مقاسات التوسيع – SP و مقاس أبتدائي -SS
بحث مقدم من قبل

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SP-Extending Modules and SS-primary modules

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Abstract

In this work we introduce SP-extending module are defined which is a generalize of extending module (CS-module) , where amodule M is unitary left R-module, and R be acommutative ring with identity. Where An module M is called SP-extending module if every non-zero submodule of M is essential in a semi-primary direct summand of M. We will also in this work provide another concept of SS-primary module. an module M is said SS-primary if every non-zero adirect summand is semi-primary. and its relationship between this concept with SP-extending module.

Keywords: Extending module, Semiprimary-extending, Primeary submodule, Directsummend, essentialsumodule.

الخلاصة

في هذا البحث نقدم تعريفاً بأن مقاسات التوسيع–SP تم تعريفها وهي عبارة عن تعميم للمقاسات التوسيع (مقاسات–CS). حيث أن M مقاس أحادي أيسر على Rو لتكن R حلقة أبدالية بمحايد. حيث يدعى المقاس M مقاس توسع–SP أذا كان كل مقاس جزئي غير صفري من M جوهري في جداء جمع داخلي شبة بتدائي. بالاضافة الى ذالك قدمنا في هذا البحث مفهوم أخر مقاس أبتدائي–SS. يقال للمقاس M مقاس |أبتدائي–SS أذا كان كل جمع داخلي غير صفري هو شبة أبتدائي. وعلاقة بين هذا المفهوم مع مقاس توسع–SP.

الكلمات المفتاحية: الوحدة الموسعة ، التمديد شبه الجزئي ، الوحدة الفرعية الأولية ، الموجة المباشرة ، الوحدة الأساسية.

1. Introduction:

Throughout this work, all commutative rings are associative with non-zero identity and all modules are unitary left R-module. An R-module M is called extending if every submodule of M is essential in adirect summand of M [1]. In section one of this paper, we introduce the concepts of SP-extending module as ageneralization of extending module. It is well known that a proper submodule A of an R-module M is called essential if $A \cap B \neq (0)$ for each anonzero submodule B of M [1]. An R-submodule Q of an R-module M is called primary submodule if for each $r \in R$ and $m \in M$, $rm \in Q$ then $r^n M \subseteq Q$ for some positive integer n[2], and also from [2] we need the definition of semi primary submodule, wher a submodule K of M is called semi primary if $\sqrt{[K:M]}$ is a prime ideal of R. In section two of this paper we will introduced the the connotation SS-primary module, some properties of SS-primary and the interconnection between SP-extending modules and extending modulea.

2. SP-Extending module:

In this section we introduced the definition of SP-extending modules and some basic properties, examples and characterization ofthis concept.

<u>Definition (2.1):</u>

An R-module M is said to be SP-extending module if for each anon-zero proper submodule K of M is essential in a semi-primary direct summand of M.

Every SP-extending module is extending module, but the converse is not true.

Proof: suppose that M is SP-extending module, and let U be anon-zero proper submodule of M. Now then there exists a semi-primary in direct summand of M say $K \in M$ such that U is essential submodule of K, so that mean there exists essential submodule in direct summand of M, hence M is extending. The converse for example: Q as aZ-module is extending, but Q is not SP-extending because the only semi-primary submodule of Q is $< 0 >$. Also for Z of a Z-module is extending but not SP-extending, we can see that the submodule $< \overline{2} >$ is essential submodule of Z but Z of aZ-module is not semi-primary submodule of Z.

Remarks and Examples(2.2):

1. Z_{10} as a Z-module is SP-extending module, since $N_1 = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\}$, $N_2 = \{\overline{0}, \overline{5}\}$ are the only anon-zero proper submodule of Z_{10} and N_1, N_2 are essential in semi-primary direct summand of Z_{10}, $Z_{10} = N_1 \oplus N_2$, where N_1 is essential in N_1 and N_2 is essential N_2
2. A Z-module Z_6 is SP-extending module, the only anon-zero submodule of Z_6 are $< \overline{2} >$,$< \overline{3} >$, where $< \overline{2} >$ is essential in $< \overline{2} >$ and $< \overline{3} >$ is essential in $< \overline{3} >$, also $< \overline{2} >$ $and < \overline{3} >$ is semi-primary and $< \overline{2} >, < \overline{3} >$ are direct summand of Z_6.
3. The uniform module is not SP-extending module, we can see that: let K be anon-zero proper submodule of M, now since M is uniform module then theonly direct summand of M are $< \overline{0} >$ and M [1]. Therefore it does not have semi-primary direct summand of M contains essential submodule K of M.
4. The concepts SP-extending module and uniform module are independent, for example Z_6 as a Z-module is SP-extending but not uniform, and Z as a Z-module is uniform but not SP-extending.
5. Let M be are module if $N \leq K \leq M$ (where N and K are submodule of M) such that N is semi-primary submodule of M and K is semi-primary submodule of M, then N is semi-primary submodule of K.

Proof: Let N be asubmodule of K, and N is semi-primary submodule of M, so $\sqrt{(N:M)}$ is a prime ideal of R, for each submodule N of M. To prove N is semi-primary of K, we must prove that $\sqrt{(N:K)}$ is a prime ideal of R, for each asubmodule N of K. Now let $rx \in \sqrt{(N:K)}$, where $r \in R$, $x \in N$, thus $r^n x^n \in (N:K)$ for some positive integer n, we need prove that either $r \in \sqrt{(N:K)}$ or $x \in \sqrt{(N:K)}$. Since N is semi-primary of M, then either $r \in \sqrt{(N:M)}$ or $x \in \sqrt{(N:M)}$. Assume that $r \in \sqrt{(N:M)}$, thus $r^n \in (N:M)$, then $r^n M \leq N$. But $K \leq M$ so $r^n K \leq N$, that is $r \in \sqrt{(N:K)}$. Now if $x \in \sqrt{(N:M)}$ thus $x^n \in (N:M)$, then $x^n M \leq N$. But $K \leq M$ so $x^n K \leq N$, that is $x \in \sqrt{(N:K)}$. Therefor N is a semi-primary of K.

Lemma(2.3):

Let M be an R-module and R is fully semi-primary ring, then every submodule N of M is semi-primary submodule.

Proof: Let K be asubmodule of M, since R is fully semi-primary ring, so $\frac{M}{K}$ is a semi-primary R-module. we must prove that $\sqrt{(K:M)}$ is prime ideal. Let $t \in R$ and $y \in M$, such that $ty \in \sqrt{K}$ that is $(ty)^n \in K$ for some n positive integers, then $t^n y^n + K = K$ so $t^n(y^n + K) = K$, that mean that either $y^n + K = K$ or $t^n \in (K:\frac{M}{K})$, it follows that either $y^n \in K$ or $t \in \sqrt{(K:\frac{M}{K})}$. Hence either $y \in \sqrt{K}$ or $t^n \frac{M}{K} \subseteq K$, so $t^n M \subseteq K$. That is $t \in \sqrt{(K:M)}$, that mean is prime ideal, therefore K is semi-primary submodule of M.

Proposition(2.4):

Let M be a semi-simple R-module and R be fully semi-primary. Then M is SP-extending R-module.

Proof: Let N be essential submodule of M, and let M be asemi-simple R-module, then N is a direct summand of M[1]. Since R is fully semi-primary ring, then by (lemma2.3) N is semi-primary submodule. Then N is essential in semi-primary direct summand od M. Hence M is SP-extending R-module.

Proposition(2.5):

Let M be SP-extending R-module and K be adirect summand of M. Then K is uniform submodule.

Proof: Let L be anon-zero proper submodule of adirect summand K of M, then there exists a semi-primary direct summand S of M, where L is essential submodule in S, since K and S is adirect summand of M, so that mean $K \oplus T = M$ and $S \oplus N = M$ for some T and N are a submodule of M, since $M \cap M = M$, then $(K \oplus T) \cap (S \oplus N) = M$, subsequently $M = (K \cap S) \oplus (T \cap N)$. Now let $x \in K$, thus $x = e + d$ where $e \in (K \cap S)$ and $d \in (T \cap N)$, so that $x - e = d$, then $x - e \in K \cap T \cap N = 0$, that is $x = e \in (K \cap S)$ that mean $K \leq (K \cap S)$, thus $K = K \cap S$, and that mean $K \leq S$. Now let G be anon-zero submodule of K, then G is a submodule of S, so $G \cap L \neq 0$, and since L is essential in S. Then L is essential in K. Therefore K is uniform.

Remark(2.6):

We can note that that from prop.(2.5) and Remark and Example(2.2, (4)) adirect summand of a SP-extending module is not necessarily is SP-extending.

Recall that asubmodule N of an module M is called closed if it has no proper essential extension in M[5][1]. Now we have anR-module M is extending if and only if every closed submodule is adirect summand.

Now we have the following proposition for SP-extending module.

Proposition(2.7):

An R-module M is SP-extending if and only if every closed submodule of M is a semi-primary direct summand.

Proof: Let M be a SP-extending module, let K be aclosed submodule of M. Then K is essential in a semi-primary direct summand N of M, but K is closed submodule of M, then $K = N$, that mean K is a semi-primary direct summand.

Conversely: Let K be aclosed submodule in a semi-primary direct summand of M, then M is extending module by [1]. Now let W be anon-zero submodule of M and let Y is adirect summand of M. Since M is extending then W is essential in Y, hence Y is closed, and so that Y is semi-primary direct summand of M. Hence W is essential in semi-primary direct summand of M. Therefor M is SP-extending module.

Collarly(2.8):

The extending module is SP-extending if every closed submodule is semi-primary direct summand.

The next theorem gives us many characterization of SP-extending module.

Proposition(2.9):

Let M be an R-module. Then the following condition are equivalent:

1- M is a SP-extending module.
2- Every closed submodule of M is a semi-primary direct summand.
3- If W is adirect summand of the injective hull of M, then $W \cap M$ is a semi-primary direct summand.

Proof: (1) ⇨ (2) It is clear that by proposition (2.7).

(2) ⇨ (3) Let W be adirect summand of E(M), where (E(M) is denoted of injective hull) so $E(M) = W \oplus U$ for same submodule U of E(M). We a claim that $W \cap M$ is a closed in M.

assume that $W \cap M \leq_e$N(where $\leq_e$is denoted for essential submodule), where N is a submodule of M. Now let $0 \neq x \in N$, $0 \neq w \in W$ and $0 \neq u \in U$, then $x = w + u$, since M$\leq_e$E(M) and $u \in U$, then there exists $r \in R$ such that $0 \neq ru \in M$ so $rx = rw + ru$, then $rw = rx - ru \in W \cap M \leq N$ and thus $ru = rx - rw \in U \cap N$, since $W \cap M \leq_e$N, then $0 = (W \cap M) \cap U$ is essential in $N \cap U$, that is $N \cap U = 0$, therefor $ru = 0$ thus is a contradiction. Hence $W \cap M$ is closed in M, then according to the assumption $W \cap M$ is a semi-primary direct summand of M.

(3) ⇨ (1) Let W be asubmodule of M, then $W \oplus Y \leq_e$M, where Y is a relative complement of W[6]. Since M$\leq_e$E(M), then $W \oplus Y \leq_e$E(M)[6]. Thus $E(M) = E(W \oplus Y) = E(W) \oplus E(Y)$, and since E(W) is direct summand of E(M), then by hypothesis $E(W) \cap M$ is a semi-primary direct summand of M, and since $W = W \cap M \leq_e E(W) \cap M$, then $W \leq_e E(W)$. Therefore W is essential in a semi-primary direct summand of M.

Proposition(2.10):

Every direct summand of SP-extending module is SP-extending module.

Proof: Let N be adirect summand of M, then $M = N \oplus W$, where W is a submodule of M. Now let K is a submodule of N, then K is a submodule of M. But M is SP-extending module, then K is semi-primary of M, so by[Remark and Example (2.2),5] then K is a semi-primary of N. and let L be a submodule of K, such that L$\leq_e$N, we need to prove that L$\leq_e$K, so let $0 \neq T$ is a submodule of N. Hence $L \cap T \neq 0$, where L$\leq_e$N. Thus L$\leq_e$K. Now we must prove that K is direct summand of N, then by definition of SP-extending module K is direct summand of M, so $M = K \oplus U$ where U is a submodule of M. Since $N = N \cap M =$ (K⊕U), then $N = K \oplus (N \cap U)$ that is K is adirect summand of N. Therefor N is SP-extending.

Proposition(2.11):

Let M be an R-module, then M is SP-extending module if and only if for asubmodule K of M, there is adirect decomposition $M = U_1 \oplus U_2$ such that K is asubmodule of U_1 where U_1 is semi-primary submodule of M and $K + U_2 \leq_e M$.

Proof: Let K be a submodule of M and assume that M is SP-extending module. Then K $\leq_e$ N, where N is a semi-primary direct summand of M, so $M = N \oplus N_1$ where N_1 is a submodule of M, it is clear that $K \oplus N_1 \leq_e N \oplus N_1 = M$ because K $\leq_e$ N and $N_1 \leq_e N_1$[6]. Therefor $K \oplus N_1 \leq_e M$.

Conversely: Let N be a submodule of M, then by hypothesis the adirect decomposition $M = U_1 \oplus U_2$ such that N is a submodule of U_1, where U_1 is semi-primary submodule of M, and $N \oplus U_2 \leq_e M$, we must prove that N$\leq_e U_1$. Now let W be anon-zero submodule of U_1, that is W is a submodule of M. Since $N \oplus U_2 \leq_e M$, then $W \cap (N \oplus U_2) \neq 0$, so let $w = n + u$ where $w \in W, n \in N$ and $u \in U_2$, then $u = w - n \in U_1 \cap U_2 = 0$, hence $n = w \in W \cap N \neq 0$. Therefor N$\leq_e U_1$, thus M is SP-extending module.

Proposition(2.12):

Let M be finitely generated and faithful multiplication module. then M is SP-extending module if and only if R is SP-extending R-module.

Proof: Suppose that M is SP-extending module, and I is an ideal in R, since M is multiplication then $K = IM$, where K is a submodule of M, and since M SP-extending module then there exists semi-primary direct summand N such that K $\leq_e$ N. Now by [2, Theorem 3.1] then $N = PM$ for same semi-primary ideal of R, that is IM $\leq_e PM$, so I is asubmodule of P. Then there exists a submodule G of M such that $M = G \oplus PM = AM \oplus PM$ for some A is an ideal of R, so that $M = AM \oplus PM = (A \oplus P) = RM$. Then $A \oplus P = R$[7]. Hence P is a semi-primary direct summand in R, we must prove that I $\leq_e P$, now let B be anon-zero subideal of P, assume that $(I \cap B)M = 0$, then $(I \cap B)M = IM \cap BM = 0$. But BM is a submodule of PM and IM

is essential in PM, then BM= 0 , But M is faithful $B \subseteq annM = 0$, hence $A = 0$. Which is acontradiction. Thus I $\leq_e P$, then R is SP-extending R-module.
Conversely: is similarly

3. SS-primary module

In this section it can be seen the concept of SP-extending modules and extending modules are equivalent when setting the condition every direct summand is semi-primary. This gives us to introduce and study this condition under the class of semi-simple and others types of modules.

Definition (2.1):
An R-module M is called SS-primary module if every non-zero proper direct summand of M is semi-primary. Also the ring R is called SS-primary ring if R is SS-primary R-module.

Examples(3.2):
1. The Z_2 as aZ-module is SS-primary, because has no non-zero proper direct summand of Z_2.
2. The Z_{12} as aZ-module is not SS-primary Z-module, we can see that: Let $K =< \overline{6} >$, then $K + < \overline{4} > =< \overline{6} > + < \overline{4} > = Z_{12}$, hence $K =< \overline{6} >$ is adirect summand but not semi-primary, because the $\sqrt{(< \overline{6} >: Z_{12})} = \sqrt{6z}$ is not prime. Therefore Z_{12} is not SS-primary Z-module.

Proposition(3.3):
Let M be an R-module, and R is a fully semi-primary ring, then every R-module is a SS-primary module.
Proof: Let M be an R-module, over R a fully semi-primary ring, then every submodule of M is semi-primary by [lemma. 2.3]. Then M is SS-primary R-module.

Proposition(3.4):
Let M be SS-primary module. Then M is extending module if and only if M is SP-extending module.
Proof: Let K be anon-zero submodule of M and let M is extending module, then K is essential in adirect summand U of M. Since M is a SS-primary, then U is a semi-primary direct summand of M, that mean M is SP-extending module.
Converse: It is clear.

Proposition(3.5):
Let M be asemi-simple R-module. Then M is SP-extending module if and only if M is SS-primary module.
Proof: Assume that M is a SP-extending module and let U be proper submodule direct summand of M, then U is closed submodule of M. Then by (proposition 2.7) U is semi-primary of M. Therefor M is SS-primary R-module.
Converse: Assume that M is SS-primary R-module, and let W be aproper submodule of M, and since M is asemi-simple then W is direct summand of M. But M is SS-primary then W is a semi-primary in M. Also since N$\leq_e$N. Therefore M is SP-extending module.

Proposition(3.6):
Every direct summand of SS-primary module is SS-primary.
Proof: Let N be a submodule direct summand of M, and M is SS-primary, then N is semi-primary of M. Now let K is a submodule of N, then K is a submodule of M, we must prove that K is direct summand of N, then by definition of SS-primary module K is direct summand of M, so $M = K \oplus U$ where U is a submodule of M. Since $N = N \cap M = N \cap (K \oplus U)$, then $N = K \oplus (N \cap U)$ that is K is adirect summand of N. Since M is SS-primary then K is semi-

primary of M, so by[Remark and Example (2.2),6] then K is a semi-primary of N. Therefor N is SS-primary.

Conclusions:

- Every SP-extending modules is extending modules, but the converse is not true.
- Every SS-primary module is SP-extending modules.
- Every direct summand of SP-extending module is SP-extending module.
- If M is SS-primary module. Then every extending module is SP-extending module.
- Every direct summand of SS-primary module is SS-primary.

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