would be labeled 2, and this is the upper bound.

2) It is obvious from (1).
3) Again, it is clear from (1).
4) There are two cases as follows.

   Case 1) If any vertex is labeled by 0 or 1, then the other vertex must be labeled by 2 according to the definition of modern Roman domination.

   Case 2) If each vertex is labeled by 3, then the weight is not a minimum and this is a contradiction with G having (MRDF). Thus, the result is obtained.
5) It is clear by definition of modern Roman domination.
6) If v has a degree less than 2, then we cannot label this vertex by⊆. Since this vertex must be adjacent to two vertices at least. Thus, the proof is complete.
7) Since, any vertex (place) can protect its self from external attacks, is labeled by 2or 3.It is clear that the minimum of these labels is 2.

Theorem2.3.*The modern Roman domination of P_n is*

$$\gamma_{mR}(P_n) = n + \left\lceil \frac{n}{3} \right\rceil$$

Proof: Let v_1 be the first vertex in path P_n, then $f(v_1) \neq 0$, by Remark 2.2(6). Thus, there are three cases to labelv_1,$f(v_1) = 3,2,1$, as follows.

Case 1: If $f(v_1) = 3$, so there are two cases for labeling vertex v_2

a) If $f(v_2) = 0$ then $f(v_3) = 2$, therefore the sum of labels of the three vertices is equal to 5.

b) If $f(v_2) = 1$, then the minimum label value for v_3 is 1, again the sum of labels for the three vertices equals 5.

Case2: If $f(v_1) = 2$, so there are two cases about labeling vertex v_2:

a) Similar to a) in Case 1.

b) If $f(v_2) = 1$, then the minimum value to label v_3 is 1, the summation of the three vertices is 4. The minimum label of the forth vertex is 2. Therefore, the summation of the four vertices is 6.

Case 3: If $f(v_1) = 1$, then the minimum labeling of the vertex v_2 is 2, and we can label the vertices v_3 and v_4 by the same label which is equal to 1. Therefore, the summation of the four vertices equal to 5.

From the above cases, it is clear that case three gives the minimum weight of the summation. Thus, the suitable labels to get the minimum weight are as follows.

$f(v_1) = 1$, $f(v_2) = 2$,$f(v_3) = 1$, $f(v_4) = 1$,$f(v_5) = 2$, $f(v_6) = 1$, $f(v_7) = 1$, $f(v_8) = 2$ …..

Thus, $\gamma_{mR}(P_2) = n + \lceil \frac{n}{3} \rceil$.

Proposition 2.4. *For* $n \geq 3$,$\gamma_{mR}(C_n) = \begin{Bmatrix} 5, & if\ n = 4 \\ n + \lceil \frac{n}{3} \rceil, & if\ n \neq 4 \end{Bmatrix}$.

Proof. If $n = 4$, the vertices would be labeled by $\{0,2,0,3\}$, it is clear that this labeling is minimum otherwise, the same technique in Proposition 2.3 can be used to get the result.

Proposition 2.5. *For*$n \geq 1$, $\gamma_{mR}(K_n) = \begin{Bmatrix} n+1, if\ n \leq 3 \\ 5, \quad if\ n \geq 4 \end{Bmatrix}$

Proof. From Remark 2.2 (4 and 7), one can conclude that $\gamma_{mR}(K_1) = 2$, and $\gamma_{mR}(K_2) = 3$. Now, in complete graph of order 3, it is clear that the labeling$\{1,1,2\}$for its vertices is the minimum weight of summation. Thus, $\gamma_{mR}(K_3) = 4$. Eventually, if $n \geq 4$, then the two vertices say v_1, and $v_2$can be labeled by $f(v_1) = 2$,$f(v_2) = 3$ and $f(v_i) = 0, \forall i = 3,4, \dots, n$. Obviously, this labeling is the minimum weight to gain γ_{mR}.

Proposition 2.6. *For* $n \geq 3, \gamma_{mR}(S_n) = n + 1$.

Proof. There are $n - 1$ pendants in star. These vertices cannot take the label 0 according to Remark 2.2(6). Thus, the possible label to these vertices is one. Thus, the possible label to the center of star is two. Therefore, the required is obtained.

The star graph S_n is a complete bipartite $K_{1,n-1}$.

The general formula for a complete bipartite graph $K_{m,n}$ is determined by the next proposition where $n, m \geq 2$.

Theorem 2.7. *For a complete bipartite graph* $K_{m,n}$, *let* $p = min\{m, n\}$:$n, m \geq 2$ then

$$\gamma_{mR}(K_{m,n}) = \begin{cases} 5, & if\ p = 2 \\ 8, & if\ p = 3 \\ 9, & if\ p = 3 \\ 10, & if\ p \geq 5 \end{cases}.$$

Proof. Let X and Y be partite sets with $|Y| = m$, and $|X| = n$. To calculate the modern Roman domination four cases are obtained as follows.

Case1: If $p = 2$, then $f(v_1) = 2, f(v_2) = 3$, where $v_1, v_2 \in X$ and $f(v_i) = 0, \forall\ v_i \in Y$. It is clear that under these labels, the weight is the minimum.

Case2: If $p = 3$, then $f(v_1) = 2, f(v_2) = 3$, and $f(v_3) = 1$, where $v_1, v_2, v_3 \in X$. Thus, vertex v_3 must be joined with at least one vertex of label 2 in set Y.

$f(v_i) = 0$ for the other vertices in the set Y.

Case3: If $p = 4$, then $f(v_1) = 2, f(v_2) = 3$, $f(v_3) = 1$, and $f(v_4) = 1$ where $v_1, v_2, v_3, v_4 \in X$. In the same manner in Case 2in set Yone vertex is labeled by 2.

Case4: If $n \geq 5$, then $f(v_1) = f(v_3) = 2, and f(v_2) = f(v_4) = 3$, where $v_1, v_2 \in X$ and $v_3, v_4 \in Y$with$f(v_i) = 0$,for the other vertices in sets X and Y.

It is obvious that the weight in each case above is the minimum. Thus, the result is obtained.

Theorem 2.8. $\gamma_{mR}(W_n) = \begin{cases} 2\left\lfloor\frac{n-1}{3}\right\rfloor + 4, if\ n - 1 \equiv 1(mod\ 3) \\ 2\left\lceil\frac{n-1}{3}\right\rceil + 3, if\ n - 1 \not\equiv 1(mod\ 3) \end{cases}$.

Proof. It is known that the wheel graph $W_n$isa join of two graphs "$K_1 + C_{n-1}$". Let $v_1$bethe vertex which represents the graph K_1 and the vertices of the cycle of order $n - 1$ are $v_2, v_3, v_4, \ldots, v_{n-1}$. If vertex $v_1$is labeled by 3 and the other vertices are labeled as follows $f(v_2) = 0, f(v_3) = 2$, $f(v_4) = 0$, $f(v_5) = 0$, and $f(v_6) = 2$and so on…. There are two cases depend on the number of vertices in the cycle as follows.

Case1: If $n-1\equiv 1(mod\ 3)$, then according to the labeling scheme mentioned above $f(v_{n-2})=0$ and $f(v_2)=0$. Thus,v_{n-1} cannot be labeled by 0 since this vertex is not adjacent to any vertex of label2. Thus, the label of vertex v_{n-1} must be 1. By simple calculation, it is clear that that the number of vertices which are labeled by 2equals to$\lfloor\frac{n-1}{3}\rfloor$. Therefore, the weight is equal to2$\lfloor\frac{n-1}{3}\rfloor+4$.

Case2: If $n-1\not\equiv 1(mod\ 3)$, then by the way of labeling mentioned above every vertex in the cycle is labeled. Thus, the number of vertices which are labeled by 2,equals to$\left\lceil\frac{n-1}{3}\right\rceil$. Thus, the weight is $2\left\lceil\frac{n-1}{3}\right\rceil+3$.

According to this labeling technique, the weight is the minimum. So, the required result is obtained.

Proposition 2.9. *If G is a graph of order $n$having a modern Roman domination$\gamma_{mR}(G)$, then*

1) *$V_0\neq\emptyset$, then $\Delta(G)\geq 2$.*
2) *If $n\geq 4$, and $daim(G)=2$, then $\gamma_{mR}(G)=5$.*
3) *If $|V_2|=\emptyset$, then the graph G has noisolated vertex.*
4) *If $|V_2|=|V|$, then G is null graph.*

Proof.

1) It is clear.
2) Let $u,v\in V$ such that $d(u,v)=diam(G)=2$, let $f(u)=2\ and f(v)=3$,and $f(v_i)=0,\forall v_i\in V,u\neq v_i\neq v$. It is clear that,$\forall v_i\in V,u\neq v_i\neq v$, $v_i$is adjacent to two vertices $u and v$, so the result is obtained.
3) It is obvious.
4) If there is at least one edge, then one of the ends of this edge can be labeled by 1, since G has a modern Roman domination. Thus, there is no edge, therefore G is a null graph.

Proposition 2.10. *$V_3\neq\emptyset$ if and only if for all $v\in V_3$, there exist $u\in V_2$ such that $N(u)\cap N(v)\neq\emptyset$.*

Proof.$\leftarrow$ It is obvious.

$\rightarrow$ Suppose that $V_3\neq\emptyset$, then there is at least one vertex that belongs to set V_3 say v_1. Now, if $N(v_1)\cap N(u)=\emptyset$ for all$u\in V_2$, then there is no vertex in set V_0 adjacent to vertex v_1. Therefore, this vertex can be labeled by a value less than or equal to

2.Thiscontradicts with the minimum weight. Thus, there is at least one vertex $u \in V_2$ such that $N(v_1) \cap N(u) \neq \emptyset$.

3. Domination number variations

In this section, some operations are discussed as deleting a vertex or deleting an edge. This actually represents on the ground a loss of a site from the sites of the battlefield or loss of access to the site. Describing how to remedy this loss, will determine the impact on the required strength, and will ensure the protection of the battlefield in full.

Proposition 3.1.*Let Gbe a graph then$\gamma_{mR}(G - v) \leq \gamma_{mR}(G)$.*

Proof. When any vertex is deleted, a modern Roman domination must decrease or at least stays stable.

Theorem 3.2. If G is a graph then$\gamma_{mR}(G - e) \geq \gamma_{mR}(G)$ where$e = uv \in E(G)$.

Proof. There are three cases that can be obtained if an edge is deleted depending on the degree of it is the end vertices as follows.

Case 1: If $\deg(u), \deg(v) \geq 3$, then the labels of vertices in$G - e$stay the samewhich means that $\gamma_{mR}(G - e) = \gamma_{mR}(G)$, since all conditions still work.

Case 2: If $\deg(u)\ or \deg(v) \leq 2$ say u, then in$G - e$, the vertex $u$cannot be labeled by 0, so if $f(u) = 0$ in Gthen $\gamma_{mR}(G - e) > \gamma_{mR}(G)$, otherwise $\gamma_{mR}(G - e) \geq \gamma_{mR}(G)$. Also, if $\deg(u)\ or\ \deg(v)$ equals 1 say v. Again, if $f(v) = 1$ in G, then in $G - e$, vertex $v$then cannot be labeled by 1 or 0. Thus, in general $\gamma_{mR}(G - e) \geq \gamma_{mR}(G)$. Therefore, for all cases above, we get the result.

Proposition 3.3. If G has a modern Roman domination, then $V_3 \neq \emptyset$ if and only if for all $v \in V_3$, there exist $u \in V_2$ such that $N(u) \cap N(v) \neq \emptyset$.

Proof.← It is obvious.

→ Suppose that $V_3 \neq \emptyset$, then there is at least one vertex that belongs to its say v_1. Now, if $N(v_1) \cap N(u) = \emptyset$ for all $u \in V_2$, then there is no vertex in set V_0that adjacent to the vertex v_1. Therefore, this vertex can be labeled by a value less than or equal to 2. This is a contradiction with minimum weight. Thus, there is at least one vertex $u \in V_2$ such that,$N(v_1) \cap N(u) \neq \emptyset$.

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الهيمنة الرومانية الحديثة في البيانات

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المستخلص

في هذا البحث نقدم نموذج جديد للهيمنة البيانية. فدالة الهيمنة الرومانية الحديثة على البيان $G = (V;E)$ هي دالة معلمة $f: V(G) \to \{0,1,2,3\}$ حيث ان كل رأس معلم بالرمز 0 يسيطر على رأسين احدهما يكون معلم ب 2 والأخر معلم ب 3. وكل راس معلم ب 1 يسيطر على رأس معلم ب 2 او 3. وزن دالة الهيمنة الرومانية الحديثة f هو $w(f) = \sum_{v \in V} f(v)$. وعدد *الهيمنة الرومانية الحديثة* $\gamma_{mr}(G)$ هو اقل القيم $f(V) = \sum_{v \in V} f(v)$ على جميع قيم الدوال للبيان G. قدمنا في هذا البحث بعض الخواص الجديدة.