On Convergence Uncertain Sequence in Measure

Zainab Hayder Abid AL-Aali

Department of Mathematics and Computer Applications

College of Science -Al-Muthana University

Abstract

We introduce the definitions of uncertain measure and convergence in measure , also we proved some properties of convergence in measure for uncertain sequence.

Key words

uncertain measure , uncertain sequence and convergence in measure.

1-Introduction

Uncertainty theory was founded by Liu [1] in 2007 and refined by Liu [3] in 2010. The first basic concept of uncertainty theory is an uncertain measure which is defined as a set function $\mu : F \to R$, satisfies the following axioms:

Axiom 1.(**Normality Axiom**) : $\mu(\Omega) = 1$ for the universal set Ω .

Axiom 2.(**Self-Duality Axiom**): $\mu(A) + \mu(A^c) = 1$ for any event A .

Axiom 3.(**Countable Subadditivity Axiom**): For every countable sequence of events$\{A_n\}$, we have

$$\mu(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu(A_n)$$

An uncertainty space is a triple (Ω, F, μ) where Ω is a set, F is a σ-field on Ω and μ is a uncertain measure on F . In 2007, Liu proposed the fourth axiom of uncertainty theory called product measure axiom.

Axiom4.(**Product Measure Axiom**): Let(Ω_k, F_k, μ_k)be uncertainty spaces for $k = 1,2,\cdots,n$. Then the product uncertain measure μ is an uncertain measure on the product σ - filed $F_1 \times F_2 \times \cdots \times F_n$ satisfying :

$$\mu(\prod_{k=1}^{n} A_k) = \min\{\mu_k(A_k) : A_k \in F_k, k = 1,2,....n\}$$

That is, for each event $A \in F = F_1 \times F_2 \times \cdots \times F_n$, we have

$$\mu(A) = \begin{cases} \alpha, & \alpha > 0.5 \\ 1-\beta, & \beta > 0.5 \\ 0.5, & o.w \end{cases}$$

where
$$\alpha = \sup\{\min\{\mu_k(A_k)\} : k = 1,2,\cdots,n\} : A_1 \times A_2 \times \cdots \times A_n \subseteq A\},$$
$$\beta = \sup\{\min\{\mu_k(A_k)\} : k = 1,2,\cdots,n\} : A_1 \times A_2 \times \cdots \times A_n \subseteq A^c\}$$

The concept of uncertain variable was introduce by Liu [1] as a measurable function from an uncertainty space (Ω, F, μ) to the set of real numbers.

Example (1.1)[3,4]

Suppose that $u : R \to R$ is a nonnegative function satisfying $\sup\{u(x) + u(y) : x \neq y\} = 1$

Then for any set A of real numbers, the set function

$$\mu(A) = \begin{cases} \sup\{u(x) : x \in A\}, & \text{if } \sup\{u(x) : x \in A\} < 0.5 \\ 1 - \sup\{u(x) : x \in A^c\}, & \text{if } \sup\{u(x) : x \in A\} \geq 0.5 \end{cases}$$

is an uncertain measure on R.

2-Properties of Convergence in Measure

Definition (2.1)[4]

Let $\{x_n\}$ be sequence of uncertain variables with finite expected value. we say that the sequence $\{x_n\}$ convergence in mean to x if $\lim_{n\to\infty} E\{|x_n - x|\} = 0$

Definition (2.2)[2,4]

Let $\{x_n\}$ be sequence of uncertain variables . we say that the sequence $\{x_n\}$ convergence in measure to x if $\lim_{n\to\infty} \mu\{|x_n - x| \geq \varepsilon\} = 0$. For every $\varepsilon > 0$.

Proposition (2.3)

If $\lim_{n\to\infty} x_n = x$ (in measure) , $\lim_{n\to\infty} x_n = y$ (in measure) , Then $x = y$.

Proof:

Suppose $\lim_{n\to\infty} \mu\{|x_n - x| \geq \varepsilon\} = 0$, $\lim_{n\to\infty} \mu\{|x_n - y| \geq \varepsilon\} = 0$

$$\lim_{n\to\infty}\mu\{|x-y|\geq\varepsilon\}=\lim_{n\to\infty}\mu\{|x-x_n+x_n-y|\geq\varepsilon\}\leq\lim_{n\to\infty}\mu\{|x_n-x|\geq\varepsilon\}+\lim_{n\to\infty}\mu\{|x_n-y|\geq\varepsilon\}$$

$$=0$$

Then, $\lim_{n\to\infty}\mu\{|x-y|\geq\varepsilon\}=0$.

Proposition (2.4)

If $\lim_{n\to\infty}x_n=x$ (in measure), $\lim_{n\to\infty}y_n=y$ (in measure), Then $\lim_{n\to\infty}(ax_n+by_n)=ax+by$.

Where $a,b\ \varepsilon\ \Re$.

Proof:

Suppose $\lim_{n\to\infty}\mu\{|x_n-x|\geq\varepsilon\}=0$, $\lim_{n\to\infty}\mu\{|y_n-y|\geq\varepsilon\}=0$

$$\lim_{n\to\infty}\mu\{|(ax_n+by_n)-(ax+by)|\geq\varepsilon\}=\lim_{n\to\infty}\mu\{|a(x_n-x)+b(y_n-y)|\geq\varepsilon\}$$

$$\leq|a|\lim_{n\to\infty}\mu\{|x_n-x|\geq\varepsilon\}+|b|\lim_{n\to\infty}\mu\{|y_n-y|\geq\varepsilon\}$$

$$=0$$

Then $\lim_{n\to\infty}\mu\{|(ax_n+by_n)-(ax+by)|\geq\varepsilon\}=0$

Proposition (2.5)

If $\lim_{n\to\infty}x_n=x$ (in measure), $\lim_{n\to\infty}y_n=y$ (in measure), Then $\lim_{n\to\infty}(ax_n-by_n)=ax-by$.

Where $a,b\ \varepsilon\ \Re$.

Proof:

Suppose $\lim_{n\to\infty}\mu\{|x_n-x|\geq\varepsilon\}=0$, $\lim_{n\to\infty}\mu\{|y_n-y|\geq\varepsilon\}=0$

$$\lim_{n\to\infty}\mu\{|(ax_n-by_n)-(ax-by)|\geq\varepsilon\}=\lim_{n\to\infty}\mu\{|a(x_n-x)+b(y-y_n)|\geq\varepsilon\}$$

$$\leq|a|\lim_{n\to\infty}\mu\{|x_n-x|\geq\varepsilon\}+|b|\lim_{n\to\infty}\mu\{|y_n-y|\geq\varepsilon\}$$

$$=0$$

Then $\lim_{n\to\infty}\mu\{|(ax_n-by_n)-(ax-by)|\geq\varepsilon\}=0$

Proposition (2.6)

If $\lim_{n\to\infty} x_n = x$ (in measure), then $\lim_{n\to\infty} |x_n| = |x|$ (in measure).

Proof:

Suppose $\lim_{n\to\infty} \mu\{|x_n - x| \ge \varepsilon\} = 0$

$\lim_{n\to\infty} \mu\{||x_n| - |x|| \ge \varepsilon\} \le \lim_{n\to\infty} \mu\{|x_n - x| \ge \varepsilon\} = 0$.

Then, $\lim_{n\to\infty} \mu\{||x_n| - |x|| \ge \varepsilon\} = 0$

Proposition (2.7)

If $\lim_{n\to\infty} x_n = x$ (in measure), $\lim_{n\to\infty} y_n = y$ (in measure), Then $\lim_{n\to\infty} |x_n + y_n| = |x + y|$.

Proof:

Suppose $\lim_{n\to\infty} \mu\{|x_n - x| \ge \varepsilon\} = 0$, $\lim_{n\to\infty} \mu\{|y_n - y| \ge \varepsilon\} = 0$

$$\begin{aligned}\lim_{n\to\infty} \mu\{||x_n + y_n| - |x + y|| \ge \varepsilon\} &\le \lim_{n\to\infty} \mu\{|(x_n + y_n) - (x + y)| \ge \varepsilon\} \\ &\le \lim_{n\to\infty} \mu\{(|x_n - x| + |y_n - y|) \ge \varepsilon\} \\ &\le \lim_{n\to\infty} \mu\{|x_n - x| \ge \varepsilon\} + \lim_{n\to\infty} \mu\{|y_n - y| \ge \varepsilon\} \\ &= 0\end{aligned}$$

Then $\lim_{n\to\infty} \mu\{||x_n + y_n| - |x + y|| \ge \varepsilon\} = 0$

Proposition (2.8)

If $\lim_{n\to\infty} x_n = x$ (in measure), $\lim_{n\to\infty} y_n = y$ (in measure), Then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.

Proof:

Suppose $\lim_{n\to\infty} \mu\{|x_n - x| \ge \varepsilon\} = 0$, $\lim_{n\to\infty} \mu\{|y_n - y| \ge \varepsilon\} = 0$

$$\begin{aligned}\lim_{n\to\infty} \mu\left\{\left|\frac{x_n}{y_n} - \frac{x}{y}\right| \ge \varepsilon\right\} = \lim_{n\to\infty} \mu\left\{\left|\frac{yx_n - xy_n}{y_n y}\right| \ge \varepsilon\right\} &= \lim_{n\to\infty} \mu\left\{\left|\frac{x_n y - x_n y_n + x_n y_n - xy_n}{y_n y}\right| \ge \varepsilon\right\} \\ &\le \lim_{n\to\infty} \mu\left\{\left(\left|\frac{x_n y - x_n y_n}{y_n y}\right| + \left|\frac{x_n y_n - xy_n}{y_n y}\right|\right) \ge \varepsilon\right\}\end{aligned}$$

$$= |x_n| \lim_{n\to\infty} \mu\left\{\left|\frac{y_n - y}{y_n y}\right| \geq \varepsilon\right\} + \lim_{n\to\infty} \mu\left\{\left|\frac{x_n - x}{y}\right| \geq \varepsilon\right\}$$

$$= 0$$

Then, $\lim_{n\to\infty} \mu\left\{\left|\frac{x_n}{y_n} - \frac{x}{y}\right| \geq \varepsilon\right\} = 0$

Corollary (2.9)

If $\lim_{n\to\infty} x_n = x$ (in measure), $\lim_{n\to\infty} y_n = y$ (in measure), Then $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$.

proof:

Suppose $\lim_{n\to\infty} \mu\{|x_n - x| \geq \varepsilon\} = 0$, $\lim_{n\to\infty} \mu\{|y_n - y| \geq \varepsilon\} = 0$

$$\lim_{n\to\infty} \mu\left\{\left|\frac{1}{y_n} - \frac{1}{y}\right| \geq \varepsilon\right\} = \lim_{n\to\infty} \mu\left\{\left|\frac{y - y_n}{y_n y}\right| \geq \varepsilon\right\} = \lim_{n\to\infty} \mu\left\{\left|\frac{y_n - y}{y_n y}\right| \geq \varepsilon\right\} = \left|\frac{1}{y_n y}\right| \lim_{n\to\infty} \mu\{|y_n - y| \geq \varepsilon\} = 0$$

Then, $\lim_{n\to\infty} \mu\left\{\left|\frac{1}{y_n} - \frac{1}{y}\right| \geq \varepsilon\right\} = 0$

3-Cauchy sequence in measure

Definition (3.1)[5]

Let G be set of all uncertain variables and let $\{x_n\}$ ε G. Then the sequence $\{x_n\}$ is called Cauchy sequence in measure if $\lim_{m,n\to\infty} \mu\{|x_m - x_n| \geq \varepsilon\} = 0$

Theorem (3.1)

If $\lim_{n\to\infty} x_n = x$ (in measure), Then $\{x_n\}$ is Cauchy sequence in G .

Proof:

Since $\lim_{n\to\infty} x_n = x$ (in measure), for any $\varepsilon > 0$, there exists a positive integer K . Such that $\mu|x_m - x| \leq \frac{\varepsilon}{2}$, $\mu|x_n - x| \leq \frac{\varepsilon}{2}$

When $m, n > K$. Thus , we have

$$\mu|x_m - x_n| \leq \mu|x_m - x + x - x_n| \leq \mu|x_m - x| + \mu|x_n - x| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

The theorem is proved.

Proposition (3.2)

If $\lim_{n\to\infty} x_n = x$ (in mean), Then $\{x_n\}$ is Cauchy sequence in G .

Proof:

Since $\lim_{n\to\infty} x_n = x$ (in mean), for any $\varepsilon > 0$, there exists a positive integer K . Such that $E|x_m - x| \le \frac{\varepsilon}{2}$, $E|x_n - x| \le \frac{\varepsilon}{2}$

When $m, n > K$. Thus , we have

$$E|x_m - x_n| \le E|x_m - x + x - x_n| \le E|x_m - x| + E|x_n - x| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

The theorem is prove.

Theorem (3.3)

If $\{x_n\}$ is Cauchy sequence in G , then there exists an uncertain variable $x \ \varepsilon\ G$. such that $\lim_{n\to\infty} x_n = x$ (in measure).

Proof:

Let $\{x_n\}$ is Cauchy sequence in G . By Markov's inequality , we have

$\mu\{|x_m - x_n| \ge \varepsilon\} \le \frac{E\{|x_m - x_n|\}}{\varepsilon} \to 0$. Thus, $\{x_n\}$ is Cauchy sequence in measure

So that there exists $x \ \varepsilon\ G$. Such that $\lim_{m,n\to\infty} \mu\{|x_m - x_n| \ge \varepsilon\} = 0$.

4-Conclusions

In this paper we make comparison between the convergence in mean square for uncertain sequence in [5] and this paper and we found there are some properties is a achieve as convergence in measure.

Reference

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الخلاصة

قدمنا تعريف القياس غير المؤكد (اللايقيني) و التقارب في القياس , كذلك برهنا بعض خواص التقارب في القياس للمتتابعات غير المؤكدة (اللايقينية).