On Locally S −prime and Locally S −Primary Submodules

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Abstract

Throughout this article, we present locally S −prime, locally S −primary and locally S-semiprime submodules, as generalizations of S −prime, S −primary and S −semiprime submodules respectively. We investigate some properties and characterizations of these modules. For a multiplication module, the concepts of $P(N)$ −locally primary and locally S −primary are equivalent. Finally, we give the following result, if M is multiplication module, then K is locally primary submodule, if there exists a $P(N)$ −locally primary ideal of R such that $K = IM$ and $M \neq IM$. We provided that, every locally S −semiprime submodule of multiplication module is the intersection of some locally S −prime submodule.

Keyword. Multiplication module, $S(N)$ −Locally prime, S −prime, S −semiprime and S −primary submodule.

1. Introduction

The localization of a module is a development to present denominators in a module for a ring. All the more decisively, it is a methodical approach to develop another module M_P out of a given module M containing algebraic fractions $\frac{m}{s}$, where the denominators s go in a given multiplicative system P of R. The system has turned out to be fundamental, especially in algebraic geometry, as the connection amongst modules and parcel hypothesis. Localization of a module generalizes localization of a ring. The localization of rings and modules have important role in module theory.

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In this paper, we utilize the localization for generalizing the concepts of S −prime and S −primary submodule. The localization were investigated by many authors for example ([1], [2]).

It is well known that prime submodules play an important role within the theory of modules over commutative rings. To this point there was a variety of studies in this issue. For numerous researches you'll look ([3], [4], [5], [6], [7], [8]). One of the main interests of many researchers is to generalize the notion of prime submodule with the aid of using different ways. As an instance, $\mathrm{S(N)}$ −locally prime which is a generalization of prime, was first introduced and studied in [9]. If B , $\mathrm{C} \leq \mathrm{M}$, then the set $(\mathrm{B{:}\,C}) = \{\mathrm{r} \in \mathrm{R{:}\,rC} \in \mathrm{B}\} \leq \mathrm{R}$. If $\mathrm{N} \leq \mathrm{M}$, then N is said to be prime in M, if whenever $\mathrm{rm} \in \mathrm{N}$, for $\mathrm{m} \in \mathrm{M}$ and $\mathrm{r} \in \mathrm{R}$, then either $\mathrm{m} \in \mathrm{N}$ or $\mathrm{r} \in (\mathrm{N{:}\,M})$ and N is said to be primary submodule in M if $\mathrm{rm} \in \mathrm{N}$, for $\mathrm{m} \in \mathrm{M}$ and $\mathrm{r} \in \mathrm{R}$, then either $\mathrm{m} \in \mathrm{N}$ or $\mathrm{r^n} \in (\mathrm{N{:}\,M})$ [8], [10], [11], [12], [13], [14]. Feller and Swokowski [12] calls a module as a prime module if $(0{:}\,\mathrm{M}) = (0{:}\,\mathrm{N})$ or equivalently, $\{0\}$ is a prime submodule in M. Feller and Swokowski showed that an R −module M is prime if and only if either M is torsion-free or M non-singular. More results on prime and primary submodule were investigated in ([15], [16], [17], [18]).

Gungoroglu [19] was introduced the notion of S −prime and S −strongly prime submodule. If M is an R −module and $End(M)$ denoted the ring of R −endomorphisms of M, then a submodule N of M as an S −prime submodule (S −strongly prime submodule), if whenever $f(m) \in N$, for $f \in End(M)$ and $m \in M$, then either $m \in N$ or $f(M) \subseteq N$ (if whenever $f(M) \in N$, for $f \in End(M)$ and $m \in M$, then $m \in N$) and he showed that every $S-$ prime (S −strongly prime) submodule are prime (strongly prime) submodule. Alhashmi and Dakheel [20] were introduced S −primary submodule, they called a submodule N of M as an S −primary submodule if whenever, $f(m) \in N$, for $f \in End(M)$ and $m \in M$, then either $m \in N$ or $f^n(M) \subseteq N$ for some positive integer n, they provided that a submodule N of M is S −prime if and only if $\big(N{:}\,f(M)\big) = \big(N{:}\,f(K)\big)$, for any every $f \in End(M)$ and $N \subset K$. If $n = 2$, then N is said to be semiprime submodule. Alhashmi and Dakheel [20] showed that a submodule is S −prime if and only if it is both S −semiprime and S −primary submodule in M.

In this article, we present the ideas of locally S −prime, locally S −semiprime and locally S −primary submodule as generalizations of S −prime, S −semiprime and S −primary submodule. If $N < M$, then it is called locally S −prime, if N_P is S −prime in M_P for every maximal ideal $P < R, S(N) \subseteq P$. If $\{0\}$ is locally S −prime submodule, then M is said to be locally S −prime module which is an extension of prime module. Give N to be a locally S −prime submodule of a R −module M. On the off chance that K is a submodule of M with the end goal that $K \subseteq N$, at that point N/K is a locally S −prime submodule of M/K. Likewise, we give that each maximal submodule of an augmentation module is a locally S −prime submodule. N be a submodule of M. A submodule N of M is called locally S −semiprime, where N_p is a S-semiprime submodule of M_p, for each maximal perfect P of R. The crossing point of any group of S −semiprime is S −semiprime. All the more for the most part, a legitimate submodule N of a

R −module M is said to be locally S −prime submodule of M, if N_P is a S −prime submodule of M_P, for each maximal perfect P of R, with $P(N) \subseteq P$. In the event that {0} is locally S −prime submodule, at that point M is said to be locally S −prime module which is an expansion of prime module. We give that to an increase module, the ideas of $P(N)-$ locally prime and locally S −prime are proportionate. At long last, we give the accompanying outcome, if M is a loyal duplication module, at that point K is locally prime submodule if and only if there exists a $P(N)-$ locally prime ideal of R with the end goal that $K = IM$ and $M \neq IM$.

All through this paper, R denotes a commutative ring with identity and modules M are unitary left R −modules. For a module M, $Prad(M)$ and $Z(M)$ are the prime radical and the singular submodules of M. If S is a multiplicative closed system, then M_S is an R_S −module which is called the localization (quotient) of M at S [5]. If P is a prime ideal in R, then $R - S$ forms a multiplicative closed system, then we denote M_P for the localization of M at $R - S$. If $f: M \to N$ is a homomorphism, then we denote the homomorphism extension $f_S: M_S \to N_S$, where it is defined by $f_S\left(\frac{m}{s}\right) = \frac{f(m)}{s}$, for $m \in M$ and $s \in S$. It is well-known that $Hom_R(M,N)_S \cong Hom_{R_S}(M_S,N_S)$. An element $r \in R$ is called prime to N if $rm \in N$, for $m \in M$, then $m \in N$[1], thus $r \in R$ is not prime to N if $rm \in N$ for some $m \in M - N$. We indicate the arrangement of all components of R that are not prime to N by $S(N)$ and $P(N)$ is the arrangement of all components $r \in Rm$ for which r isn't prime to N. A module M is said to be multiplication module if for each submodule N of M there exists a ideal I in R with the end goal that $N = IM$ [15].

2. Locally S −prime and Locally S −primary

In this section we introduce Locally S −prime and Locally S −primary submodule as generalizations of S −prime and S −primary submodules. If M is an R −module and $End(M)$ denoted the ring of R −endomorphisms of M, then Gungoroglu [19] calls a submodule N of M as an S −prime submodule (S −strongly prime submodule), if whenever $f(m) \in N$, for $f \in End(M)$ and $m \in M$, then either $m \in N$ or $f(M) \subseteq N$ (if whenever $f(m) \in N$, for $f \in End(M)$ and $m \in M$, then $m \in N$) and he showed that every $S-$ prime (S −strongly prime) submodule are prime (strongly prime) submodule.

Definition 2.1. If $N < M$, then N is called locally S −prime, if N_P is S −prime submodule of M_P for every maximal ideal P in R with $S(N) \subseteq P$.

Proposition 2.2. If N is S −prime in a module M, then $N$is locally S −prime.

Proof. Let N be an S −prime submodule, we must show that N is locally S −prime. Let $f_P \in End(M)_P$ such that $f_P\left(\frac{m}{s}\right) \in N_P$, then there exists $f \in End(M)$, such that $f_P\left(\frac{m}{s}\right) = \frac{f(m)}{s}$, then $\frac{f(m)}{s} \in N_P$,then there exists $r \notin P$ such that $rf(m) \in N$, then $f(rm) \in N$, so $rm \in N$ or $f(rM) \in N$, therefore $rM \subseteq N$ or $m \in N$ or $rf(M) \subseteq N$. Hence $rM \subseteq N$ or $m \in N$ or $rf(M) \subseteq N$. But

$S(N) \subseteq P$ gives that $m \in N$ or $f(M) \subseteq N$, then $\frac{m}{s} \in N_P$ or $f_P(M_P) \subseteq N_P$. Thus N is locally S −prime submodule.

In view of the above theorem, we conclude that every S −prime submodule is locally S −prime, but the converse is not hold, for instance, if $\mathrm{M} = \mathrm{Z}_5 \oplus \mathrm{Z}_7$ as a Z −module, consider $\mathrm{N} = \{0\} \oplus \mathrm{Z}_7$, then N is not S −prime. To show N is locally S −prime:

Since M is semisimple, then End(M) is regular, consequently the localization of End(M) over every maximal ideal is a field. Suppose that $\left(\frac{m}{s}, \frac{n}{t}\right) \neq (0,0)$ and $f\left(\frac{m}{s}, \frac{n}{t}\right) \in N_P$, then $\left(\frac{m}{s}, \frac{n}{t}\right) \in f^{-1}(N_P)$, but $f^{-1}(N_P)$ is maximal submodule and M_P has only two maximal submodule, then $f^{-1}(N_P) = N_P$ or $f^{-1}(N_P) = (Z_5)_P \oplus \{0\}_P$. If $f^{-1}(N_P) = (Z_5)_P \oplus \{0\}_P$, then $\frac{n}{t} = 0$. If $(0, \frac{n'}{t}) = f(\frac{m}{s}, \frac{n}{t}) = f(\frac{m}{s}, 0) = (0,0)$, then we get that $\left(\frac{m}{s}, \frac{n}{t}\right) = (0,0)$, which is contradiction. Thus $f^{-1}(N_P) = N_P$.

Proposition 2.3. Let $N < M$, then that following are equivalent:

1- N is $S(N)$ −locally prime submodule.
2- N is locally S −prime submodule.

Proof. $(1 \Rightarrow 2)$ Suppose that N is $S(N)$ −locally prime submodule, then N_P is a prime submodule in M_P and since M is cyclic, then M_P is also cyclic. Thus N_P is S −prime. Hence N is locally S −prime. $(2 \Rightarrow 1)$ Assume that N is locally S −prime submodule, this implies N_P is S −prime in M_P, then N_P is prime in M_P. Hence N is $S(N)$ −locally prime submodule.

Corollary 2.4. Let N be a locally S −prime in M, then $(N_P : M_P)$ is an S −prime ideal in R_P,for each maximal $P < R$.

If M is an R −module, we denote $T(M)$ for the torsion submodule of M which is defined by $T(M) = \{m \in M; rm = 0 \text{ for some } 0 \neq r \in R\}$. It is easy to show that $T(M)_P = T(M_P)$, then we have the following consequence results $T(M) = M$ if and only if $T(M_P) = M_P$ and $T(M) = 0$ if and only if $T(M_P) = 0$.

Proposition 2.5. If R is an integral domain and Mbe a nonzero torsion module, then M has no locally S −prime submodule.

Proof. Since M is torsion module, then M_P is also torsion module. Now, since R is an integral domain, then R_P is a field, then M is divisible, so M_P has no S −prime submodule. Hence it has no locally S −prime submodule.

Proposition 2.6. Let M be a module over an integral domain, if $T(M) \neq M$ and $\ker f \subseteq T(M)$ for all $0 \neq f \in End(M)$, then $T(M)$ is a locally S −prime submodule, where $T(M)$ is the torsion submodule of M.

Proof. Let $h\left(\frac{m}{s}\right) \in T(M_P)$, where $h \in End(M_P)$ and $\frac{m}{s} \in M_P$. If $h = 0$, then $h(M_P) = 0 \in T(M_P)$ and we are done. Now, let us assume that $h \neq 0$, since $h\left(\frac{m}{s}\right) \in T(M_P)$, so there exists $0 \neq \frac{x}{t} \in R_P$, with $\frac{x}{t} h\left(\frac{m}{s}\right) = h\left(\frac{x}{t}\frac{m}{s}\right) = 0$, then $\frac{x}{t}\frac{m}{s} \in \ker h(M_P) \subseteq T(M_P)$. Hence $\frac{xm}{ts} \in T(M_P)$, this implies that there exists $0 \neq \frac{r}{t_1} \in R_P$ such that $\frac{r}{t_1}\left(\frac{xm}{ts}\right) = \left(\frac{rx}{t_1 t}\right)\frac{m}{s} = 0$. Hence $\frac{m}{s} \in T(M_P)$ and $\frac{rx}{t_1 t} \neq 0$.

Proposition 2.7. Let N be a maximal submodule of M. If N is a fully invariant, then N is locally S −prime submodule.

Proof. If N is a maximal fully invariant M, then N_P is also maximal fully invariant in M_P. Suppose that $f\left(\frac{m}{s}\right) \in N_P$, where $f \in End(M_P)$. If $\frac{n}{s} \notin N_P$, then $M_P = N_P + (Rm)_P \subseteq N_P$. Now, $f(M_P) = f(N_P) + f((Rm)_P) \subseteq N_P$. Hence N is locally S −prime submodule.

Proposition 2.8. Let N be fully invariant of M. If $(N{:}M) = (N{:}f(K))$ for all $N \subset K$, for all $f \in End(M\ )$, then N is locally S-prime submodule of M.

Proof. Let $h(\frac{m}{s}) \in N_p$,where $h \in End(M_p)$ and $\frac{m}{s} \in M_p$ and suppose that $\frac{m}{s} \notin N_p$, we must prove that $\mathrm{h}(\mathrm{M_p}) \subseteq N_p$ .Now , $N_p \subset N_p + (Rm)_p$, hence by assumption $(N{:}M) = (N{:}h(K))$, this implies that $(N_p{:}M_p) = (N_p{:}h(K_p))$, but $1 \in (N_p{:}h(N_p){:}(Rm)_p$, since $h(N_p) + h(Rm)_p \subseteq N_p$ .Thus $1 \in (N_p : h(M_p)$ which implies that $h(M_p) \subseteq N_p$.

Proposition 2.9. Let N be a locally S-prime submodule of an R −module M, then $(N{:}f(M)) = (N{:}f(K))$, for all $N \subset K$ and for all $f \in End(M)$.

Proof. Let N be a locally S-prime and let K be a submodule of M containing N properly.If $f \in End(M)$ then $f_p \in End(M_p)$ and clearly $(N{:}f(M)) \subseteq (N{:}f(K))$ then $\left(N_p : f_p(M_p)\right) \subseteq \left(N_p : f_p(K_p)\right)$. Since $N \subset K$ then $N_p \subseteq K_p$, there exsist $\frac{x}{s} \in K_p$ and $\frac{x}{s} \notin N_p$ .Assume $\frac{r}{t} \in \left(N_p{:}f_p(K_p)\right)$, this implies that $\frac{r}{t} f_p\left(\frac{x}{s}\right) \in N_p$ .Now, define $h_p : M_p \to M_p$ by $h_p\left(\frac{x}{s}\right) = \frac{r}{t} f_p\left(\frac{x}{s}\right)$ for all $x \in M$.Clearly ,$h_p \in End(M_p)$, also $h_p\left(\frac{x}{s}\right) = \frac{r}{t} f_p\left(\frac{x}{s}\right) \in N_p$, but N_p is an S-prime submodule of M_p and $\frac{x}{s} \notin N_p$, thus $h_p(M_p) \subseteq N_p$. This implies that $\frac{r}{t} f_p(M_p) \subseteq N_p$ and hence $\frac{r}{t} \in (N_p{:}f_p(N_p)$.

Theorem 2.10. Let N be fully invariant in M, then N is a locally S-prime in M if and only if $(N{:}f(M)) = (N{:}f(K))$, for every $f \in End(M)$.

Proposition 2.11. Let $\phi \in End(M)$ and N be a fully invariant locally S-prime of an $R -$ module $\phi(M) \not\subset N$, then $\phi^{-1}(N)$ is also locally S-prime submodule of M.

Proof. First, we must prove that $\phi_p^{-1}(N_p)$ is a proper submodule of M_p. Suppose that $\phi_p^{-1}(N_p) = M_p$, then $\phi_p(M_p) \subseteq N_p$, hence $\phi(M) \subseteq N$ which is a contradiction. Now, let $f_p\left(\frac{m}{s}\right) \in \phi_p^{-1}(N_p)$, where $f_p \in End(M_p)$ and $\frac{m}{s} \in M_p$. If $\frac{m}{s} \notin \phi_p^{-1}(N_p)$, then $\phi_p\left(\frac{m}{s}\right) \notin N_p$, which implies that $\frac{m}{s} \notin N_p$, since N is fully invariant, then N_p is also fully invariant. We only have to show that $f_p(M_p) \subseteq \phi_p^{-1}(N_p)$. Since $f_p\left(\frac{m}{s}\right) \in \phi_p^{-1}(N_p)$, then $(\phi_p \circ f_p\left(\frac{m}{s}\right) = \phi_p(f_p\left(\frac{m}{s}\right) \in N_p$, but N_p is S-prime submodule of M_p and $\frac{m}{s} \notin N_p$, therefore $(\phi_p \circ f_p)(M_p) \subseteq N_p$. This implies that $f_p(M_p) \subseteq \phi_p^{-1}(N_p)$.

Proposition 2.12. Let K be a fully invariant submodule contained in N such that $\frac{N}{K}$ is a locally S-prime submodule of $\frac{M}{K}$, then N is a locally S-prime submodule of M.

Proof. Suppose that $\frac{N}{K}$ is locally S-prime in $\frac{M}{K}$, then $\frac{N_p}{K_p}$ is an S-prime of $\frac{M_p}{K_p}$. To show N_p is an S-prime submodule of M_p, we must show that $f_p\left(\frac{m}{s}\right) \in N_p$, where $f_p \in End\ (M_p)$ and $\frac{m}{s} \in M_p$, if $\frac{m}{s} \notin N_p$, then $f_p(M_p) \subseteq N_p$. Let $g\colon \frac{M_p}{K_p} \to \frac{M_p}{K_p}$ by $g\left(\frac{x}{s} + K_p\right) = f_p\left(\frac{x}{s}\right) + K_p$ for all $f_p \in End(M_p)$ and $\frac{x}{s} \in M_p$, where $\frac{x}{s}, \frac{y}{t} \in M_p$, this means $\frac{x}{s} - \frac{y}{t} \in K_p$. Let $\frac{x}{s} + K_p = \frac{y}{t} + K_p$, then $f_p\left(\frac{x}{s} - \frac{y}{t}\right) \in f_p(K_p) \subseteq K_p$, since K_p is a fully invariant in M_p. This implies that $f_p\left(\frac{x}{s}\right) - f_p\left(\frac{y}{t}\right) \in K_p$. Thus, $f_p\left(\frac{x}{s}\right) + K_p = f_p\left(\frac{y}{t}\right) + K_p$. Now $\left(\frac{m}{s} + K_p\right) = f_p\left(\frac{m}{s}\right) + K_p \in \frac{N_p}{K_p}$, but $\frac{N_p}{K_p}$ is S-prime in $\frac{M_p}{K_p}$ and $\frac{m}{s} + K_p \notin \frac{N_p}{K_p}$ hence $g\left(\frac{M_p}{K_p}\right) \subseteq \frac{N_p}{K_p}$, thus $\frac{(f_p(M_p)+K_p)}{K_p} \subseteq \frac{N_p}{K_p}$, which means $f_p(M_p) + K_p \subseteq N_p$ and $f_p(M_p) \subseteq f_p(M_p) + K_p \subseteq N_p$, so $f_p(M_p) \subseteq N_p$. Thus N is a locally S-prime in M.

Proposition 2.13. Let $f\colon M \to M'$ be an epimorphism, where M, M' are $R-$modules and $M'$is $M-$projective. Suppose that N is a locally S-prime in M' such that $kerf \subseteq N$, then $f(N)$ is a locally S-prime.

Proof. Suppose that $f_p(N_p) = M'_p$, since f is an epimorphism, then f_p is also an epimorphism , thus $f_p(N_p) = f_p(M_p)$, hence $M_p = N_p + (kerf)_p$, therefore $M_p = N_p$,which is a contradiction. Hence $f_p(N_p)$ is a proper submodule of M'_p. Now, let $h \in End(M'_p)$ such that $h\left(\frac{m'}{s}\right) \in f_p(N_p)$, $\frac{m'}{s} \in M'_p$ and $\frac{m'}{s} \notin f_p(N_p)$, we have to show that $h_p(M'_p) \subseteq f_p(N'_p)$. Since f_p is an epimorphism and $\frac{m'}{s} \in M'_p$, then there exists $\frac{m}{s} \in M_p$ such that $f_p\left(\frac{m}{s}\right) = \frac{m'}{s} \notin f_p(N_p)$, thus $\frac{m}{s} \notin N_p$. Since M' is an $M-$projective module, then M_p is also M_p-projective module , hence there exists a homomorphism $k_p\colon M'_p \to M_p$ such that $f_p \circ k_p = h_p$. Clearly, $f_p \circ k_p \in$

$End\left(M_p\right)$. Now, we have $f_p\left(\left(k_p \circ f_p\right)\left(\frac{m}{s}\right)\right)=\left(f_p \circ k_p\right)\left(f_p\left(\frac{m}{s}\right)\right)=h_p\left(\frac{m'}{s}\right) \in f_p(N_p)$ and since $(kerf)_p \subseteq N_p$, we get $\left(k_p \circ f_p\right)\left(\frac{m}{s}\right) \in N_p$ but N_p is S-prime and $\frac{m}{s} \notin N_p$, therefore $\left(k_p \circ f_p\right)\left(M_p\right) \subseteq N_p$ and hence $k_p\left(M_p'\right) \subseteq N_p$. Thus $f_p\left(k_p\left(M_p'\right)\right) \subseteq f_p\left(N_p\right)$, which implies that $h_p\left(M_p'\right) \subseteq f_p(N_p)$.

Theorem 2.14. If N is locally S-prime and K is a submodule of M such that $K \subseteq N$, then $\frac{N}{K}$ is locally S-prime in $\frac{M}{K}$ and $\frac{M}{K}$ is an M −projective module.

Proposition 2.15. Suppose that K is locally S-prime in M and $N \leq M$, which is M −projective, then either $N \subseteq K$ or $K \cap N$ is a locally S-prime submodule of N.

Proof. If $N \nsubseteq K,$ then $K \cap N < N$ and hence $(K \cap N)_p \subset N_p$. Let $f \in End(N),$ then we get $f_p \in End\ (N_p)$ and $\frac{x}{s} \in N_p$ with $f_p\left(\frac{x}{s}\right) \in K_p \cap N_p$. Suppose that $\frac{x}{s} \notin K_p \cap N_p$, then $\frac{x}{s} \notin K_p$, we must show that $f_p(N_p) \subseteq K_p \cap N_p$. Consider $i_p : N_p \to M_p$ inclusion map, since N_p is M_p −injective module, then there exists $h_p: M_p \to N_p$, such that $h_p \circ i_p = f_p$. Clearly, $h_p \in End(M_p)$. On the other hand $f_p\left(\frac{x}{s}\right)=\left(h_p \circ i_p\right)\left(\frac{x}{s}\right)=h_p\left(\frac{x}{s}\right) \in K_p$. Since K_p is an S-prime and $\frac{x}{s} \notin K_p$, hence $h_p\left(M_p\right) \subseteq K_p$. Also $f_p\left(N_p\right)=(h \circ i\,)_p\left(\frac{x}{s}\right)=h_p(N_p) \subseteq N_p$ and $f_p\left(N_p\right)=h_p\left(N_p\right) \subseteq h_p(M_p) \subseteq K_p$. Therefore $f_p\left(N_p\right) \subseteq K_p \cap N_p$

Proposition 2.16. Suppose that N is a maximal submodule of a multiplication module M, then N is locally S-prime.

Proof. If N is maximal submodule of a multiplication M, so N_p is maximal M_p. Since M and M_p are multiplication modules, so we get $N=(N\colon M)M$ then $N_p=((N\colon M)M)_p=(N\colon M)_p M_p=\left(N_p\colon M_p\right)M_p$ and thus for every $f_p \in End(M_p)$ we have $f_p\left(N_p\right)=(N_p\colon M_p)f_p\left(M_p\right) \subseteq N_p$,this implies that N_p is a fully invariant submodule of M_p, hence N_p is a maximal fully invariant. Therefore, N_p is S-prime in M_p, so N is locally S-prime.

Lemma 2.17. Suppose that M is a non-zero multiplication, then $\{0\}$ is a locally $S(N)$ −locally prime.

Proof. (⇒) Suppose that $\{0\}$ is a locally S-prime, then $\{0\}_p$ is an S-prime submodule of M_p, hence prime, which implies that $\{0\}$ is $S(N)$ −locally prime.

(⇐) Assume that $\{0\}$ is $S(N)$ −locally prime means that $\{0\}_p$ is prime, but we have M_p is a multiplication module then $\{0\}$ is an S-prime submodule of M.

Definition 2.18. If $\{0\} < M$ is locally S-prime, then M is called locally S-prime module.

Theorem 2.19. If $N < M$ and M multiplication M, then N is $S(N)$ −locally prime submodule of M if and only if it is locally S-prime submodule of M.

Definition 2.20. If $N < M$, then N is called locally S-semiprime if N_p is an S-semiprime submodule of M_p, for each maximal ideal P of R.

Proposition 2.21. Suppose that $M < N$, then N is locally semiprime if and only if, whenever $f_p^n\left(\frac{m}{s}\right) \in N_p$ for some $f_p \in End(M_p)$, $\frac{m}{s} \in M_p$ and $n \geq 2$, then $f_p\left(\frac{m}{s}\right) \in N_p$.

Proof. Use mathematical induction on the positive integer $n \geq 2$. The proposition is true for $n = 2$ by definition. Suppose that it is true for $n-1$, means that $f_p^{n-1}\left(\frac{m}{s}\right) \in N_p$, then $f_p\left(\frac{m}{s}\right) \in N_p$. Now, suppose that $f_p^n\left(\frac{m}{s}\right) \in N_p$, then $f_p^2(f_p^{n-2}\left(\frac{m}{s}\right) \in N_p$, which implies that $f_p^{n-1}\left(\frac{m}{s}\right) = f_p(f_p^{n-2}\left(\frac{m}{s}\right) \in N_p$. Thus $f_p\left(\frac{m}{s}\right) \in N_p$.

Proposition 2.22. If N is locally S-semiprime in M, then it is $S(N)$ −locally semiprime.

Proof. Suppose that N is locally semiprime, then N_p is an S-semiprime submodule of M_p, hence semiprime. Thus N is $S(N)$ −locally semiprime.

Proposition 2.23. If M is a module, then:

1- Any locally S-prime submodule of M is locally S-semiprime.
2- If $N = \cap N_\alpha$ for all $\alpha \in \Lambda$, where each N_α is locally S-prime submodule of M, then N is locally S-semiprime .

Proposition 2.24. Let M be a non-zero multiplication R −module, then $\{0\}$ is a locally semiprime if and only if it is locally S-semiprime.

Proof. Suppose that $\{0\}$ is a locally semiprime submodule of M, this implies that $\{0\}_p$ is a semiprime submodule of M_p. Now, let $f_p^2\left(\frac{m}{s}\right) = 0_p$, for some $f_p \in End(M_p)$ and $\frac{m}{s} \in M_p$. Since M_p is a multiplication module, then $(Rf(m))_p = (IM)_p$, hence $R_p f_p\left(\frac{m}{s}\right) = I_p M_p$, for some I_p of R_p. Now, $I_p R_p f_p(\frac{m}{s}) = I_p^2 M_p$, which implies that $I_p f_p(\frac{m}{s}) = I_p^2 M_p$. Thus $I_p(f_p^2(\frac{m}{s}) = I_p^2 f_p(M_p)$, but $f_p^2\left(\frac{m}{s}\right) = 0_p$, hence $I_p^2\left(f_p(M_p)\right) = 0_p$, then $I_p f_p(M_p) = 0_p$. Also $I_p f_p(\frac{m}{s}) \subseteq I_p f_p(M_p)$, therefore $I_p f_p(\frac{m}{s}) = 0_p$, hence $I_p^2 M_p = 0_p$, then $I_p M_p = 0_p$, hence $R_p f_p\left(\frac{m}{s}\right) = 0_p$, therefore $f_p\left(\frac{m}{s}\right) = 0_p$. Thus $f_p\left(\frac{m}{s}\right) \in \{0\}_p$.

Also $I_p f_p(\frac{m}{s}) \subseteq I_p f_p(M_p)$, therefore $I_p f_p(\frac{m}{s}) = 0_p$, hence $I_p^2 M_p = 0_p$, then $I_p M_p = 0_p$, hence $R_p f_p\left(\frac{m}{s}\right) = 0_p$, therefore $f_p\left(\frac{m}{s}\right) = 0_p$. Thus $f_p\left(\frac{m}{s}\right) \in \{0\}_p$.

Definition 2.25. Suppose that M is a module, if $\{0\}$ is a locally S-semiprime submodule of M, then M is called locally S-semiprime module.

Theorem 2.26. If $0 \neq M$ is multiplication module and $N < M$, then N is locally semiprime if and only if it is locally S-semiprime.

Proof. Suppose that $N < M$. Since M is a multiplication module, then M_p is also multiplication. Now, $\left(\frac{M}{N}\right)_p = \frac{M_p}{N_p}$ is a multiplication module. Clearly, N_p is a zero of a module $\frac{M_p}{N_p}$, assume that N_p is semiprime and since $\frac{M_p}{N_p}$ is amultiplication module, then N_p is an S-semiprime and hence, N is locally S-semiprime.

Corollary 2.27. Every locally S-semiprime submodule of multiplication module is the intersection of some locally S-prime submodule.

Proposition 2.28. Let $f: M \to M^{\prime}$ be an epimorphism. If N is locally S-semiprime submodule of M, such that $ker f \subseteq N$, then $f(N)$ is locally S-semiprime submodule of $M^{\prime}$, whenever $M^{\prime}$ is an M −projevtive module.

Proof. Clear that $f(N)$ is a proper submodule of $M^{\prime}$, $f(N)_p$ is also proper in $M_p^{\prime}$. Now, let $h_p^2\left(\frac{m}{s}\right) \in f_p(N_p)$, where $h_p \in End\ (M_p')$ and $\frac{m^{\prime}}{s} \in M_p^{\prime}$, we must show that $h_p\left(\frac{m^{\prime}}{s}\right) \in f_p(N_p)$.

Since f is an epimorphism, then f_p is also epimorphism, so for all $\frac{m^{\prime}}{s} \in M_p^{\prime}$ there exists $\frac{m}{s} \in M_p$ such that $f_p\left(\frac{m}{s}\right) = \frac{m^{\prime}}{s}$. We have $M^{\prime}$ is $M -$ projective, then $M_p^{\prime}$ is also M_p −projective, then there exists a homomrphism $k_p: M_p^{\prime} \to M_p$ such that $f_p \circ k_p = h_p$.

Now, $h_p^2\left(\frac{m}{s}\right) = h_p(h_p\left(\frac{m^{\prime}}{s}\right) \in f_p(N_p)$, this implies that $\left(f_p \circ k_p \circ f_p \circ k_p \circ f_p\right)\left(\frac{m}{s}\right) \in N_p$, but N_p is S-semiprime, then $(k_p \circ f_p)(\frac{m}{s}) \in N_p$ and hence $h_p\left(\frac{m^{\prime}}{s}\right) \in f_p\left(N_p\right)$.

Corollary 2.29. Suppose that N,$K \leq M$, such that $K \subseteq N$ such that N is locally S-semiprime, then $\frac{N}{K}$ is locally S-semiprime, where $\frac{M}{K}$ is M −projective.

Definition 2.30. If $N < M$, then N is said to be locally S −primary submodule of M, if N_P is S −primary in M_P, for every maximal ideal P of R, with $P(N) \subseteq P$.

It clear that every locally S −prime submodule is locally S −primary submodule.

Proposition 2.31. If N is S −primary submodule of M, then N is $P(N)$ −locally primary submodule.

Proof. Suppose that N is locally S −primary, this implies that N_P is an S −primary submodule of M_P. If $\frac{r}{s} \in R_P$ and $\frac{m}{t} \in M_P$ with $\frac{r}{s}\frac{m}{t} \in N_P$. Let $\frac{m}{t} \notin N_P$, define $f: M_P \to M_P$ by $f\left(\frac{x}{t_1}\right) = \frac{r}{s}\frac{x}{t_1}$ for all $\frac{x}{t_1} \in M_P$. Clearly, $f \in End(M_P)$ and $f\left(\frac{m}{t}\right) = \frac{r}{s}\frac{m}{t} \in N_P$, but N_P is S −primary and $\frac{m}{t} \notin N_P$, then there exists a positive integer $f^n(M_P) \subseteq N_P$, then $\left(\frac{r}{s}\right)^n M_P \subseteq N_P$. Consequently, $\left(\frac{r}{s}\right)^n \in (N_P: M_P)$. Thus N is a $P(N)$ −locally primary.

Proposition 2.32. Suppose that $0 \neq M$ is a multiplication module, then $\{0\}$ is a $P(N)$ −locally primary if and only if it is locally S −primary.

Proof. Let $\{0\}$ be a $P(N)$ −locally primary, then $\{0\}_P$ is a primary submodule in M_P and hence S −primary. So, $\{0\}$ is a locally S −primary submodule in M. The converse is obvious.

Definition 2.33. If M is a nonzero R −module and zero submodule of M is a locally S −primary submodule in M, then M is said to be locally S −primary module.

Theorem 2.34. Suppose that M is a multiplication module, then N is $P(N)$ −locally primary if and only if it is locally S −primary.

Proof. Clearly, N is the zero of $\frac{M}{N}$. Since, N is $P(N)$ −locally primary, then locally S −primary and the converse is clear.

Proposition 2.35. If $f: M \to M'$ is an epimorphism and $N < M$ is a locally S −primary such that $\ker f \subseteq N$, then $f(N)$ is a locally S −primary, where M' is projective module.

Proof. Suppose that N is locally S −primary, then $f(N) < M'$. Now, N_P is an S −primary submodule of M_P, we must show that $f_P(N_P)$ is S −primary. Let $h_P\left(\frac{m'}{s}\right) \in f_P(N_P)$, where $h_P \in End(M'_P)$ and $\frac{m'}{s} \in M'_P$. Suppose that $\frac{m'}{s} \notin f(N_P)$, since f_P is an epimorphism and $\frac{m'}{s} \in M'_P$, then there exists $\frac{m}{s} \in M_P$ such that $f_P\left(\frac{m}{s}\right) = \frac{m'}{s}$. Consider the following diagram, since $\frac{m'}{s} \notin f_P(N_P)$ and Since $M_p^{'}$ is an M_p − projective and $\frac{m^{'}}{s} \notin f_p(N_p)$, then there exists a homomorphism k_p such that $f_p \circ k_p = h_p$. Now, $h_p\left(\frac{m^{'}}{s}\right) \in f_p(N_p)$, this implies that $(f_p \circ k_p)\left(\frac{m^{'}}{s}\right) \in f_p(N_p)$ and hence $(f_p \circ k_p)\left(f(\frac{m}{s})\right) \in f_p(N_p)$, but $(kerf)_p \subseteq N_p$, then $(k_p \circ f_p)(\frac{m}{s}) \in N_p$, but N_p is an S-primary submodule of M_p and $\frac{m}{s} \notin N$, then there exists a positive integer n such that $(k_p \circ f_p)^n(M_p) \subseteq N_p$. Therefore $f_p[(k_p \circ f_p)^n(M_p)] \subseteq f_p(N_p)$, which implies that $h^n(M_p^{'}) \subseteq f_p(N_p)$.

Corollary 2.36. If N is a locally S-primary submodule of M, then for any $K_p \subseteq N_p$, we have $\frac{N}{K}$ is a locally S-primary submodule of $\frac{M}{K}$, whenever $\frac{M}{K}$ is an M −projective module.

Proposition 2.37. Suppose that N is a proper submodule of M, then N is a locally S-primary and locally S-semiprime if and only if it is a locally S-prime.

Proof. Let N be locally S-primary and locally S-semiprime, then N_p is an S-pimary and S-semiprime submodule of M_p. To show N_p is an S-prime, let $f_p\left(\frac{m}{s}\right) \in N_p$, we must show that $f_p\left(M_p\right) \subseteq N_p$. Since N_p is an S-primary submodule of M_p and $\frac{m}{s} \notin N_p$, then $f^n\left(M_p\right) \subseteq N_p$ for some positive integer, but N_p is an S-semiprime, hence $f_p\left(M_p\right) \subseteq N_p$. Conversely is clear.

Corollary 2.38. A module M is locally S-primary and locally S-semiprime it is locally S-prime.

Proposition 2.39. If N is primary submodule of M, then N is $P(N)$ −locally primary.

Proof. Suppose that P is maximal ideal of R, $P(N) \subseteq P$ and N is a primary submodule of M. Clear that $rad(N:M) \subseteq P(N) \subseteq P$ and N_p is a proper submodule of M_p. Now, let $\frac{rx}{sp} \in N_p$, for $\frac{r}{s} \in R_p$, where $s,p \notin P$ and $\frac{x}{p} \in M_p$, then $qrx \in N$, for some $q \notin P$ and since N is primary and $q \notin (N:M)$, then $q^n \notin (N:M)$, we get $rx \in N$. Hence $x \in N$ or $r^nM \subseteq N$, which implies that either $\frac{x}{p} \in N_p$ or $\left(\frac{r}{p}\right)^n M_p = (r^nM)_p \subseteq N_p$. Hence N is $P(N)$ −locally primary.

Proposition 2.40. Let $K < M$, where M is a faithful multiplication Rmodule and R is commutative ring with identity, then K is $P(N)$ −locally primary submodule of M.

Proof. Since R_P is a local ring with the unique maximal ideal I_P, then K_P is primary submodule with $K_P = I_PM_P$ and $M_P \neq I_PM_P$. Hence K is $P(N)$ −locally primary.

Lemma 2.41. Let $N < M$, then $\left(rad(N:M)\right)_P \subseteq P(N_P)$.

Proof. $\left(rad(N:M)\right)_P = rad(N_P:M_P)$. If $\frac{r}{s} \in rad(N_P:M_P)$, then $\left(\frac{r}{s}\right)^n M_P \subseteq N_P$, for some positive integer n, then there exists $\frac{m}{t} \in M_P \backslash N_P$ such that $\left(\frac{r}{s}\right)^n \frac{m}{t} \in N_P$, so $\frac{r}{s} \in P(N_P)$. Hence $\left(rad(N:M)\right)_P \subseteq P(N_P)$.

Lemma 2.42. Suppose that M_i is an R_i −modules, for $i = 1,2$, then for the module $M = M_1 \times M_2$ as an $R_1 \times R_2$ −module we have the following:

1- If N_i is $P(N_i)$ −locally primary submodules of M_i, for $i = 1,2$, then $N_1 \times M_2$ and $M_1 \times N_2$ are $P(N_1 \times N_2)$ −locally primary submodule of M.
2- If $N_1 \times N_2$ is $P(N_1 \times N_2)$ −locally primary submodule of M, then N_i is $P(N_i)$ −locally primary submodules of M_i.

Proof. Let N_i be $P(N_i)$ −locally primary in M_i, then $(N_i)_P$ is a primary submodule in $(M_i)_P$. If $\left(\frac{r_1}{s_1},\frac{r_2}{s_2}\right)\left(\frac{m_1}{t_1},\frac{m_2}{t_2}\right) \in (N_1)_P \times (M_2)_P$, then $\left(\frac{r_1 m_1}{s_1 t_1},\frac{r_2 m_2}{s_2 t_2}\right) \in (N_1)_P \times (M_2)_P$, so $\frac{r_1 m_1}{s_1 t_1} \in (N_1)_P$, since $(N_1)_P$ is primary submodule in $(M_1)_P$, then $\frac{m_1}{t_1} \in (N_1)_P$ or $\left(\frac{r_1}{t_1}\right)^n (M_1)_P \subseteq (N_1)_P$. If $\frac{m_1}{t_1} \in (N_1)_P$, then $\left(\frac{m_1}{t_1},\frac{m_2}{t_2}\right) \in (N_1)_P \times (M_2)_P$, otherwise $\left(\frac{r_1}{s_1}\right)^n (M_1)_P \subseteq (N_1)_P$, then $\left(\frac{r_1}{s_1},\frac{r_2}{s_2}\right)^n M_P \in (N_1)_P \times (M_2)_P$, then $N_1 \times M_2$ is $P(N_1 \times N_2)$ −locally primary submodule of M. Similarly, we can get the second part.

Proposition 2.43. Suppose that $N,L \leq M$, then

1- $N_P \subseteq Prad(N_P)$.
2- $Prad(N \cap L)_P \subseteq Prad(N)_P \cap Prad(L)_P$.
3- $Prad(Prad(N)_P) = Prad(N)_P$.

Proposition 2.44. Let M be an R −module and K be a primary completely irreducible submodule containing $N \cap L$, where N and L are submodules of M, then K is $P(N)$ −locally primary completely irreducinle. Furthermore, $Prad(N_P \cap L_P) = Prad(N_P) \cap Prad(L_P)$.

It is clear that every multiplication R −module has a maximal submodule and every proper submodules contains in a maximal submodule [14]. So, let R_P be the localization of R, then R_P is a local ring and M_P is local module.

Proposition 2.45. Suppose that M be a faithful multiplication R −module, where R is a commutative ring with identity, then K is locally primary submodule if and only if there exists an $P(N)$ −locally primary ideal of R such that $K = IM$ and $M \neq IM$.

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CONFLICT OF INTERESTS.
There are non-conflicts of interest

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