# Weak $\omega$-Homeomorphisms

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## Abstract.

In this paper we use the weak $\omega-$continuities defined in [1] to define the weak $\omega-$homeomorphisms, and introduce some rather simple related relations as theorems.

## 1.Introduction

First let us recall some definitions and results from [1] needed as a background for our results.

Through out this paper , $(X,T)$ stands for topological space. Let $(X,T)$ be a topological space and $A$ a subset of $X$. A point $x$ in $X$ is called ***condensation point*** [1] of $A$ if for each $U$ in $T$ with $x$ in $U$, the set $U \cap A$ is un countable In 1982 the $\omega-$closed set was first introduced by Hdeib, H. Z. in [2], and he defined it as: $A$ is $\boldsymbol{\omega}-$***closed*** set if it contains all its condensation points and the $\boldsymbol{\omega}-$***open*** set is the complement of the $\omega-$closed set. The union of all $\omega-$ open sets contained in $A$ is the $\boldsymbol{\omega}-$***interior*** of A and will denoted by $int_{\omega}(A)$. In 2009 in [3] T. Noiri, A. Al-Omari and M. S. M. Noorani introduced and investigated new notions called $\alpha-\omega-$open, $pre-\omega-$open, $b-\omega-$open and $\beta-\omega-$open sets which are weaker than $\omega-$open set. Let us introduce these notions in the following definition:

**Definition 1.1.** [3] A subset $A$ of a space $X$ is called

**1.** $\boldsymbol{\alpha-\omega-}$***open*** if $A \subseteq int_{\omega}\left(cl\left(int_{\omega}(A)\right)\right)$ and the complement of the $\alpha-\omega-$open set is called $\boldsymbol{\alpha-\omega-}$***closed*** set.

**2.** ***pre*** $\boldsymbol{-\omega-}$***open*** if $A \subseteq int_{\omega}\left(cl(A)\right)$ and the complement of the $pre-\omega-$ open set is called $\boldsymbol{pre-\omega-}$***closed*** set.

**3.** ***b*** $\boldsymbol{-\omega-}$***open*** if $A \subseteq int_{\omega}\left(cl(A)\right) \cup cl\left(int_{\omega}(A)\right)$ and the complement of the $b-\omega-$open set is called $\boldsymbol{b-\omega-}$***closed*** set.

**4.** $\boldsymbol{\beta-\omega-}$***open*** if $A \subseteq cl\left(int_{\omega}\left(cl(A)\right)\right)$ and the complement of the $\beta-\omega-$ open set is called $\boldsymbol{\beta-\omega-}$***closed*** set.

In [3] T. Noiri, A. Al-Omari and M. S. M. Noorani introduced relationships among the weak $\omega-$open sets by the lemma below:

**Lemma 1.2.** [3] In any topological space:

**1**. Any open set is $\omega-$open.

**2**. Any $\omega-$open set is $\alpha-\omega-$open.

**3**. Any $\alpha-\omega-$open set is $pre-\omega-$open.

**4**. Any $pre-\omega-$open set is $b-\omega-$open.

**5**. Any $b-\omega-$open set is $\beta-\omega-$open.

The converse is not true [3].

**Definition 1.3.** [3] A space $(X,T)$ is called a ***door space*** if every subset of $X$ is either open or closed.

**Lemma 1.4.** [3] If $(X,T)$ is a door space, then every $pre-\omega-$open set is $\omega-$open.

**Lemma 1.5.** [1] If $(X,T)$ is a door space, then every $\beta-\omega-$open set is $b-\omega-$open.

**Definition 1.6.** [3] A subset $A$ of a space $X$ is called

**1**. An $\boldsymbol{\omega - t -}$***set***, if $int(A) = int_\omega(cl(A))$.

**2**. An $\boldsymbol{\omega - B -}$***set***, if $A = U \cap V$, where $U$ is an open set and $V$ is an $\omega - t -$ set.

**3**. An $\boldsymbol{\omega - t_\alpha -}$***set***, if $int(A) = int_\omega(cl(int_\omega(A)))$.

**4**. An $\boldsymbol{\omega - B_\alpha -}$***set***, if $A = U \cap V$, where $U$ is an open set and $V$ is an $\omega - t_\alpha -$ set.

**5**. An $\boldsymbol{\omega -}$***set***, if $A = U \cap V$, where $U$ is an open set and $int(V) = int_\omega(V)$.

**Definition 1.7.**[1] Let $(X,T)$ be topological space. It said to be satisfy

**1**. The $\boldsymbol{\omega -}$***condition*** if every $\omega -$open set is $\omega -$set.

**2.** The $\boldsymbol{\omega - B_\alpha -}$***condition*** if every $\alpha - \omega -$open set is $\omega - B_\alpha -$set.

**3.** The $\boldsymbol{\omega - B -}$***condition*** if every $pre - \omega -$open is $\omega - B -$set.

**Lemma 1.8.** [3] For any subset $A$ of a space $X$, we have

**1.** A is open if and only if $A$ is $\omega -$open and $\omega -$set.

**2.** A is open If and only if $A$ is $\alpha - \omega -$open and $\omega - B_\alpha -$set.

**3.** A is open if and only if $A$ is $pre - \omega -$open and $\omega - B -$set.

**Definition 1.9.** [1] A function $f:(X,\sigma)\to(Y,\tau)$ is called $\boldsymbol{\omega}$-*continuous* ( resp. $\boldsymbol{\alpha-\omega}$-*continuous* , $\boldsymbol{pre\ -\omega-}$*continuous*, $\boldsymbol{b\ -\omega-}$*continuous and* $\boldsymbol{\beta-\omega-}$*continuous* ), if for each $x\in X$ , and each $\omega-$open ( resp. $\alpha-\omega-$open, $pre\ -\omega-$open, $b\ -\omega-$open and $\beta-\omega-$open ) set $V$ containing $f(x)$ , there exists an $\omega-$open ( resp. $\alpha-\omega-$open, $pre\ -\omega-$open, $b\ -\omega-$open and $\beta-\omega$-open,) set $U$ containing $x$, such that $f(U)\subset V$.

**Proposition 1.10.** Let $(X,\sigma)$ and $(Y,\tau)$ be two topological spaces. A map $f:(X,\sigma)\to(Y,\tau)$ is $\omega-$continuous ( resp. $\alpha-\omega-$continuous, $pre\ -\omega-$continuous, $b\ -\omega-$continuous and $\beta-\omega-$continuous ) if and only if for each $\omega-$open ( resp. $\alpha-\omega-$open, $pre\ -\omega-$open, $b\ -\omega-$open and $\beta-\omega-$open ) set $V$ in $Y$, $f^{-1}(V)$ is $\omega-$open ( resp. $\alpha-\omega-$open, $pre\ -\omega-$open, $b\ -\omega-$open and $\beta-\omega-$open ) set in $X$.

Using the above proposition we can get the following corolory:

**Corollary 1.11.** Let $(X,\sigma)$ and $(Y,\tau)$ be two topological spaces. A map $f:(X,\sigma)\to(Y,\tau)$ is $\omega-$continuous ( resp. $\alpha-\omega-$continuous, $pre\ -\omega-$continuous, $b\ -\omega-$continuous and $\beta-\omega-$continuous ) if and only if for each

$\omega-$closed ( resp. $\alpha-\omega-$closed, $pre\ -\omega-$ closed, $b\ -\omega-$ closed and $\beta-\omega-$ closed) set $V$ in $Y$, $f^{-1}(V)$ is $\omega-$ closed ( resp. $\alpha-\omega-$ closed, $pre\ -\omega-$ closed, $b\ -\omega-$ closed and $\beta-\omega-$ closed) set in $X$.

**Lemma 1.12.** [1] Let $(X,\sigma)$ and $(Y,\tau)$ be two topological spaces such that $X$ satisfies the $\omega-B_{\alpha}-$condition, and $f:(X,\sigma)\longrightarrow(Y,\tau)$ be a map. If $f$ is $\alpha-\omega-$continuous then it is $\omega-$continuous.

**Lemma 1.13.** [1] Let $(X,\sigma)$ and $(Y,\tau)$ be two topological spaces such that $X$ is door space, and $f:(X,\sigma)\longrightarrow(Y,\tau)$ be a map.

**1**. If $f$ is $pre-\omega-$continuous then it is $\alpha-\omega-$continuous.

**2**. If $f$ is $\beta-\omega-$continuous then it is $b-\omega-$continuous.

**Lemma 1.14.**[1] Let $(X,\sigma)$ and $(Y,\tau)$ be two topological spaces that satisfy the $\omega-$condition then the map $f:(X,\sigma)\longrightarrow(Y,\tau)$ is continuous if and only if it is $\omega-$continuous.

**Lemma 1.15.**[1] Let $(X,\sigma)$ and $(Y,\tau)$ be two topological spaces that satisfy the $\omega - B_\alpha -$condition then the map $f:(X,\sigma) \longrightarrow (Y,\tau)$ is continuous if and only if it is $\alpha - \omega -$continuous.

**Lemma 1.16.**[1] Let $(X,\sigma)$ and $(Y,\tau)$ be two topological spaces that satisfy the $\omega - B -$condition then the map $f:(X,\sigma) \longrightarrow (Y,\tau)$ is continuous if and only if it is $pre - \omega -$continuous.

**Lemma 1.17.**[1] Let $(X,\sigma)$ and $(Y,\tau)$ be two door topological spaces and $f:(X,\sigma) \longrightarrow (Y,\tau)$ be a map. Then

**1.** $f$ is $pre - \omega -$continuous if and only if it is $\omega -$continuous.

**2.** $f$ is $\beta - \omega -$continuous if and only if it is $b - \omega -$continuous.

## 2.Weak types of $\omega$-homeomorphisms

**Definition 2.1.** A map $f:X \longrightarrow Y$ between two topological spaces $X$and $Y$ is called $\omega -$***homeomorphism*** (resp. $\alpha - \omega -$***homeomorphism***, $pre - \omega -$***homeomorphism***,$b - \omega -$***homeomorphism***, and $\beta - \omega -$

***homeomorphism*** ) if it is bijective , and bi$-\omega-$continuous ( resp. bi $-\alpha-\omega-$continuous, bi$-pre-\omega-$continuous, bi$-b-\omega-$continuous, and bi$-\beta-\omega-$continuous). Bi$-\omega-$continuity of $f$ means that $f$ and its inverse are $\omega-$continuous.

**Remark 2.2.** Using Lemma 1.12, Lemma 1.13, Lemma 1.14, Lemma 1.15, Lemma 1.16, and Lemma 1.17, we can prove theorems similar to Lemma 1.12, Lemma 1.13, Lemma 1.14, Lemma 1.15, Lemma 1.16, and Lemma 1.17, for the weak types of $\omega-$homeomorphisms. Also we can summarize these theorems by a figure like Figure 1.3 in [1], ( see p.33 of [1] )

To prove our next theorem we need the following results and definitions from [1].

**Lemma 2.3.**[1] An $\omega-$closed ( resp. $\alpha-\omega-$closed, $pre-\omega-$closed, b $-\omega-$closed and $\beta-\omega-$closed) subset of $\omega-$compact ( resp. $\alpha-\omega-$compact, $pre-\omega-$compact, $b-\omega-$compact and $\beta-\omega-$compact ) subspace is $\omega-$compact ( resp. $\alpha-\omega-$compact, $pre-\omega-$compact, $b-\omega-$compact and $\beta-\omega-$compact ).

Where $\omega-$compact $\alpha-\omega-$compact, $pre\ -\omega-$compact, $b\ -\omega-$compact and $\beta-\omega-$compact sets have the same definition of the compact sets by replacing the open sets by $\omega-$open $\alpha-\omega-$ open, $pre\ -\omega-$ open, $b\ -\omega-$ open and $\beta-\omega-$ open sets respectively.

**Lemma 2.4.** [1] Let $f:X\longrightarrow Y$ be an $\omega-$continuous ( resp. $\alpha-\omega-$ continuous, $pre\ -\omega-$ continuous, $b-\omega-$ continuous and $\beta\ -\omega-$ continuous) map from the $\omega-$compact ( resp. $\alpha-\omega-$compact, $pre\ -\omega-$ compact, $b-\omega-$compact and $\beta\ -\omega-$compact ) space $X$ onto a topological space $Y$. Then $Y$ is $\omega-$compact ( resp. $\alpha-\omega-$compact , $pre\ -\omega-$compact, $b-\omega-$compact and $\beta\ -\omega-$compact ) space.

**Definition 2.5.** [1] Let $\mathrm{X}$ be a topological space. And for each $\mathrm{x}\neq \mathrm{y}\in \mathrm{X}$, there exist two disjoint sets $\mathrm{U}$ and $\mathrm{V}$ with $\mathrm{x}\in \mathrm{U}$ and $\mathrm{y}\in \mathrm{V}$, then $\mathrm{X}$ is called:

1. $\omega-\mathrm{T}_2$ space if $\mathrm{U}$ is open and $\mathrm{V}$ is $\omega-$open sets in $\mathrm{X}$.
2. $\alpha-\omega-\mathrm{T}_2$ space if $\mathrm{U}$ is open and $\mathrm{V}$ is $\alpha-\omega-$open sets in $\mathrm{X}$.
3. $\omega^{*}-\mathrm{T}_2$ space if $\mathrm{U}$ and $\mathrm{V}$ are $\omega-$open sets in $\mathrm{X}$.

**4**. $\alpha-\omega^{\star}-T_2$ space if $U$ is $\omega-$open and $V$ is $\alpha-\omega-$open sets in $X$.

**5**. $pre-\omega-T_2$ space if $U$ is open and $V$ is $pre-\omega-$open sets in $X$.

**6**. $pre-\omega^{\star}-T_2$ space if $U$ is $\omega-$open and $V$ is $pre-\omega-$open sets in $X$.

**7**. $b-\omega-T_2$ space if $U$ is open and $V$ is $b-\omega-$open sets in $X$.

8. $b-\omega^{\star}-T_2$ space if $U$ is $\omega-$open and $V$ is $b-\omega-$open sets in $X$.

**9**. $\beta-\omega-T_2$ space if $U$ is open and $V$ is $\beta-\omega-$open sets in $X$.

**10**. $\beta-\omega^{\star}-T_2$ space if $U$ is $\omega-$open and $V$ is $\beta-\omega-$open sets in $X$.

**Lemma 2.6.[1]** For any topological space.

**1**. Every $\omega-$compact subset of $\omega-T_2$ space is closed.

**2**. Every $\alpha-\omega-$compact subset of $\alpha-\omega-T_2$ space is closed.

**3**. Every $\omega-$compact subset of $\omega^{\star}-T_2$ space is $\omega-$closed.

**4**. Every $\alpha-\omega-$compact subset of $\alpha-\omega^{\star}-T_2$ space is $\omega-$ closed.

**5**. Every $pre-\omega-$compact subset of $pre-\omega-T_2$ space is closed.

**6**. Every $pre^{-\omega}$-compact subset of $pre-\omega^{\star}-T_2$ space is $\omega-$ closed.

**7**. Every $b-\omega-$compact subset of $b-\omega-T_2$ space is closed.

**8**. Every $b^{-\omega}-$compact subset of $b-\omega^{\star}-T_2$ space is $\omega-$ closed.

**9**. Every $\beta-\omega-$compact subset of $\beta-\omega-T_2$ space is closed.

**10**. Every $\beta-\omega-$compact subset of $\beta-\omega^{\star}-T_2$ space is $\omega-$closed.

Now we can introduce our main results in the following theorem:

**Theorem 2.7.** Let $(X,\tau)$ and $(Y,\sigma)$ are two topological spaces, and $f:X\longrightarrow Y$ be a bijective map.

**1.** If $f$ is $\omega-$continuous, $X$ is $\omega-$compact, and $Y$ is $\omega-T_2$, then $f$ is $\omega-$ homeomorphism.

**2.** If $f$ is $\omega-$continuous, $X$ is $\omega-$compact, and $Y$ is $\omega^{\star}-T_2$, then $f$ is $\omega-$ homeomorphism.

**3.** If $f$ is $\alpha-\omega-$continuous, $X$ is $\alpha-\omega-$compact, and $Y$ is $\alpha-\omega-T_2$, then $f$ is $\alpha-\omega-$ homeomorphism.

**4.** If $f$ is $\omega-$continuous, $X$ is $\alpha-\omega-$compact, and $Y$ is $\alpha-\omega^{\star}-T_2$, then $f$ is $\alpha-\omega-$ homeomorphism.

**5.** If $f$ is $pre-\omega-$continuous map, $X$ is $pre-\omega-$compact, and $Y$ is $pre-\omega-T_2$, then $f$ is $pre-\omega-$ homeomorphism.

**6.** If $f$ is $\omega-$continuous map, $X$ is $pre-\omega-$compact, and $Y$ is $pre-\omega^{\star}-T_2$, then $f$ is $pre-\omega-$ homeomorphism.

**7.** If $f$ is $b-\omega-$continuous map, $X$ is $b-\omega-$compact, and $Y$ is $b-\omega-T_2$, then $f$ is $b-\omega-$ homeomorphism.

**8.** If $f$ is $b-\omega-$continuous map, $X$ is $b-\omega-$compact, and $Y$ is $b-\omega^{\star}-T_2$, then $f$ is $b-\omega-$ homeomorphism.

**9.** If $f$ is $\beta-\omega-$continuous map, $X$ is $\beta-\omega-$compact, and $Y$ is $\beta-\omega-T_2$, then $f$ is $\beta-\omega-$ homeomorphism.

**10.** If $f$ is $\beta-\omega-$continuous map, $X$ is $\beta-\omega-$compact, and $Y$ is $\beta-\omega^{\star}-T_2$, then $f$ is $\beta-\omega-$ homeomorphism.

**Proof of (1):**

We shall prove that the image of any $\omega-$closed set in $X$ under $f$ is $\omega-$closed in $Y$. By Corollary 1.11, this gives continuity of $f^{-1}$. If $A$ is $\omega-$closed in $X$, then by Lemma 2.3 we have $A$ is $\omega-$compact. And by Lemma 2.4 we have $f(A)$ is $\omega-$compact. Since $Y$ is $\omega-T_2$, so by Lemma 2.6 $f(A)$ is closed in $Y$, which implies $f(A)$ is $\omega-$closed.

The proofs of the other cases are similar to that proof with simple modifications $\square$

### 3. Notes remarks and open problems

Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a family of sets. The ***Cartesian product*** $\Pi_\alpha A_\alpha$ is the set of all maps $c:\Lambda \to \cup_\alpha A_\alpha$, having the property for each $\alpha \in \Lambda$**;** $c(\alpha) \in A_\alpha$.

An element $c \in \Pi_\alpha A_\alpha$ is generally written $\{a_\alpha\}_{\alpha\in\Lambda}$ indicating that $c(\alpha) = a_\alpha$ for each $\alpha$, with $a_\alpha \in A_\alpha$.

Let $\{X_i, Y_i\}_{i=1}^n$ be a family of topological spaces. If we have a family of $\omega-$ homeomorphisms $f_i: X_i \to Y_i$, $i = 1, \dots, n$. We can raise many questions which represent the start point of many problems:

Does $F = (f_1, f_2, \dots, f_n): \Pi_{i=1}^n X_i \to \Pi_{i=1}^n Y_i$, represent $\omega-$homeomorphism? How can we define the $\omega-$continuity of $F$? Are the properties $cl\,(A \times B) = cl(A) \times cl(B)$ and $int(A \times B) = int(A) \times int(B)$, true for the weak $\omega-$interiors and weak $\omega-$closures?

Also we can study the other types of weak $\omega-$homeomorphisms.

### References

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