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Robust Full State Feedback Controller Design Using H_∞ for Inverted Pendulum System

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Abstract— *This paper presents the design of a full state feedback H_∞ controller to an inverted pendulum system. The nonlinear and linearized models of the system are obtained. The main goal of the proposed controller is to maintain the pendulum in the upright position and achieve a desirable tracking for the cart position. To achieve desirable tracking properties an integral term is added. The robustness of the proposed controller is examined when a 20% variation in the parameters of system is considered.*

Index Terms— *Inverted pendulum, robust controller, H_∞control, state feedback controller.*

I. INTRODUCTION

The inverted pendulum is classified as a very important control problem which for the last few decades the researchers worldwide have been trying to solve. The inverted pendulum being an inherently unstable system is often used as a benchmark for verifying the effectiveness and performance of a control algorithm [1], [2]. The inverted pendulum system is the most suitable test-bed for a wide range of classical and contemporary control technique design. Inverted pendulums have a broad prospects for utilization and development since they are widely applied in various fields such as delicate devices processing, semiconductors, artificial intelligence, robot control technology, aviation docking control technology, missiles interception control systems, gesture control in satellite circling and general industrial applications, perpendicularity control in rocket launching [3]. A single rod Inverted Pendulum includes a rod pivoted freely, placed above a motor driven cart. There is no existence of a resultant force on the rod due to the position of the rod which is centered above the motionless cart. It can stay this way indefinitely from principle view, but from a practical side of view it never does. In case of shifting of the rod away from equilibrium, it would cause an increase in the forces pushing the rod away from equilibrium point, illustrating that the upright equilibrium point is unstable. If the rod was not affected by external forces, it would rest to equilibrium point, hanging down [4], [5]. In H_∞ control, there are no constraints in working with the pure difference between the current and desired state, and there is no need to just multiply by few chosen numbers. After setting up a matrix, it's possible to do things and then use optimization (a mathematical procedure) to find the values in the matrix that give the required results. The art of H_∞ is setting up the correct optimization problem; picking the values is not a choice, they come out of the optimization. H_∞ can handle much more complicated systems and make defining the problem in terms of the results possible at the end, so there's less iteration. Recently, H_∞ optimization is very famous in the theory of control. This is generally because it generalizes the success of the design techniques in the frequency domain from SISO to MIMO systems. The H_∞theory allows to incorporate robustness requirements, disturbance attenuation and performance properties into one optimization issue which leads directly to measurement feedback controllers. Approaching the H_∞problem solution began in the frequency domain and state-space realizations mainly in computational devices [6].The control of inverted pendulum contains three segments. The swing-up control of inverted pendulum, which is widely researched, is considered as the first segment [7]-[9]. Inverted pendulum stabilization is the second [10], [11]. Inverted pendulum control tracking is the third

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[12]. The most useful controls for a wide range of real time applications are stabilization and tracking. The control of inverted pendulum has many problems that need solving, such as the swing up and the catch of the pendulum to the up-right unstable position from its stable position, and then during disturbances the balance of the pendulum at the up-right position, and moreover moving the cart to a specific place on the rail [13], [14]. Several control algorithms are already implemented in the field of stabilizing the inverted pendulum. In control systems, it is common that the proportional integral derivative (PID) controllers are used a lot [15]. These controllers design, however, is generally done using tuning approaches like the Ziegler-Nichols method, which does not ensure a good loop robustness, and to meet the design constraints is very difficult [16]. Sliding mode control, which is widely used for the control of under actuated systems, has the potential problem of chattering, which is a high frequency oscillation presents during the control [17]. A neural network based motion control of cart inverted pendulum system is proposed in [18], the controller is developed for wheeled inverted pendulum models like SEGWAY, which is an example of the inverted pendulum system.

II. SYSTEM MATHMATICAL MODEL

The inverted pendulum system is composed of the horizontal movement electrical cart and the single inverted pendulum supported by it, as shown in FIG. 1. The pendulum is constrained to move on the X axis, and the horizontal displacements of the electrical cart are the base for control action. The goal of controlling the inverted pendulum is to keep it in this upright position when it starts with some nonzero angle off the vertical position because of the external interference [19]. The rod mass is presumed to be zero and that the cart mass is donated as (M) and at the upper end of the inverted pendulum the point mass is donated as (m) to develop a mathematical model for the system. On the cart, there is an x-directed external force, $u(t)$, and on the point mass a gravity force acts. FIG. 1 defines the chosen coordinate system, where $x_c(t)$ represents the cart position and $\theta(t)$ is the tilt angle to the vertical direction. The external force on the system that gives a force balance in the x-direction must be equal to the mass times acceleration of the cart plus the mass times the x-directed acceleration of the point mass.

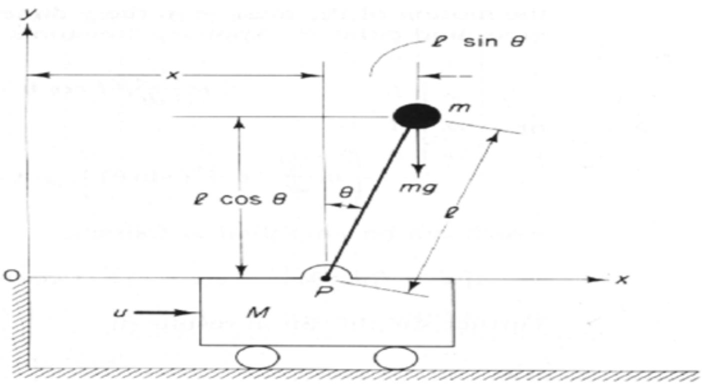

FIG. 1. SYSTEM OF INVERTED PENDULUM.

Using the second law of newton to the x direction [4]:

$$M\frac{d^2 x_c}{dt^2} + m\frac{d^2 x_G}{dt^2} = u \quad (1)$$

Where, the coordinates, (x_G ,y_G), are given by the point mass center of gravity (time-dependent). The location of center of gravity of the pendulum mass for the point mass is defined as:

$$x_G = x_c + l \sin\theta \text{ ; And } y_G = l \cos\theta \quad (2)$$

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Where (l) is length of the pendulum rod. Substitution of Eq. (2) into Eq. (1) gives:

$$M\frac{d^2 x_c}{dt^2} + m\frac{d^2}{dt^2}(x_c + l\ sin\,\theta) = u \tag{3}$$

Noticing the next definitions ,

$$\frac{d}{dt}\ sin\,\theta\ = (cos\,\theta)\,\dot{\theta}\ \ ;\text{And}\ \ \frac{d^2}{dt^2} sin\,\theta = -(sin\,\theta)\dot{\theta}^2 + (cos\,\theta)\ddot{\theta},$$

$$\frac{d}{dt} cos\,\theta = -(sin\,\theta)\dot{\theta}\ ,\text{And}\ \ \frac{d^2}{dt^2} cos\,\theta = -(\,cos\,\theta)\dot{\theta}^2 - (sin\,\theta)\ddot{\theta} \tag{4}$$

Eq. (3) can be defined as:

$$(M + m)\ddot{x}_c - ml(sin\,\theta)\,\dot{\theta}^2 + ml(cos\,\theta)\,\ddot{\theta} = u \tag{5}$$

Using Newton's second law for the rotational motion, gives:

$$[m\frac{d^2}{dt^2}(x_c + lsin\theta)]\ lcos\,\theta - [\,m\frac{d^2}{dt^2}(lcos\,\theta)]lsin\theta = mg\ lsin\theta \tag{6}$$

This can be simplified to:

$$m\{\ddot{x}_c - l(sin\,\theta)\,\dot{\theta}^2 + l\big(cos\,\theta)\,\ddot{\theta}\big\}l\ \ cos\,\theta - m\{-l\big(cos\,\theta)\,\dot{\theta}^2 - l\,(sin\,\theta)\,\ddot{\theta}\big\}l\,sin\,\theta = \ mgl\ \ sin\,\theta \tag{7}$$

Eq. (6) further simplified to:

$$m\ddot{x}_c\,cos\,\theta + ml\,\ddot{\theta} = mg\ \ sin\,\theta \tag{8}$$

Eq. (5) and Eq. (8) both considered as nonlinear differential equations. Inverted pendulum is required to stay vertical, then $\theta(t)$ and $\dot{\theta}(t)$ are minor quantities such that $sin\,\theta = \theta$, $\cos\theta = 1$ and $\theta\dot{\theta}^2 = 0$. Then Eq. (5) and Eq. (8) can be transformed to linear equations:

$$(M + m)\ddot{x}_c + ml\,\ddot{\theta} = u \tag{9}$$

$$m\ddot{x} + ml\ddot{\theta} = mg\theta \tag{10}$$

Rearranging Eq. (9) and Eq. (10):

$$Ml\ddot{\theta} = (M + m)g\theta - u \tag{11}$$

$$M\ddot{x}_c = u - mg\theta \tag{12}$$

The state variables are defined as $x_1 = \theta,\ x_2 = \dot{\theta},\ x_3 = x_c,\ x_4 = \dot{x}_c$, then consequently and from Equations (11) and (12), model of the state space expressed as the following:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ \frac{M+m}{Ml}g & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ -\frac{m}{M}g & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ -\frac{1}{Ml}\\ 0\\ \frac{1}{Ml}\end{bmatrix}u \tag{13}$$

$$y = [0\ \ 0\ \ 1\ \ 0]\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} \tag{14}$$

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III. CONTROLLER DESIGN

The controller design starts with building a model that is simple enough to facilitate the controller design while at the same time capturing the critical dynamic characteristics. The design strategy was to use a simple linear model for controller design and an uncertainty model to capture un-modeled dynamics to the extent possible [20]. The aim of the controller design for the inverted pendulum system is to minimize the worst case error that occurs if the system is perturbed by exogenous perturbations. It is an example of a robust control issue. H_∞ control is employed since it enables the designer to shape the closed-loop sensitivity functions via the appropriate selection of cost function weighting polynomials. The emphasis is on the selection of the weighting functions for stability in the presence of un-modeled time delay, or modeling uncertainty introduced by Pade approximation of delay. H_∞ synthesis provides a framework for the control of linear system from the concepts of H_∞ norm and frequency. It allows designing dynamic controller using the available measure by taking into considerations the stability of robust and the specifications of performances at the same time [21]. FIG. 2 represents a Full State feedback H_∞ control system.

Assume that

$$M= \begin{vmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ I & 0 & 0 \end{vmatrix} \quad (15)$$

Where A, B_1, B_2, C_1 and D_{12} matrices are defined in the following:

$$\dot{x} = Ax + B_1 d(t) + B_2 u(t) \quad (16)$$

$$e(t) = C_1 x(t) + D_{12} u(t) \quad (17)$$

$$y(t) = x(t) \quad (18)$$

The following assumptions are made:

1- (A,B_1) and (A,B_2) are stabilizable .
2- (C_1,A_1) is detectable.
3- $C_1^T D_{12}=0$ and $D_{12}^T D_{12}=I$.

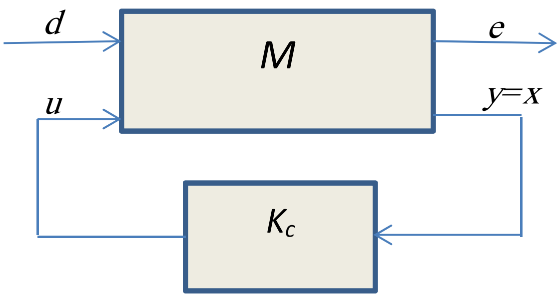

FIG. 2. FULL STATE FEEDBACK CONTROL SYSTEM.

The condition

$$\|T_{ed}(s)\|_{H_\infty} < \gamma \quad (19)$$

Assume that the worst-case disturbance *d(t)* and optimal control *u(t)* have the following structures.

$$d(t) = K_d x(t) \quad \text{and} \quad u(t) = K_c x(t) \quad (20)$$

Then

$$e(t) = (c_1 + D_{12} K_c) x(t) \quad (21)$$

Using assumption 3,

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$$e^T e = x^T(C_1^T C_1 + K_c^T K_c)x \quad (22)$$

Therefore,

$$J = \int_0^\infty x^T \left(C_1^T C_1 + K_c^T K_c - \gamma^2 K_d^T K_d\right) x dt \quad (23)$$

From Equation (16) and Equation (20)

$$\dot{x} = (A + B_1 K_d + B_2 K_c)x \quad (24)$$

Hence, under the assumption that Equation (24) is stable

$$J = x^T(0)Px(0) \quad (25)$$

Where P satisfies the Lyapunov equation [22].

$$P(A + B_1 K_d + B_2 K_c) + (A + B_1 K_d + B_2 K_c)^T P + C_1^T C_1 + K_c^T K_c - \gamma^2 K_d^T K_d = 0 \quad (26)$$

The maximization condition of J, Equation (25) respecting to K_d is

$$\nabla_{k_d} P = 0 \quad (27)$$

The gradient matrix $\nabla_{k_d} P$ as the following:

$$(\nabla_{k_d} P)_{ij} = \frac{\partial P}{\partial K_{d_{ij}}} \quad (28)$$

From Equation (26),

$$\nabla_{k_d} P(A + B_1 K_d + B_2 K_c) + B_1^T P + B_1^T P + (A + B_1 K_d + B_2 K_c)^T \nabla_{k_d} P - 2\gamma^2 K_d = 0 \quad (29)$$

Using Equation (27), yields:

$$K_d = \frac{1}{\gamma^2} B_1^T P \quad (30)$$

Similarly, the minimization condition of J, Equation (25), respecting K_c is

$$\nabla_{k_c} P = 0 \quad (31)$$

Therefore, it can be derived that

$$K_c = -B_2^T P \quad (32)$$

Substituting Equations (30) and (32) into Equation (26),

$$PA + A^T P + C_1^T C_1 - P\left[B_2 B_2^T - \frac{1}{\gamma^2} B_1 B_1^T\right] P = 0 \quad (33)$$

The condition $\|T_{ed}(s)\|_{H_\infty} < \gamma$ is satisfied and provided

1. $u(t) = K_c x(t)$ where K_c is given by Equation (32).
2. $P \geq 0$.

(34)

3. The matrix $(A + B_1 K_d + B_2 K_c)$ is stable.

Using the iterative procedure, one can find the minimum value γ of such that

$$\|T_{ed}(s)\|_{H_\infty} < \gamma_{min} \quad (35)$$

Equations (13) and (14) give the inverted pendulum system's state-space representation. It is wanted to control the position of the cart and keep the inverted pendulum upright as much as possible, for instance, by moving the cart in a step fashion. It is necessary to build a type 1 servo system to control the position of the cart. There is no integrator in the inverted pendulum system above the cart. The position signal y is fed back to the input and an integrator is inserted in the feed forward path, as shown in FIG. 3.
Referring to FIG. 3, the equations for the Inverted Pendulum Control System are:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$u = -K_c x + K_i \int e(t)dt$$
$$e = r - y = r - Cx \quad (36)$$

Where r represents a reference input and by a trial and error method, it is found that a suitable value of K_i is -3.8

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and K_c= [K_1 K_2 K_3 K_4].

In this work, the next numerical values are assumed for *M*, *m*, and *L*:

$$M = 2\ kg\ ,\quad m = 0.1\ kg,\quad L = 0.5\ m,\quad and\ \ g = 9.81\ m/s^2$$

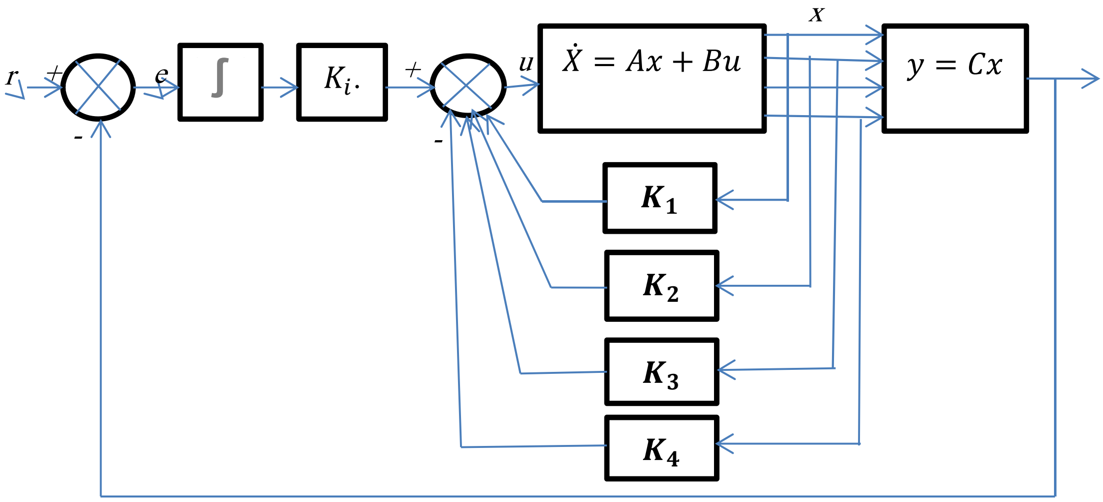

FIG. 3. THE INVERTED PENDULUM CONTROL SYSTEM.

IV. RESULTS AND DISCUSSION

FIG. 4 shows the response of the uncontrolled system. It is clear that the system is unstable. After applying the proposed full state feedback H_∞controller it is seen that the proposed controller can stabilize the system as shown in state trajectories of FIG. 5. The weighting matrices *Q* and *R* are set by trial and error trials with about 5 to be:

$$Q = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix},\qquad R= \begin{bmatrix} -100 & 0 \\ 0 & 1 \end{bmatrix} \tag{37}$$

The resulting *P* matrix is:

$$P = \begin{bmatrix} 8\,66.9807 & 179.7087 & 192.2023 & 181.0290 \\ 179.7087 & 38.3209 & 40.8017 & 38.2013 \\ 192.2023 & 40.8017 & 115.4152 & 61.6033 \\ 181.0290 & 38.2013 & 61.6033 & 53.3195 \end{bmatrix} \tag{38}$$

with $\gamma = 10$

The resulting state feedback gain vector is:

$$Kc = [-83.6611\ \ -18.9172\ \ -10.0000\ \ -11.7713] \tag{39}$$

It is shown that the proposed state feedback H_∞ controller can achieve a desirable time response with t_r of 4 sec., t_s of 8 sec. and *Mp* of 2% as shown in FIG. 6. The control action response is shown in FIG. 7. It shows that a low control effort has been achieved. On the other hand, to examine the robustness of the system to parameters variation, a random variation of $\pm$20% in system parameters (M, m, L) has been considered. FIG. 8 shows the cart response for this case.

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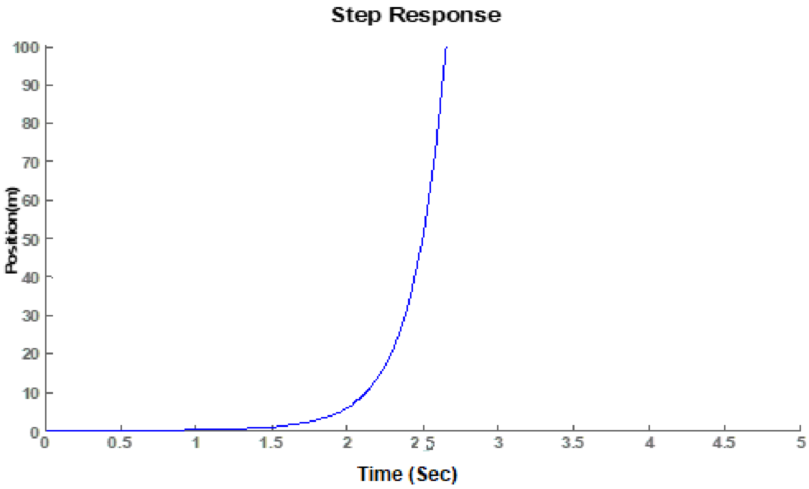

FIG. 4. UNCONTROLLED TIME RESPONSE FOR THE CART POSITION (X3).

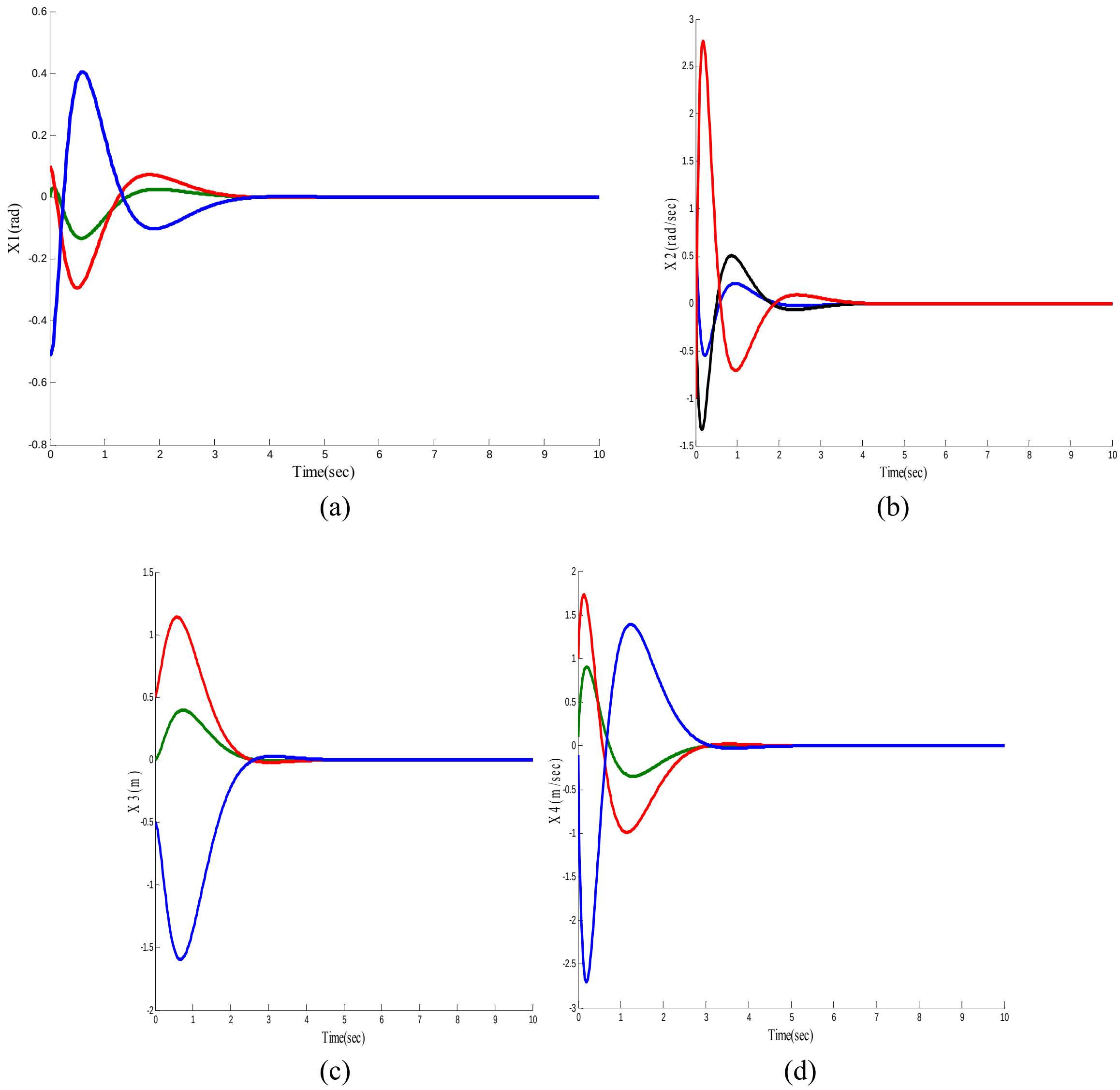

FIG. 5. STATE TRAJECTORIES OF THE CONTROLLED SYSTEM WITH DIFFERENT INITIAL CONDITIONS A): PENDULUM ANGLE X1(RAD), B): PENDULUM ANGULAR SPEED X2(RAD/S), C): CART POSITION X3(M) AND D): CART SPEED X4(M/S).

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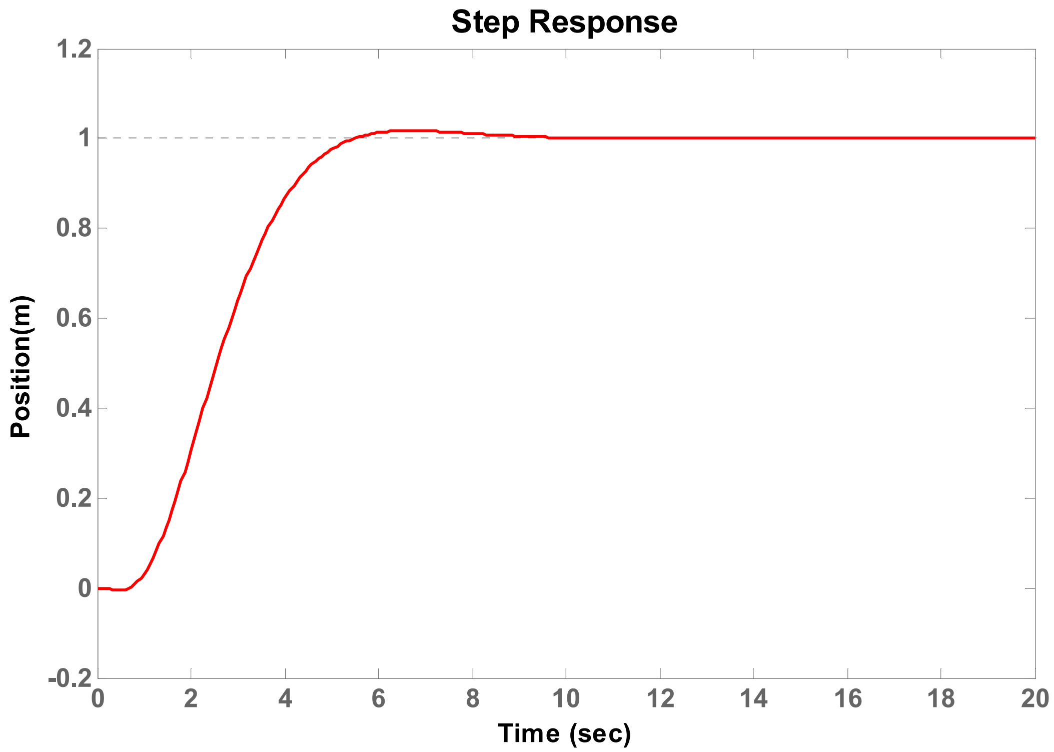

FIG. 6. CONTROLLED TIME RESPONSE FOR THE CART POSITION (X3).

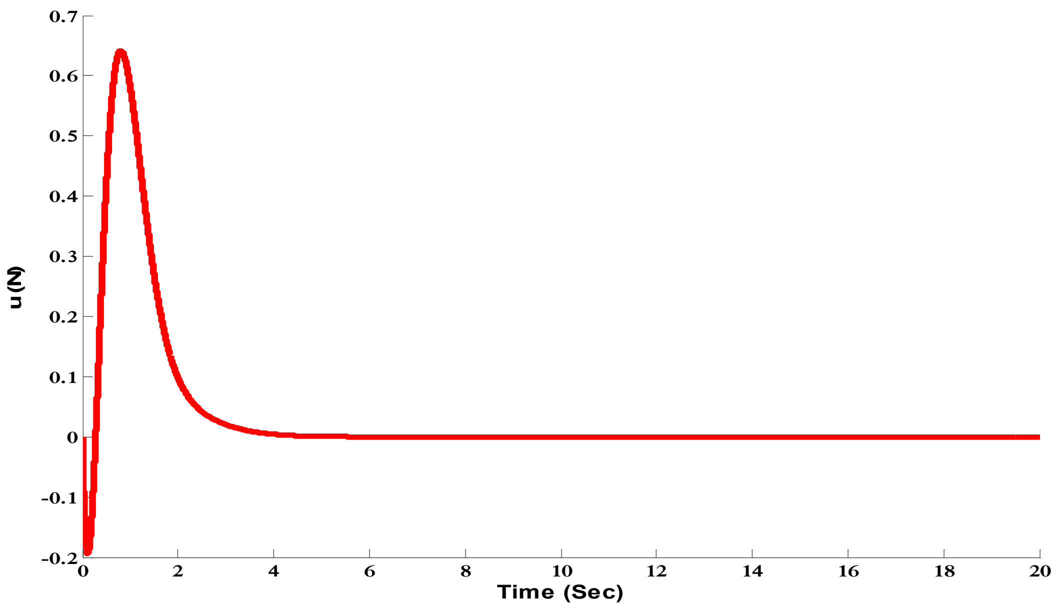

FIG. 7. CONTROL ACTION RESPONSE.

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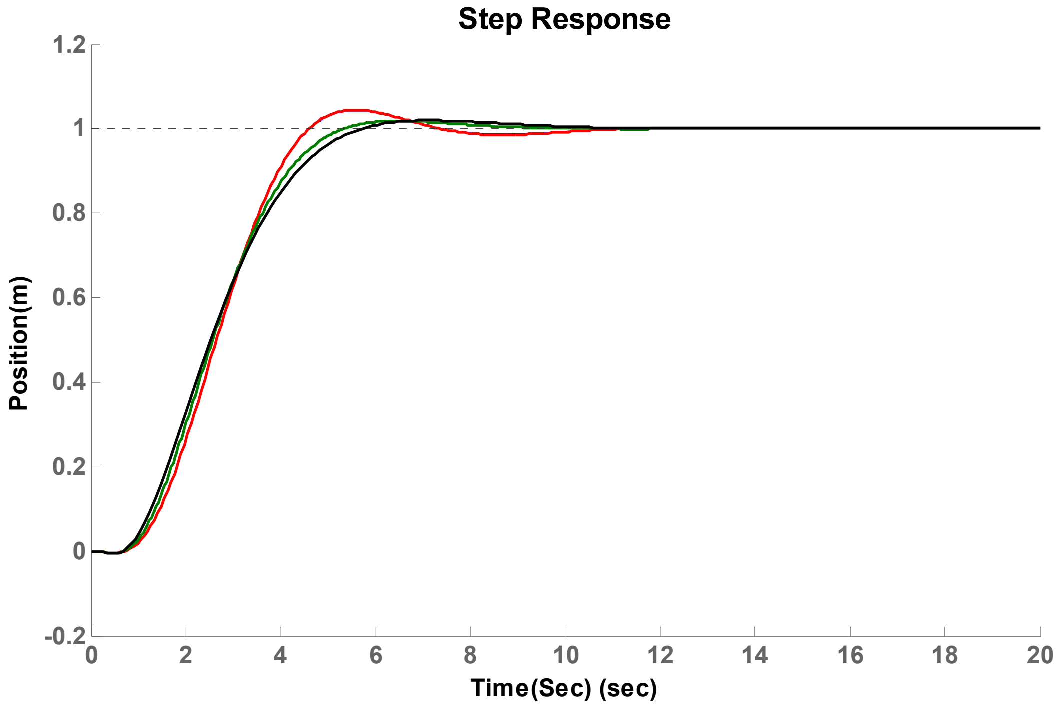

FIG. 8. CART RESPONSE WITH A RANDOM VARIATION OF ∓20% IN SYSTEM PARAMETERS.

V. CONCLUSION

In this work, the design of an H_∞ controller based on the state feedback has been presented. The inverted pendulum system which is inherently unstable, nonlinear and uncertain system was considered to clarify the efficiency of the proposed controller. The full state feedback H_∞ controller has achieved the required stabilization for the system. To improve the tracking properties an integral term has been added to the system. The proposed controller has achieved a more desirable time response. The robustness of the proposed controller has been examined when a random variation of $\pm 20\%$ in system parameters has been considered. To achieve the tracking for the cart position, it was seen that the proposed controller has compensated the variation in system parameters effectively.

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