Q-modules

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Abstract

Let R be a commutative ring with identity and all R-modules are unitary. A ring R is called q-ring if every ideal of R is quasi-injective . In this work, we introduce the concept of Q-module as a generalization for the concept of q-ring. An R-module M is said to be a Q-module if every R-submodule of M is quasi-injective. We characterize such modules and study their properties . Relationships between Q-modules and other classes of modules are given. Q-modules are studied over artinian ring. Endomorphisms rings of Q-modules are examined.

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1-Basic properties of Q-modules.

Jain, Mohamed and Singh in [9] called a ring R is q-ring if every right ideal of R is quasi-injective. In this section, we generalize the concept of q-ring to modules. Also we introduce some properties of Q-modules and give some characterizations of Q-modules. J(R) will denote the Jacodson radical of R. For an R-module M, ann(M)=$\{r\in R : rM=0\}$, and End(M) is the endomorphisms ring of M.

Before we give the definition of Q-module , we must recall the concept of quasi-injective module. An R-module M is called quasi-injective if for each monomorphism f: N $\rightarrow$ M, where N is an R-submodule of M, and homomorphism g: N $\rightarrow$ M, there is a homomorphism h: M $\rightarrow$ M such that $h\circ f=g$[6].

Definition 1.1: An R-module M is called a Q-module if every R-submodule of M is quasi-injective.

Examples and remarks 1.2:

1-Z_n is a Q-module over Z for each positive integer $n\geq 2$.

2-Everey semi-simple R-module is a Q-module.

3-Z_{p^∞} is a Q-module over Z .

4-Q(the set of all rational numbers) as a Z-module is not a Q-module, but it is a qusi-injective Z-module.

5- Every submodule of Q-module is a Q-module.

6-The inverse image of Q-module is not necessary a Q-module. In fact, Z_2 is a Q-module over Z and Z is not a Q-module over Z.

7- The direct sum of Q-modules need not be a Q-module as the following example shows: Z_2

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and Z_4 are Q-modules over Z. Since $Z_2\oplus Z_4$ is not quasi injective, $Z_2\oplus Z_4$ is not a Q-module over Z. Furthermore, Z_4 is a Q-module over Z, but $Z_4\oplus Z_4$ is not a Q-module over Z.

Before we give the following proposition we need the following lemma which appeared in [12,P.244,Ex. 27 A].

Lemma 1.3: An R-module M is quasi-injective if and only if M is a quasi- injective R/I-module for any ideal I $\subseteq$ann(M).

The proof of the following proposition follows from Lemma 1.3.

Proposition 1.4: M is a Q-module over R if and only if M is a Q-module over R/I for any ideal I of R contained in ann(M).

Recall that an R-submodule N of M is called fully invariant if f(N) $\subseteq$N, for all f $\in$End(M) [21].The following lemma shows that the fully invariant submodule inherits the quasi-injectivity of the module.

Lemma 1.5: Every fully invariant R-submodule of a quasi –injective R-module is quasi-injective.

An R-submodule N of M is called essential if $N\cap L\neq 0$ for every non-zero R-submodule L of M[11].

Theorem 1.6: Let M be an R-module .The following statements are equivalent:

1-M is a Q-module.

2-M is quasi-injective and every essential R-submodule of M is fully invariant in M.

3-Every essential R-submodule of M is quasi-injective .

Proof: (1)$\Rightarrow$(2).Suppose that M is a Q-module. Thus M is quasi-injective. Let N be an essential R-submodule of M . Hence E(N)=E(M)[18,Prop.2.22]. Because M is a Q-module , N is quasi-injective . It follows that N is a fully invariant R-submodule of E(N)=E(M)[10, Th.6.74]. Hence N is a fully invariant R-submodule of M.

(2) $\Rightarrow$(3).It follows from Lemma 1.5.

(3) $\Rightarrow$(1). Since M is an essential R-submodule of M , so M is quasi-injective. Let N be a proper non-zero R-submodule of M. Thus $N\oplus C$ is an essential R-submodule of M, where C is the relative intersection complement of N[11]. Therefore, $N\oplus C$ is a quasi-injective R-submodule of M. By [12, Prop.6.73], N is quasi-injective . ▪

A nonzero R-submodule N of M is called quasi-invertible if Hom(M/N,M)=0[13].

Theorem 1.7: Let M be an R-module such that J(End(M))=0. Then M is a Q-module if and if only if every quasi-invertible R-submodule of M is quasi-injective.

Proof: The only if part is trivial. Suppose that every quasi-invertible R-submodule of M is quasi-injective. It is clear that M is a quasi-invertible R-submodule of M, thus M is quasi-injective. Let N be a proper non-zero R-submodule of M. Then $N\oplus C$ is an essential R-submodule of M , where C is the relative intersection complement of N[11]. By[13 ,Th.3.8,Ch.1], $N\oplus C$ is quasi-invertible and by hypothesis, $N\oplus C$ is quasi-injective. Whence N is quasi-injective[12,Prop.6.73]. Therefore M is a Q-module. ▪

Proposition 1.8: Let M be an R-module such that every non-zero R-submodule of M contains a copy of R. Then M is a Q-module if and only if every R-submodule of M is injective.

Proof: Suppose that M is a Q-module and N is an R-submodule of M . Then N is quasi-injective.Let I be an ideal of R and let f: I $\to$N be an R-homomorphism. Because N is quasi-injective , there exists h: N $\to$N such that $h|_I$ =f, that is , h is an extension of f. Since R is contained in N ,by Baer's criterion[12], N is injective. The converse is clear. ▪

Corollary 1.9: Let R be an integral domain and M is a torsion free R-module . Then M is a Q-module if and only if every R-submodule of M is injective.

Recall that an R-module M is said to be divisible if M=rM for every r of R which is not a zero-divisor [12]. It is well-known that if R is a principal ideal domain and M is an R-module , then M is injective if and only if M is divisible[18, Th.2.8].

Proposition 1.10: Let R be a principal ideal domain and M is a torsion free R-module. Then the following statements are equivalent:

1-M is a Q-module.
2-Every R-submodule is injective.
3-Every R-submodule is divisible.

An R-module M is called uniform if every non-zero R-submodule of M is essential[8]. In Ex. and Remark 1.2(7), we have seen that a direct sum of Q-modules need not be a Q-module . In the following two propositions, we discuss conditions under them a direct sum of Q-modules is a Q-module.

Proposition 1.11: Let M_1 and M_2 be uniform R-modules such that Hom(M_i,M_j)=0 for all distinct i, j , $1\leq$ i, j$\leq$ 2. Then M=$M_1 \oplus M_2$ is a Q-module if and only if M_1 and M_2 are Q-modules and M is quasi-injective.

Proof: Suppose that M_1 and M_2 are Q-modules and M is quasi-injective . Let N be an R-submodule of M and let f:A $\to$N be a homomorphism, where A is an R-submodule of N. f can be extended to an endomorphism g of M .Then N = π_1(N) $\oplus$ π_2(N) , where π_i :M $\to$ M_i is the projection homomorphism for each i=1,2. Since M_i is a Q-module for each i=1,2, π_i (N) is a quasi-injective R-submodule of M_i. Because M_i is uniform for each i=1,2, thus π_i (N) is an essential R-submodule of M_i. Hence π_i (N) is an essential quasi-injective R-submodule of M_i and consequently π_i (N) is a fully invariant R-submodule of E(π_i (N))=E(M_i)[6,Corollary 19.3] [18, Prop. 2.22].It follows that π_i (N) is a fully invariant R-submodule of M_i. From the assumption, Hom(M_i ,M_j)=0 for all distinct i,j,$1\leq$ i,j ≤ 2, we get End(M)=End(M_1) $\oplus$End(M_2)[11, p.87] and hence g(N)=h(π_1(N)) $\oplus$k(π_2(N))$\subseteq$ π_1(N) $\oplus$ π_2(N) $\subseteq$N, where h $\in$End(M_1) and k$\in$End(M_2). From above argument, we have g(N) $\subseteq$N and $g|_A$ =f which proves that N is quasi-injective. Therefore M is a Q-module. The other direction follows from Ex.and Remark 1.2(5). ▪

Proposition 1.12: Let M_1 and M_2 be R-submodules such that Hom(M_i, M_j)=0 for all distinct i, j,$1 \leq$i, j ≤ 2, and every direct summand of E(M_i) is fully invariant for each i=1,2. Then M= $M_1 \oplus M_2$ is a Q-module if and only if M_1 and M_2 are Q-modules and M is quasi-injective.

Proof: The only if part follows from Ex.and Remark 1.2(5). To prove if part , let N be an R-submodule of M and f:A $\to$N be a homomorphism , where A is an R-submodule of N. Then N=π_1(N) $\oplus$ π_2(N) , where π_i :M $\to$$M_i$ is the projection homomorphism for each i=1,2 . Since M_i is a Q-module for each i=1,2, thus π_i (N) is quasi-injective . It follows that π_i (N) is a fully invariant R-submodule of E(π_i (N))[6,Corollary 19.3]. Because E(π_i (N)) is a direct summand of E(M_i)[18, Prop.2.22] and by hypothesis, E(π_i (N)) is a fully invariant R-submodule of E(M_i), thereforeπ_i (N) is a fully invariant R-submodule of E(M_i) which means that π_i (N) is a fully invariant R-submodule of M_i. Now, f can be extended to an endomorphism g of M. Since Hom (M_i,M_j) = 0 for all distinct i,j,$1\leq$i,j$\leq$2, so End (M)= End (M_1) $\oplus$ End (M_2) [11,p.87]. Thus g(N) = h (π_1(N)) $\oplus$ k (π_2(N))$\subseteq$ π_1(N) $\oplus$ π_2(N) $\subseteq$N, where h $\in$End (M_1) and k$\in$End (M_2). Consequently, we have g(N) $\subseteq$ N and $g|_A$ =f which implies that N is quasi-injective and hence M is a Q-module. ▪

2- Q-modules and quasi-injective modules.

As we have mentioned in section one that every Q-module is quasi-injective, the converse is not true in general (Ex. and Remak1.2(4)). In this section, we give sufficient conditions on a quasi-injective module to be a Q-module.

Recall that an R-module M is called duo if every R-submodule of M is fully invariant [21].

Proposition 2.1: If M is a quasi-injective duo R-module , then M is a Q-module .

Proof: It follows from Lemma 1.5. ▪

Corollary 2.2: If M is duo R-module , then M is a Q-module if and only if M is qasi-injective .

An R-submodule N of M is said to be satisfy Baer criterion if for each R-homomorphism f: N$\to$M, there exist r$\in$ R such that f(x)=rx for all x$\in$ N, and an R-module M is said to be satisfy Baer criterion if every R-submodule of M satisfies Baer criterion[1]. It is easily seen that every R-submodule satisfying Baer criterion is fully invariant.

Corollary 2.3: Let M be a Baer criterion R-module. Then M is a Q-module if and only if M is quasi-injective.

Recall that an R-submodule N of M is annihilator if N=ann_M (I) for some ideal I of R[1].

Corollary 2.4: Let M be an R-module in which all of its R-submodules are annihilator . Then M is a Q-module if and only if M is quasi-injective.

Proof: The only if part is easy . Suppose that M is quasi-injective and N is an R-submodule of M, and hence $N=ann_M(I)$ for some ideal I of R. Let $f\in$ End(M).Thus $0=f(IN)=If(N)$ which means that $f(N) \subseteq ann_M(I)=N$. It follows that N is a fully invariant R-submodule of M. .By Corollary 2.2, M is a Q-module. ▪

It is easily seen that M is a duo R-module if and only if every cyclic R-submodule of M is fully invariant in M .Thus from Corollary 2.2 ,we have the following result .

Corollary 2.5: Let M be an R-module such that every cyclic R-submodule of M is fully invariant in M. Then M is a Q-module if and only if M is quasi-injective.

The converse of Prop.2.1 is not true . The following example indicate that.

Example 2.6: $M=Z_3\oplus Z_3$ is a Q-module over Z , but M is not duo. In fact , $N=\{(x, x): x \in Z_3\}$ is not a fully invariant Z-submodule of M.

The following proposition give another condition on a quasi-injective R-module to be a Q-module.

Proposition 2.7: Let M be an R-module in which each R-submodule is of the form f(K) ,where K is a fully invariant R-submodule of M and f is an idempotent endomorphism of M. Then M is a Q-module if and only if M is quasi-injective.

Proof: Suppose that M is quasi-injective . Let N be an R-submodule of M. There exists a fully invariant R-submodule K of M and an idempotent endomorphism f of M such that N=f(K). By Lemma 1.5,K is quasi-injective, and $M=f(M) \oplus (I-f)(M)$, where I: $M\rightarrow M$ is the identity endomorphism of M. Since K is fully invariant, thus $K=f(K)\oplus(I-f)(K)$. Because K is quasi-injective, N=f(K) is quasi-injective [12,Prop.6.73]. Therefore M is a Q-module. The converse is direct. ▪

Under certain condition on the elements of quasi-injective R-module, we get another characterization of Q-module.

Proposition 2.8: Let M be an R-module such that for all x and y in M, $y \notin Rx$ implies $ann(x) \not\subset ann(y)$. Then M is a Q-module if and only if M is quasi-injective.

Proof: The only if part is direct. Suppose that M is not a Q-module , so by Corollary 2.5, there is a cyclic R-submodule Rx of M which is not fully invariant in M, that is there exists $f \in$ End(M) such that $f(Rx) \not\subset Rx$. This means that there exists $y\in Rx$ such that $f(y) \notin Rx$ .Let y=rx, where $r\in R$. Thus $ann(x) \subseteq ann(y) \subseteq ann(f(y))$. This contradicts our hypothesis. Therefore every cyclic R-submodule of M must be fully invariant in M. By Corollary 2.5, M is a Q-module. ▪

Recall that an R-submodule N of M is said to be closed in M if and only if N has no proper essential extension in M [8]. It is known that if M is a quasi-injective R-module and N is a closed R-submodule of M , then N is quasi-injective [8]. From this fact , we conclude the following result .

Proposition 2.9: Let M be an R-module such that every R-submodule of M is closed . Then M is a Q-module if and only if M is quasi-injective.

3-Q-modules and cohopfian (hopfian) modules.

Recall that An R-module is called cohopfian(hopfian) if every injective (surjective) endomorphism is isomorphism [3].

Proposition 3.1: Let M be a cohopfian Q-module over R and let N be an R-submodule of M. Then E(N) and N are cohopfian.

Proof: Because M is a cohopfian Q-module ,E(M) i ٢٨٩ cohopfian [3, Th.8]. Let N be an R-submodule of M.By [18,Prop.2.22], E(N) is a direct summand of E(M). But E(M) is cohopfian, so E(N) is cohopfian [3,Th.6]. Since N is quasi-injective and E(N) is cohopfian, so N is cohopfian. ▪

Corollary 3.2: Let M be an R-module. If M is a Q-module and M_P is cohopfian for each prime ideal P of R., then every R-submodule of M is cohopfian.

Proof: By [14, Prop. 2.3], M is a cohopfian Q-module and hence every R-submodule of M is cohopfian (Prop.3.1). ▪

Corollary 3.3: Let M be an R-module. If M is a Q-module and M_P is artinian for each prime ideal P, then every R-submodule of M is cohopfian.

Proof: By [14,Prop.2.4], M_P is cohopfian and by Corollary 3.2 every R-submodule of M is cohopfian. ▪

Proposition 3.4: Let M be a Q-module over R and let N be an R-submodule of M . Then N is cohopfian if and only if E(N) is cohopfian.

Proof: It follows from[3, Th.8]. ▪

An R-module M is said to be perfect if it satisfies the descending chain condition on cyclic R-submodule. Equivalently, an R-module M is perfect if M satisfies the descending chain condition on a finitely generated R-submodule[6].

Proposition 3.5: Let M be a finitely generated perfect R-module . If M is a Q-module , then every R-submodule of M is cohopfian.

Proof: Let $f:M \rightarrow M$ be a monomorphism. Therefore $M \supseteq f(M) \supseteq f^2(M) \supseteq f^3(M) \supseteq \ldots$is descending chain of finitely generated R-submodules of M. Because M is perfect , so $f^n(M)=f^{n+1}(M)$ for some $n \in Z$. But f is monomorphism, hence M=f(M) which means that f is epimorphism. Whence M is cohopfian. By Prop. 3.1, every R-submodule of M is cohopfian. ▪

Proposition 3.6: Let M be a non zero R-module and let M be a Q-module . If N is a non-zero R-submodule of M, then N is not cohophian if and only if for any positive integer n, $N=N_n \oplus (\overset{n}{\underset{i=i}{\oplus}} C_i)$, where $N \cong N_n$ and $C_i \neq 0$ for all i=1,2,…,n.

Proof: It follows from [3, Prop.1]. ▪

Corollary 3.7: Let M be an R-module .If M is a Q-module and N is a hopfian R-submodule of M , then N is cohopfian.

Proof: To prove that N is cohopfian , let $f: N \rightarrow N$ be a monomorphism and $f(N)\neq N$. Consequently, f(N) is a proper R-submodule of N .Thus f^{-1}: $f(N) \rightarrow N$ is isomorphism. Since M is a Q-module, so N is quasi-injective .Whence there exists an extension homomorphism g: $N\rightarrow N$ such that $g|_{f(N)}=f^{-1}$. But f^{-1} is an isomorphism, so $N=I(f(N))\oplus \ker g = f(N)$

$\oplus$ kerg [11,Corollary 3.4.10].Because $g\big|_{f(N)} = f^{-1}$, g is onto and kerg $\neq$ 0.It follows that g is not one-to-one and hence N is not hopfian, a contradiction. Therefore N is cohopfian. ▪

It is known that every finitely generated module is hopfian . From this fact, we have the following result.

Corollary 3.8: Every finitely generated R-submodule of Q-module is cohopfian.

Recall that an R-module M has finite dimensional(Goldie dimension) provided that M contains no infinite independent families of non-zero R-submodule [8].

Corollary 3.9: Let M be an R-module and let N be an R-submodule of M. If M is a Q-module and N has finite dimensional then N is cohopfian.

Proof: Suppose that N is not cohopfian, then by Prop.3.6,for any positive integer n, $N = N_n \oplus (\overset{n}{\underset{i=i}{\oplus}} C_i)$, where $N \cong N_n$ and $C_i \neq 0$ for all i=1,2,…,n. This implies that N contains infinite independent families of non-zero R-submodule , a contradiction. Therefore N is cohopfian.

Recall that an R-Module M is cancellation (or direct sum cancellation) whenever M $\oplus$ H $\cong$ M $\oplus$ K, implies H $\cong$ K. Equivalently if M $\oplus$ H = B $\oplus$ K with M $\cong$ B implies H $\cong$ K for any R-modules K,H[3].

Proposition 3.10: Let M be an R-module . If M is a Q-module and N is an R-submodule of M, then N is cancellation if and only if N is cohopfian.

Proof: It follows from [3,Prop.5] . ▪

Recall that an R-module M is directly finite provided that M is not isomorphic to any proper direct summand of itself[8].

Proposition 3.11: Let M be a non-zero R-module . If M is a Q-module and N is a non-zero R-submodule of M , then N is directly finite if and if N is cohopfian.

Proof: Suppose that N is directly finite and N is not cohopfian, By Prop.3.6, for any positive integer n, $N = N_n \oplus (\overset{n}{\underset{i=i}{\oplus}} C_i)$, where $N \cong N_n$ and $C_i \neq 0$ for all i=1,2,…,n. This contradicts that N is directly finite. Hence N is cohopfian.

Conversely; suppose that N is cohopfian and N is not directly finite , that is, $N=N_1 \oplus C$ with $N \cong N_1$,C $\neq$ 0. Since $N \cong N_1$, there exists an isomorphism f: $N \rightarrow N_1$. Let j: $N_1 \rightarrow N_1 \oplus C$ be the injection homomorphism. j is monomorphism, but it is not epimorphism. Thus $j \circ f$:N $\rightarrow$ N is monomorphism, but it is not epimorphism. Therefore N is not cohopfian. This contradicts our hypothesis. Consequently, N is directly finite. ▪

4- Q-modules over artinian rings.

Recall that an R-module M is finendo if M is finitely generated over End(M) [6].

The following proposition gives a necessary condition for Modules over artinian ring to be Q-modules.

Proposition 4.1: If R is an artinian ring , then every R-submodule of Q-module is finendo.

Proof: It is follows from [6, Th.19.16A]. ▪

Proposition 4.2: If M is an R-module such that every R-submodule of M is injective modulo annihilator, then M is a Q-module .

Proof: Let N be an R-module of M , thus N is an injective R/ann(M)-submodule of M . This implies that N is a quasi-injective R/ann(M)-submodule of M, and hence N is quasi- injective R-submodule of M(Lemma ٢٩٠ 1.3) . Therefore M is a Q-module . ▪

The converse of Prop. 4.2 is not true in general as the following example shows : Let M= $\underset{p}{\oplus} Z_p$ be a Z-module , where p is a prime number . M is a semi-simple Z-module and hence a Q-module . But M is not injective over Z/ann(M) $\cong$ Z.

In the following proposition , we give a condition under which the converse of Prop. 4.2 is true.

Proposition 4.3: Let R be an artinian ring and M is an R-module . Then M is a Q-module if and only if every R-submodule of M is injective modulo annihilator.

Proof: Suppose that M is a Q-module . Let N be an R-submodule of M . Since M is a Q-module over artinian ring , then N is a finendo quasi-injective R-submodule of M (Prop. 4.1). Therefore N is injective modulo annihilator [6,Th.19.14A].The converse follows from Prop. 4.2. ▪

An R-submodule N of M is said to satisfy the double annihilator condition with respect to M if N=ann_M (ann_R (N)) [6].

Theorem 4.4 : Let R be an artinian ring and let M be an R-module. Then the following statements are equivalent:

1-M is a Q-module

2-M is quasi-injective and every R-submodule of M satisfies the double annihilator condition.

3-M is quasi-injective and duo.

Proof: (1)$\Rightarrow$(2). Suppose that M is a Q-module , so M is quasi- injective . Let N be an R-submodule of M .By Prop. 4.1, N is finendo. Since M is quasi-injective and N is finendo , then N satisfies double annihilator condition [6,Th.19.10A].

(2) $\Rightarrow$(3). Let N be an R-submodule of M. Thus N satisfies the double annihilator condition. Let f$\in$ End (M). 0 =f (ann_R (N)N) = ann_R (N)f (N) and hence f (N)$\subseteq$ ann_M (ann_R (N))=N which proves that N is fully invariant.

(3) $\Rightarrow$(1).By Corollary 2.2. ▪

Recall that an R-module M is bounded if and only if ann(M)=ann(x) for some x in M [6].

Proposition 4.5: Let R be an artinian ring and let M be a Q-module over R. Then an R-submodule N of M is bounded if and only if N is cyclic.

Proof: Suppose that N is bounded, that is ,ann (N) = ann (x) for some x in N. But M is a Q-module over artinian ring , so every R-submodule of M satisfies the double annihilator condition (Th. 4.4). Therefore, N= ann_M (ann_R (N)) = ann_M (ann_R (x))=Rx.. The converse is trivial. ▪

5- Q-modules and other modules.

Recall that an R-module M is called multiplication if every R-submodule of M is of the form IM for some ideal I of R[2].The following proposition shows that the two concepts Q-module and quasi-injective module are equivalent in the class of multiplication modules.

Proposition 5.1: If M is a multiplication R-module , then M is a Q-module if and only if M is quasi-injective.

Proof: The only if part is clear . Suppose that M is quasi-injective . Let N be an R-submodule of M. Because M is multiplication , N=IM for some ideal I of R. Let $f \in$ End(M) . Thus f(N)=f(IM)=If(M) $\subseteq$ IM=N, therefore N is fully invariant in M .By virtue of Prop.2.1 , M is a Q-module . ▪

An R-module M is called distributive if $X \cap (N+L)=(X \cap N)+(X \cap L)$ for all R-submodules X,N and L of M[5].And an R-module M is called Z-regular if for every element $m \in M$,there exists $f \in M^{*}$=Hom(M,R) such that m=f(m)m[22].

The proof of the following lemma follows from [2],[5,Prop.7] and [15, Prop. 2.1].

Lemma 5.2: Let M be an R-module. M is multiplication if M is one of the following cases:

1-Cyclic.

2-finitely generated , distributive and ann(M)=0.

3-Z-regular and End(M) is commutative.

Corollary 5.3: Let M be as in Lemma 5.2. Then M is a Q-module if and only if M is quasi-injective.

Proposition 5.4: If M is a multiplication R-module and S=End(M), then M is a Q-module over R if and only if M is a Q-module over S.

Proof: Suppose that M is a Q-module over R, and let N be an R-submodule of M .Since M is multiplication, so End(M) is commutative[14,Prop.1.1]. Because N is fully invariant (see the proof of Prop.5.1), N is an S-submodule of M . Thus N is a quasi-injective R-submodule of M if and only if N is a quasi-injective S-submodule of M. Therefore M is a Q-module over S. Similarly ,we can prove the converse. ▪

Before we state the following proposition , we recall that an R-module M is called CS-module if every R-submodule of M is essential in a direct summand of M[17]. And an R-module M is said to be continuous if M is CS-module and every R-submodule of M isomorphic to direct summand of M is itself summand[7].

Proposition 5.5: Let M be a uniform multiplication R-module. Then M is a Q-module if and only if M is continuous and $M \oplus M$ is CS-module .

Proof : Suppose that M is a Q-module .Thus M is quasi-injective ,so M is continuous and $M \oplus M$ is a quasi-injective [6]. Hence $M \oplus M$ is CS-module [17].

Conversely; suppose that M is continuous and $M \oplus M$ is CS-module . Since M is a continuous uniform R-module and $M \oplus M$ is a CS-module , so M is quasi-injective [7,Lemma 1.2.3]. Therefore M is a Q-module (prop.5.1). ▪

Proposition 5.6: Let M be a uniform multiplication R-module .If M is finite composition length, then M is a Q-module if and only if $M \oplus M$ is CS-module .

Proof: Suppose that M is a Q-module . Then $M \oplus M$ is quasi-injective [6]. Therefore $M \oplus M$ is CS-module [17].

Conversely; assume that $M \oplus M$ is CS-module . Thus M is quasi-injective [7, Prop.2.1.1]. By Prop.5.1, M is a Q-module. ▪

Recall that a ring R is perfect if R has d.c.c. on principal ideals[6].

Proposition 5.7: Let R be a perfect ring and let M be a uniform cyclic R-module .Then M is a Q-module if and only if $M \oplus M$ is CS-module.

Proof: Suppose that $M \oplus M$ is CS-module. Hence M is quasi-injective [7, Prop.2.2.2]. By Corollary 5.3 , M is a Q-module. The converse is direct . ▪ ٢٩١

Recall that an R-module M is called simply embedded i. M has a simple R-submodule which is essential in M [20], and the R-module M is almost finitely generated if M is not finitely generated , but every proper R-submodule of M is finitely generated [20].

Proposition 5.8: Let M be a multiplication almost finitely generated artinian R-module . Then M is a Q-module if and only if M is simply embedded.

Proof: It follows from [20,Prop.3.1]. ▪

Theorem 5.9: Let M be a multiplication R-module and let ann(M) be a prime ideal of R. Then M is a Q-module if and only if every quasi-invertible R-submodule of M is quasi-injective .

Proof : The only if part is direct . Suppose that every quasi-invertible R-submodule of M is quasi-injective. Let N be an R-submodule of M and let C be an intersection relative complement of N. By [11], $N \oplus C$ is essential R-submodule of M. By [13 , Th.3.11,Ch.1], $N \oplus C$ is quasi-invertible and by hypothesis, $N \oplus C$ is quasi-injective. Consequently, N is quasi-injective [12, Prop.6.7.3] .Therefore M is a Q-module . ▪

Recall that an R-module M is called prime if ann(M)=ann(N) for every non-zero R-submodule N of M [4]. It is known that if M is prime then ann(M) is a prime ideal of R [4].

Corollary 5.10: Let M be a prime multiplication R-module . Then M is a Q-module if and only if every quasi-invertible R-submodule of M is quasi- injective .

In the following proposition , we give a characterization of Q-modules in the class of uniform modules .

Proposition 5.11: Let M be a uniform R-module , then M is a Q-module if and only if M is quasi-injective and duo.

Proof: It follows from Th.1.6. ▪

Corollary 5.12: Let M be an indecomposable R-module , then M is a Q-module if and only if M is quasi-injective and duo.

Proof: Suppose that M is a Q-module. Since M is indecomposable and quasi-injective , so M is uniform [12,Ex.32,P.244]. Hence by prop.5.11, M is quasi-injective and duo. The proof of the other direction follows from Prop. 2.1. ▪

Recall that An R-module M is said to be almost finitely generated if M is not finitely generated and every proper R-submodule of M is finitely generated[20]. An R-module M is called quasi-Dedekind if every non-zero R-submodule of M is quasi-invertible[13] .And the R-module M is called a chained module if the R-submodules of M are ordered by inclusion. Because almost finitely generated, quasi-Dedekind and prime chained R-modules are indecomposable (see[20, Th.11], [13, Remark 1.3,Ch.2]and[13,Th.1.15]), we get the following corollary.

Corollary 5.13: Let M be an R-module. If M is one of the following cases:
1-almost finitely generated .
2- quasi-Dedekind .
3- prime and chained.
Then M is a Q-module if and only if M is quasi-injective and duo.

Let M be an R-module . The singular R-submodule of M is the set Z(M)={m $\in$ M : ann(m) is an essential ideal of R}. M is called singular if Z(M)=M and M is called non-singular if Z(M)=0 [8].

Theorem 5.14: Let M be a non-singular R-module . Then M is a Q-module if and only if End(N) $\cong$ End(E(N)) for any R-submodule N of M.

Proof: Suppose that M is a Q-module .Let N be any R-submdule of M. Thus N is quasi-injective . Therefore we have $\frac{End(N)}{J(End(N))} \cong \frac{End(E(N))}{J(End(E(N))}$ [8, Corollary 2.17].Since N is non-singular , thus E(N) is non-singular. Again since N is non-singular ,J(End(N))=0. For if ;f $\in$ J (End(N))={f $\in$ End(N): kerf is essential R-submodule of N }[8].Thus kerf is an essential R-submodule of N .By [13,Prop.3.13,Ch.1], kerf is a quasi-invertible R-submodule of N, and hence $Hom(\frac{N}{\ker f}, N) = 0$. Therefore ,f=0. Similarly , J(End(E(N))=0. This prove that End(N) $\cong$ End(E(N)) for any R-submodule N of M .
Conversely; because M is non-singular ,then N is non-singular and hence E(N) is non-singular. Therefore J (End (E(N)) = 0 = J (End (N)). It follows that $\frac{End(N)}{J(End(N))} \cong \frac{End(E(N))}{J(End(E(N))}$. This prove that N is quasi-injective[10]. ▪

Theorem 5.15: Let M be a non-singular R-module.The following statements are equivalent:
1-M is a Q-module .
2-End(N) $\cong$ End(E(N))) for any R-submodule N of M.
3-Every quasi-invertible R-submodule of M is quasi-injective .
4-Every essential R-submodule of M is quasi-injective .

Proof: (1) $\Rightarrow$ (2). By Prop.5.14.
(2) $\Rightarrow$ (3). Every R-submodule N of M is quasi-injective (see proof of Prop.5.14).
(3) $\Rightarrow$ (4).It is clear that M is a quasi-invertible R-submodule of M, and hence M is quasi-injective. Let N be a proper essential R-submodule of M. By [13 ,Prop. 3.13], N is quasi-invertible and by hypothesis N is quasi-injective .
(4) $\Rightarrow$ (1).By Th.1.6. ▪

6-Endomorphisms rings of Q-modules.

We start this section by the following lemma which is a key for next results.

Lemma 6.1: If M is a duo R-module, then End(M) is a commutative ring.

Proof: Let f, g $\in$ End(M) and x $\in$ M .Since M is duo R-module ,then Rx is a fully invariant R-submodule of M for all x $\in$ M. Hence there exist two elements r_1 ,$r_2 \in$ R such that f(x)=r_1 x and g(x)=r_2 x. Thus f $\circ$ g(x) = f(g (x)) = f(r_2 x)= r_1 (r_2 x) = ($r_1 r_2$)x = ($r_2 r_1$)x = r_2 (r_1 x) = r_2 (f(x)) = g (f(x)) =g $\circ$ f (x). ▪

Proposition 6.2: Let M be a uniform R-module. If M is a Q-module, then M is quasi-injective and End(M) is a commutative local ring.

Proof: Since M is a uniform Q-module , so M is quasi-injective and duo (Pro.5.11). Again since M is uniform and quasi-injective , End(M) is local[12,Ex.32,P.244]. But M is duo, thus End (M) is a commutative local ring (Lemma 6.1). ▪

Proposition 6.3: Let M be an indecomposable R-module . If M is a Q-module, then M is quasi-injective ٢٩٢
and End(M) is a commutative local ring .

Proof: By Corollary 5.12 , M is quasi-injective and duo. Since M is an indecomposable quasi-injective R-module , so End(M) is a local ring [12,Ex.32, P.244]. Because M is duo , End(M) is a commutative local ring (Lemma 6.1). ▪

The converse of Prop.6.2 and Prop.63 is not true in general, for example, Q (the set of all rational numbers) as a Z-module is not a Q-module, but Q is quasi-injective and End(Q) $\cong$ Q is a field.

In the following proposition , we give a condition under which the converse of Prop. 6.2 and Prop.6.3 is true .

Proposition 6.4: Let M be a Z-regular R-module , then the following statements are equivalent:
1-M is a uniform Q-module.
2-M is quasi-injective and End(M) is a commutative local ring .
3-M is an indecomposable Q-module.

Proof: (1) $\Rightarrow$ (2). By Prop.6.2.
(2) $\Rightarrow$ (1). By [12,Ex.32,P.244], M is uniform. Since End (M) is commutative and M is Z-regular, then M is multiplication (Lemma5.2). Hence M is quasi-injective and multiplication and by corollary 5.3, M is a Q-module.
(2) $\Rightarrow$ (3). Since M is quasi-injective and End(M) is a commutative local ring, so M is indecomposable [12,Ex.32, P.244]. Because M is Z-regular, every cyclic R-submodule of M is a direct summand [22. Th.1.6]. But M is indecomposable, so M is cyclic. Again by Lemma 5.2 and its Corollary 5.3, M is a Q-module.
(3) $\Rightarrow$ (2). By Prop.6.3. ▪

Proposition 6.5: Let M be an R-module . If M is a quasi-Dedekind Q-module , then End(M) is a field .

Proof: By Corollary 5.13 , M is duo and hence End(M) is commutative (Lemma 5.1). By [13,Prop.2.1],End(M) is an integral domain . Since M is quasi-Dedekind, so M is indecomposable [13,Remark 1.2]. Because M is quasi-injective and indecomposable, End(M) is a local ring [12,Ex32, P.244].By [13,Corollary 3.5,Ch.2], J (End (M)) = 0. Therefore End (M) is a field. ▪

Proposition 6.6: Let M be a non-singular uniform R-module . If M is a Q-module, then End(M) is a field .

Proof: By Prop.6.2, End(M) is a commutative local ring. Since M is quasi-injective and non-singular, J (End (M)) = 0 (see the proof of Th.5.14). It follows that End(M) is a field. ▪

It is known that an indecomposable quasi-injective R-module is uniform and a quasi-injective R-module with End(M) is a local ring is uniform [12,Ex.32,P.244]. Form these two fact ,we get the following corollary.

Corollary 6.7: Let M be a non-singular R-module Then End(M) is a field in each of the following cases:
1-If M is an indecomposable Q-module.

2-If M is a Q-module such that End(M) is a local ring.

Proposition 6.8: Let M be a Noetherian R-module. If M is a Q-module , then End(N) is a perfect ring for all R-submodule N of M.

Proof: It follows from [16,P.253]. ▪

Proposition 6.9: Let M be an R-module such that every R-submodule of M is Z-regular . Then M is a Q-module if and only if End (N) is self injective ring for each R-submodule N of M.

Proof: It follows from [22,Th.2.5]. ▪

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مقاسات−Q

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الملخص

لتكن R حلقة تبادلية ذات عنصر محايد و كل المقاسات على R هي مقاسات أحادية. تسمى الحلقة R حلقة−q اذا كان كل مثالي في R هو شبه−اغماري.قدمنا في بحثنا هذا مفهوم المقاس−Q بصفته أعماما إلى مفهوم حلقة−q . يقال عن مقاس على الحلقة R انه مقاس−Q أذا كان كل مقاس جزئي من M هو شبه−أغماري. ميزنا تلك المقاسات ودرسنا خصيصاتها وعلاقتها بالأصناف الأخرى من المقاسات. فضلا عن ذلك درسنا مقاسات−Q على الحلقة الارتينية ودرسنا حلقات التشاكل لمقاسات−Q.