Comparing Different Algorithms to Estimate two Parameters of Weighted Rayleigh Distribution

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Abstract

This paper deals with transforming a one parameter Rayleigh distribution, to another family called weighted two parameters Rayleigh, where the weighted functions are necessary when sampling done through samples of different size, also we work of estimating its two parameters using methods of maximum likelihood and moment estimators, and percentile estimators, the comparison has been done through simulation, we need to generate the values using a program of incomplete Gamma function.

Keywords: two parameters weighted Rayleigh distribution, maximum likelihood estimators and moment estimators, percentile estimators.

1. Introduction

The expanding of probability distribution through adding a parameter of weight or a transmuted parameter is considered a failure extension of the family distribution, were this task is necessary when sampling using different sample size or the studied distribution doesn't fitting the data perfectly, so we need to add another parameter to represent the transformation of data from some known family. M. C. Jones, generating a family of distributions from it by employing two parameters whose role is to introduce skew-ness and to vary tail weight. The proposal in this paper is a simple generalization of the use of the collection of order statistic distributions associated with for this purpose; an alternative derivation of this family of distributions is as the result of applying the inverse probability integral transformation to the beta distribution. General properties of the proposed family of distributions are explored. It is argued that two particular special cases are especially attractive because they appear to provide the most tractable instances of families with power and exponential tails. Shawky Y.A.I. and Hanah Abu – Zinadah, considerd the maximum likelihood estimation of the different parameters of an exponentiated Pareto distribution. We also mainly consider five other estimation procedures and compare their performances through numerical simulations.

2. Definition

The one parameter Rayleigh with scale parameter (θ) is given by;

$$f(t) = \frac{2}{\theta^2}\, t\, e^{-\left(\frac{t}{\theta}\right)^2} \qquad t > 0 \tag{1}$$

And its mean is;

$$E(t) = \frac{\theta\sqrt{\pi}}{2}$$

But when we sampling from different sample size, we need to transform $p.d.f$ in equation (1) into new $p.d.f$ called weighted Rayleigh with two parameters (c, θ) is;

$$g(t) = \frac{t^c\, f(t)}{E(t^c)} \tag{2}$$

$$E(t^c) = \int_0^\infty t^c \ f(t)dt$$

$$= \frac{2}{\theta^2}\int_0^\infty t^{c+1}\, e^{-\left(\frac{t}{\theta}\right)^2} dt$$

Let;

$$u = \left(\frac{t}{\theta}\right)^2 \quad \rightarrow \quad \sqrt{u} = \frac{t}{\theta} \quad \rightarrow \quad t = \theta\sqrt{u} \qquad dt = \frac{\theta}{2\sqrt{u}}\, du$$

$$E(t^c) = \frac{2}{\theta^2}\int_0^\infty (\theta\sqrt{u})^{c+1}\, e^{-u}\, \frac{\theta}{2\sqrt{u}}\, du$$

$$= \theta^c \int_0^\infty (\sqrt{u})^c\, e^{-u}\ du$$

$$E(t^c) = \theta^c \int_0^\infty u^{\frac{c}{2}}\, e^{-u}\ du$$

$$= \theta^c \Gamma\left(\frac{c}{2}+1\right) \tag{3}$$

New generated two parameters weighted Rayleigh is;

$$g_w(t) = \frac{t^c\, f(t)}{E(t^c)}$$

$$= \frac{\frac{2}{\theta^2}\ t\, e^{-\left(\frac{t}{\theta}\right)^2} t^c}{\theta^c \Gamma\left(\frac{c}{2}+1\right)}$$

$$= \frac{2t^{c+1}e^{-\left(\frac{t}{\theta}\right)^2}}{\theta^{c+2}\Gamma\left(\frac{c}{2}+1\right)}$$

(4)

While the C.D.F is;

$$G(t) = \int_0^x g(t)dt = \int_0^x \frac{2t^{c+1}e^{-\left(\frac{t}{\theta}\right)^2}}{\theta^{c+2}\Gamma\left(\frac{c}{2}+1\right)} dt$$

$$G(t) = \frac{2}{\theta^{c+2}\Gamma\left(\frac{c}{2}+1\right)} = \int_0^x t^{c+1}e^{-\left(\frac{t}{\theta}\right)^2} dt$$

(5)

Let;

$$u = \left(\frac{t}{\theta}\right)^2 \quad \rightarrow \quad \sqrt{u} = \frac{t}{\theta} \quad \rightarrow \quad t = \theta\sqrt{u} \qquad dt = \frac{\theta}{2\sqrt{u}}\, du$$

$$G(t) = \frac{2}{\theta^{c+2}\Gamma\left(\frac{c}{2}+1\right)} = \int_0^{\left(\frac{x}{\theta}\right)^2} (\theta\sqrt{u})^{c+1}e^{-u}\frac{\theta}{2\sqrt{u}}\, du$$

$$G(t) = \int_0^{\left(\frac{x}{\theta}\right)^2} u^{\frac{c}{2}}e^{-u}\, du$$

$G(t) = \gamma\left(\frac{c}{2}+1, \left(\frac{t}{\theta}\right)^2\right)$ (6)

Incomplete Gamma since;

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1}\, e^{-t}\, dt$$

3. Methods of Estimation

This section deals with introducing different methods to estimate the two parameters (c: $shape$ and θ: $scale$) of two parameters weighted Rayleigh.

3.1 Maximum Likelihood Method

Let $(x_1, x_2, \dots, x_n)$ be a random sample from $p.d.f$ given in (4), then;

$$L = \prod_{i=1}^{n} g(t_i) = \frac{2^n \prod_{i=1}^{n} e^{-\sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^2}}{\theta^{n(c+2)}\Gamma^n\left(\frac{c}{2}+1\right)} \qquad (7)$$

$$\log L = n\log 2 + (c+1)\sum_{i=1}^{n}\log t_i - \sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^2 - n(c+2)\log\theta$$

$$-n\log\Gamma\left(\frac{c}{2}+1\right) \qquad (8)$$

Deriving (8) with respect to (θ, c) gives;

$$\frac{\partial \log L}{\partial \theta} = 2\frac{\sum t_i^2}{\theta^3} - \frac{n(c+2)}{\theta} = 0$$

$$2\sum t_i^2 = n\theta^2(c+2)$$

$$\hat{\theta}_{MLE} = \sqrt{\frac{2\sum t_i^2}{n(c+2)}}$$

(9)

Now;

$$\frac{\partial \log L}{\partial c} = \sum_{i=1}^{n} \log t_i - n \log \theta - \frac{n}{2} \frac{\Gamma'\left(\frac{c}{2}+1\right)}{\Gamma\left(\frac{c}{2}+1\right)} = 0$$

$$\sum_{i=1}^{n} \log t_i - n \log \theta = \frac{n}{2} \frac{\Gamma'\left(\frac{c}{2}+1\right)}{\Gamma\left(\frac{c}{2}+1\right)}$$

Where;

$$\frac{d \log \Gamma\left(\frac{c}{2}+1\right)}{dc} = \frac{1}{\Gamma\left(\frac{c}{2}+1\right)} \frac{d\Gamma\left(\frac{c}{2}+1\right)}{dc}$$

$$\hat{c}_{MLE} = \frac{2(\sum_{i=1}^{n} \log t_i - n \log \theta)\Gamma\left(\frac{c}{2}+1\right)}{n\,\Gamma\left(\frac{c}{2}+1\right)}, \qquad c > 2$$

(10)

$$\because \Gamma\left(\frac{c}{2}+1\right) = \left(\frac{c}{2}+1-1\right)! = \left(\frac{c}{2}\right)!$$

$$d\left(\frac{c}{2}\right)! = \left(\frac{c}{2}\right)\left(\frac{c}{2}-1\right)\left(\frac{c}{2}-2\right)!$$

$$= \frac{1}{\Gamma\left(\frac{c}{2}+1\right)}\left[\left(\frac{c}{2}\right)\left(\frac{c}{2}-1\right)\left(\frac{c}{2}-2\right)!\right]$$

$$d\left(\frac{c}{2}\right)! = \frac{1}{\Gamma\left(\frac{c}{2}+1\right)}\left[\left(\frac{c}{2}\right)\Gamma\left(\frac{c}{2}-1\right)\right]$$

$$\sum_{i=1}^{n} \log t_i - n \log \theta = \frac{n\left(\frac{\hat{c}}{2}\right)\Gamma\left(\frac{c}{2}-1\right)}{\Gamma\left(\frac{c}{2}+1\right)} \qquad (11)$$

From equations (10 or 11) we can find ($\hat{c}_{MLE}$) which is an implicit, function (c) and (θ) can be solved numerically.

3.2 Moment Estimators

This method requires finding;

$$E(t^r) = \int_0^\infty t^r\, g(t)dt = \int_0^\infty t^r \frac{2t^{c+1}e^{-\left(\frac{t}{\theta}\right)^2}}{\theta^{c+2}\Gamma\left(\frac{c}{2}+1\right)} dt$$

$$= \frac{2}{\theta^{c+2}\Gamma\left(\frac{c}{2}+1\right)} \int_0^\infty t^{r+c+1} e^{-\left(\frac{t}{\theta}\right)^2} dt$$

Let;

$$u = \left(\frac{t}{\theta}\right)^2 \quad \rightarrow \quad \sqrt{u} = \frac{t}{\theta} \quad \rightarrow \quad t = \theta\sqrt{u} \qquad dt = \frac{\theta}{2\sqrt{u}}\, du$$

$$E(t^r) = \frac{2}{\theta^{c+2}\Gamma\left(\frac{c}{2}+1\right)} \int_0^\infty (\theta\sqrt{u})^{r+c+1} e^{-u} \frac{\theta}{2\sqrt{u}}\, du$$

$$E(t^r) = \frac{\theta^{r+c+2}}{\theta^{c+2}\Gamma\left(\frac{c}{2}+1\right)} \int_0^\infty (u)^{\frac{r+c}{2}} e^{-u}\, du$$

$$E(t^r) = \frac{\theta^r}{\Gamma\left(\frac{c}{2}+1\right)} \int_0^\infty (u)^{\frac{r+c}{2}} e^{-u}\, du$$

$$= \frac{\theta^r}{\Gamma\left(\frac{c}{2}+1\right)} \Gamma\left(\frac{r+c}{2}+1\right)$$

$$E(t^r) = \frac{\theta^r}{\Gamma\left(\frac{c}{2}+1\right)}\Gamma\left(\frac{r+c+2}{2}\right)$$

(12)

$$E(t) = \frac{\theta}{\Gamma\left(\frac{c}{2}+1\right)}\Gamma\left(\frac{c+3}{2}\right)$$

$$E(t^2) = \frac{\theta^2}{\Gamma\left(\frac{c}{2}+1\right)}\Gamma\left(\frac{c+4}{2}\right)$$

Then from solving $\left(E(t^r) = \frac{\sum_i^n t_i^r}{n}\right)$ we obtain $(\hat{\theta}_{mom})$ and $(\hat{c}_{mom})$;

$$\hat{\theta}_{mom}\frac{\Gamma\left(\frac{c+3}{2}\right)}{\Gamma\left(\frac{c}{2}+1\right)} = \frac{\sum_{i=1}^n t_i}{n}$$

$$\hat{\theta}_{mom} = \bar{t}\left(\frac{\Gamma\left(\frac{c}{2}+1\right)}{\Gamma\left(\frac{c+3}{2}\right)}\right)$$

(13)

When $(r = 2)$ we obtain;

$$\frac{\sum_i^n t_i^2}{n} = \theta^2\left(\frac{\Gamma\left(\frac{c+4}{2}\right)}{\Gamma\left(\frac{c}{2}+1\right)}\right)$$

(14)

Solving equation (14) we get $(\hat{c}_{mom})$.

3.3 Percentile Estimators

This method depend on minimizing the total sum square between the $F(x_i)$ and its non–parameter estimator$\hat{F}(x_i)$;

$$T = \sum_{i=1}^{n} \left(F(x_i) - \hat{F}(x_i)\right)^2$$

$$\hat{F}(x_i) = \left(\frac{i - \frac{3}{8}}{n + \frac{1}{4}}\right)$$

$$T = \sum_{i=1}^{n} \left(\gamma\left(\frac{c}{2} + 1, \left(\frac{t}{\theta}\right)^2\right) - \left(\frac{i - \frac{3}{8}}{n + \frac{1}{4}}\right)\right)^2$$

Derive T with respect to $(\theta \ and \ c)$;

$$\frac{\partial T}{\partial \theta} = 2\sum_{i=1}^{n} \left(\gamma\left(\frac{c}{2} + 1, \left(\frac{t}{\theta}\right)^2\right) - \left(\frac{i - \frac{3}{8}}{n + \frac{1}{4}}\right)\right)\left[\gamma'\left(\frac{c}{2} + 1, \left(\frac{t}{\theta}\right)^2\right)\left(\frac{-2t^2}{\theta^3}\right)\right] = 0$$

$$\frac{\partial T}{\partial c} = 2\sum_{i=1}^{n} \left(\gamma\left(\frac{c}{2} + 1, \left(\frac{t}{\theta}\right)^2\right) - \left(\frac{i - \frac{3}{8}}{n + \frac{1}{4}}\right)\right)\left[\gamma'\left(\frac{c}{2} + 1, \left(\frac{t}{\theta}\right)^2\right)\left(\frac{1}{2}\right)\right] = 0$$

And applying simulation procedure to obtain $(\hat{\theta}_{PEC}, \hat{c}_{PEC})$.

4. Simulation

Table (1)

Weighted Rayleigh distribution two parameters estimators

n	Method	Experiment 1		Experiment 2		Experiment 3	
		$C = 1$	$\theta = 0.9$	$C = 2$	$\theta = 1.5$	$C = 3$	$\theta = 2$
50	MLE	1.1611	0.8858	2.2207	1.4766	3.3148	1.9661
	MOM	0.9930	0.8987	1.9925	1.4982	2.9888	2.0024
	PERCE	1.1446	0.8972	2.1936	1.4951	3.3121	1.9863
75	MLE	1.0972	0.8934	2.1329	1.4890	3.2073	1.9756
	MOM	0.9961	0.9011	1.9960	1.5016	2.9924	1.9999
	PERCE	1.1023	0.8989	2.1397	1.4978	3.1840	1.9936
100	MLE	1.0858	0.8933	2.1214	1.4869	3.1158	1.9902
	MOM	0.9963	0.9008	1.9955	1.5001	2.9984	1.9991
	PERCE	1.0816	0.8974	2.1118	1.4951	3.0886	2.0038
150	MLE	1.0548	0.8948	2.0900	1.4906	3.0864	1.9913
	MOM	0.9979	0.8995	1.9954	1.5000	2.9981	1.9998
	PERCE	1.0557	0.8969	2.0921	1.4948	3.0761	1.9987

Table (2)

Mean Square Error (MSE)

n	Method	Experiment 1		Experiment 2		Experiment 3	
		$C = 1$	$\theta = 0.9$	$C = 2$	$\theta = 1.5$	$C = 3$	$\theta = 2$
50	MLE	0.0259	0.2018e−003	0.0487	0.5498e−003	0.0991	0.0011
	MOM	0.0000	0.0016e−003	0.0001	0.0034e−003	0.0001	0.0000

	PERCE	0.0209	0.0078e−003	0.0375	0.0240e−003	0.0974	0.0002
		MOM	MOM	MOM	MOM	MOM	MOM
75	MLE	0.0094	0.4337e−004	0.0177	0.1217e−003	0.0430	0.5971e−003
	MOM	0.0000	0.0126 e−004	0.0000	0.0025e−003	0.0001	0.0000e−003
	PERCE	0.0105	0.0128e−004	0.0195	0.0050e−003	0.0339	0.0412e−003
		MOM	MOM	MOM	MOM	MOM	MOM
100	MLE	0.0074	0.4225e−004	0.0147	0.1711e−003	0.0134	0.9551e−004
	MOM	0.0000	0.0048e−004	0.0000	0.0000e−003	0.0000	0.0075e−004
	PERCE	0.0067	0.0058e−004	0.0125	0.0236e−003	0.0079	0.1464e−004
		MOM	MOM	MOM	MOM	MOM	MOM
150	MLE	0.0030	0.2715e−004	0.0081	0.8807e−004	0.0075	0.7618e−004
	MOM	0.0000	0.0024e−004	0.0000	0.0154e−004	0.0000	0.0004e−004
	PERCE	0.0031	0.0050e−004	0.0085	0.2727e−004	0.0058	0.0182e−004
		MOM	MOM	MOM	MOM	MOM	MOM

Algorithm

1. Generation the variable $z = \left(\frac{t}{\theta}\right)^2$ through inverse diversion method.
2. Calculating the variable (t), which distributed weighted Rayleigh by the equation $t = \theta \times \sqrt{z}$
3. Estimating the two parameters by;
   a. Maximum Likelihood Method.
   b. Moment Method.
   c. Divisional Estimators Method.
4. Calculating mean square error [$\sum(\hat{\theta} - \theta)^2$] for comparison between the best estimators.

Conclusion

1-the best estimator for θ when (n=50) and accordiry to values (c=1,c=2,c=3) is found to be $\hat{\theta}_{mom}$

2-when n large (n=100,150) the best one estimator is moment and then Maximum like hood and then percentile.

3- the values of random variable generated using inverse Gamma transformation due to certain program and then the parameters (C, θ)are estimated using three different methods and the results are compared using statistical measure mean square error.

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