On the b-separation axioms

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Abstract:
In this paper we present b-separation axioms by using b-open set and proved some related theorems .Where we consider T_0 ,T_1,$T_2$and T_3spaces also regular and normal spaces .

الملخص :
درسنا في هذا الموضوع بديهيات الفصل من النوع b وذلك بأستخدام المجموعات المفتوحة من النوعbوقمنا بأثباتوتوضيح بعض النظريات المتعلقة بها .كما درسنا فضاءاتT_0و T_1 وT_2 وأيضا T_3 وكذلك الفضاءات المنتظمة والسوية .

1. Introduction

Andrijevi´c [2] introduced a new class of generalized open sets called b-open sets into the fieldof topology. This class is a subset of the class of semi-preopen sets [3], i.e. a subset of atopological space which is contained in the closure of the interior of its closure. Also the classof b-open sets is a superset of the class of semi-open sets [7], i.e. a set which is contained inthe closure of its interior, and the class of locally dense sets [5] or preopen sets [1], i.e. aset which is contained in the interior of its closure. Andrijevi´c studied several fundamentaland interesting properties of b-open sets.

Definition (1.1):[2]
A subset A of a space X is called b-open if $A\subset cl(intA)\cup int(cl\ A)$. The class of all b-open sets in X will be denoted by B(X).

Example(1.2):[2]
Consider the set R of real numbers with the usual topology, and letA = $[0,1]\cup[\ ((1, 2)\cap Q)$ where Q stands for the set of rational numbers. Then A is b-open .

Proposition(1.3):
(a) The union of any family of b-open sets is a b-open set.
(b) The intersection of an open and a b-open set is a b-open set.

Proof:
The statements are proved by using the same method as in provingthecorresponding results for the other three classes of generalized open sets. (see [3])

Definition (1.4): [2]
A subset A of a space X is called b-closed if X-A is b-open.Thus A is b-closed iff $int(cl\ A) \cap cl(intA) \subset A$.

Definition (1.5):[2]
If A is a subset of a space X the b-closure of A, denoted bybcl(A), is $bcl(A)=\cap\{F:F \text{ b-closed}, F\supset A\}$. The b-interior of A, denoted bybint(A), is $bint(A)=\cup\{G:G \text{ b-open}, G\subset A\}$.

Definition (1.6):[4]

Let (X, τ) be a topological spaceand let $x\in X$, Asubset N of X is saidto be **b–nhd**ofx iff there exist a b-open set G such that$x\in G\subset N$.

Definition (1.7): [4]

The point x is a b-accumulation point of A if $\forall$ b-nhd N of x , $N\cap A\neq \varnothing$ .Then b-drive set defined by D(A)={x:x is a b-accumulation point }

Theorem(1.8):[6]

A subset of atopological space (X, τ) is b-open iff it isb–nhdof eachof its points.

Proof:

let G be a b-open subset ofX . Then for every $x \in G$, $x \in G\subset G$ and therefore G is ab-nhd of each of its points.

Conversely , let G be b – nhd of each of its point . if $G = \varnothing$, it is b-open ,If$G \neq\varnothing$,Then to each $x\in G$ there exist b-open setG_x such that $x\in G_x\subset G$ itfollowsThat$G=\cup\{G_x:x\in G\}$.

Hence G is b-open being a union of b-open sets .

2. b-T_0 space

Definition(2.1):

Atopological space $(X, \tau)$is said to be b-T_0iff given any pairof distinct pointsx,y of X , there exist b-nhd of one of them not containing the other .that is iffthere exist a b-open set G such that $x\in G, y\notin G$.

Example(2.2):

Let (X,D) be a discrete topological space and let x,y be distinct points of X . Since the space is discrete ,{x} is a b-opennhd of x which does not contain y . It follows that (X,D) is a b-T_0 space .

Theorem (2.3):

If W subspace of X (where W is open subset of X) ,Then W is b-T_0 if X is b-T_0 space .

Proof:

Let $x,y\in W$,Then $x,y\in X$. Since X is b-T_0space ,Then$\exists G$ b-open set $\ni x\in G$ $y\notin G$.

$E=G\cap W$ is b-open set in W

$x\in G\cap W =E$,$y\notin G\cap W =E$

Hence W is b-T_0 space.

Theorem(2.4):

A topological space (X, τ) is b-$T_o$iff for any distinct arbitrarypoints x,yof X, the b-closures of {x},{y}are distinct .

Proof:

$\Rightarrow$Let $x\neq y$ implies $\{x\}\neq\{y\}$, $bcl\{x\}\neq bcl\{y\}$ where x,y are points of X .Since$bcl\{x\}\neq bcl\{y\}$, There exist at least one point z of X which belongs to one ofthem,Say$bcl\{x\}$,and does not belongs to$bcl\{y\}$.We claim that $x\notin bcl\{y\}$.For let $x\in bcl\{y\}$then$bcl\{x\}\subset bcl(bcl\{y\})=bcl\{y\}$and so$z\in bcl\{x\}\subset bcl\{y\}$which contradiction.Accordingly $x\notin bcl\{y\}$ andconsequently$x\in X-bcl\{y\}$. Also since bcl{y} isb-closed,X- bcl{y}isb-open.Hence X- bcl{y}isb-opennhd of x containing y.it follows that (X, τ)isb-T_o space.

$\Leftarrow$Let (X, τ) be b-T_o space and let x,y be two distinct points of X .then we haveto show that $bcl\{x\}\neq bcl\{y\}$.since the space is b-T_o,there exist a b-open set G Containing one of them say x but notcontaining y .by definition $bcl\{y\}$ is intersection of all b-closed sets containing {y}. it follows that $bcl\{y\}\subset X-G$ .hence $x\notin X-G$ .implies that $x\notin bcl\{y\}$.thus $x\in bcl\{x\}$ but $x\notin bcl\{y\}$.it follows $bcl\{x\}\neq bcl\{y\}$.thus it is shown that in b-T_o space distinctpoints have distinct closure.

3. b-T_1 space

Definition(3.1):

Atopological spaces (X, τ) is said to be a b-T_1 spaceiffgiven anypair of distinct points x , y of X there exist two b-open sets one containingy but not x, that is , there exist b-open sets G and H such that x $\in$ G but y$\notin$ G and y$\in$H but x$\notin$H .

Theorem (3.2):

If W subspace of X (where W is open subset of X) ,Then W is b-T_1 if X is b-T_1 space .

Proof:

Let x,y$\in$W ,Then x,y$\in$X . So $\exists$ B_1,B_2 b-open sets in X $\ni$ x $\in B_1$ but y$\notin B_1$ and y$\in B_2$ but x$\notin B_2$.

$E_1 = B_1 \cap W$, $E_2 = B_2 \cap W$

So E_1,$E_2$are b-open set in W .

Then x$\in E_1$, y$\in E_2$ & x$\notin E_2$,y$\notin E_1$ therefore W is b-T_1 space .

Remark (3.3):

Every b-T_1 space is b-$T_o$space.

Proof :

Let(X, τ) is b-T_1space and let x,y any pair of distinct points of X , Then there exists there exist two b-open sets G and H such that x $\in$ G but y$\notin$ G and y$\in$H but x$\notin$H .That is H-{x}is b-open nhd of x containing y ,and G-{y}is b-open nhd of y containing x, it follows that (X, τ)is b-T_o space.

Remark (3.4):

Every T_1 space is b-T_1space.

Proof :

Let (X, τ) is said to be a T_1space,Thengiven anypair of distinct points x , y of X there exist two open sets one containingy but not x, that is , there exist open sets G and H such that x $\in$ G but y$\notin$ G and y$\in$H but x$\notin$H .And since every open set is b-open set ,Hence every T_1 space is b-T_1 space.

Theorem(3.5):

A topological space (X,τ) is b-T_1 space iff every singleton subset{x}of X is b-closed.

Proof:

$\Leftarrow$Let every singleton subset {x}of X be b-closed we haveto show that the space is b-T_1.let x,y be any two distinct point of X, then X-{x} is ab-open which contains y but doesnot contains x. similarly X-{y}is b-open which contains x but does not contains y Hence the space(X, τ) is b-T_1.

$\Rightarrow$let the space beb-T_1 and let x be any point of X.We want to show that{x} isb-closed, that is X-{x} is b-open ,Let y$\in$ X-{x} then y$\neq$x since X is b-T_1 ,there existab-open set G_y such that y$\in G_y$ but x$\notin G_y$ it follows that y$\in G_y \subset$ X-{x} hence bytheorem(1.8) ,X-{x} isb-open , {x}is b-closed .

4. b-T_2 space

Definition(4.1):

A topological space (X, τ) is said to be a b-T_2 space iff forevery pair of distinct points x,y of X there exist disjoint b-nhds N and M of x and y respectively such that $N \cap M = \emptyset$.

Theorem (4.2):

If W subspace of X (where W is open subset of X) ,Then W is b-T_2 if X is b-T_2 space .

Proof:

Let x,y$\in$W ,Then x,y$\in$X . So $\exists$ B_1,B_2such that $B_1 \cap B_2 = \emptyset$&x $\in B_1$, y$\in B_2$.Where B_1,B_2 are b-open sets in X .

$E_1 = B_1 \cap W$, $E_2 = B_2 \cap W$ are b-open subsets in W .
And $x \in E_1$, $y \in E_2$
$E_1 \cap E_2 = (B_1 \cap W) \cap (B_2 \cap W)$
$= (B_1 \cap B_2) \cap W = \emptyset \cap W = \emptyset$
Hence W is b-T_2-space .

Theorem(4.3):

Each singleton subset of b-T_2 space is b-closed .

Proof :

Let X be a b-T_2 space and let $x \in X$ .to show that {x}is closed . let y be anarbitrary point of X distinct from x .since the space is b- T_2 space ,there exist b-nhd N of y such that $x \notin N$ .It follows that y is not b- accumulationpointof {x}and consequently D({x})=∅, hence b*cl{x}*={x}then {x}is b-closed.

Remark (4.4):

Every T_2 space is a b-T_2space .

Proof :

Let (X, τ) is said to be a T_2 space iff forevery pair of distinct points x,y of X there exist disjointnhds N and M of x and y respectively such that $N \cap M = \emptyset$ .since N ,M arenhds , Then contain two open sets G ,H respectivelyand since every open set is b-open set ,Then G,H are b-open sets impilies N,M are b-nhds,HenceEvery T_2 space is a b-T_2 space .

Theorem(4.5):

Everyb-T_2 space is b-T_1space.

Proof:

Let (X, τ) bea b-T_2 space and let x,y be any two distinct points of X , since the space is b-T_2 space, there exist b-nhd Nof x and b-nhd Mof y such that$N \cap M = \emptyset$.this implies that $x \in N$ but $y \notin N$ and $y \in M$ but $x \notin N$ hence is b-T_1 space.
But the converse of Theorem(4.5) is not true in general as the following example shows:

Example (4.6):

Consider the co-finite topologyτ on an infinite set X. Then the space (X, τ) is b-T_1 . For if x is any arbitrary point of X , then by definition of τ , X-{x} is b-open (being the complement of finite set) and consequently {x} is b-closed . Thus every singleton subset of X is b-closed and hence the space is b-T_1 . But This space is not b-T_2 . For this topology no two b-open sets can be disjoints subsets of X so that $G \cap H = \emptyset$. Hence

$$(G \cap H)' = \emptyset' = X$$
$$\rightarrow G' \cup H' = X \quad . \qquad \text{[De-Morgan Law]}$$

But G′ and H′ are finite sets and so their union is also finite which is a contradiction since X is infinite . Hence no two distinct points can be separated by b-open sets . Accordingly this space is not b-T_2 .

Theorem(4.7):

Let (X, τ) be b-topological space .then the following statements areEquivalent :

a) **τ** is b-T_2 topology for X,
b) the intersection of all b-closed nhds of each point of X is singleton.

Proof:

(a)⇔ (b) .let (X, τ) be b- T_2 space and let x,y be any two points of X there exist b-open sets G andH such that $x \in G$, $y \in H$ *and* $G \cap H = \emptyset$.since$G \cap H = \emptyset$; $x \in G \subset X\text{-}H$ hence *X-H* isb-closed nhd of x which does notcontain y.since y is arbitrary ,the intersection of all b-closed nhds is the singleton{x}.

Conversely ,let {x} be the intersection of all b-closednhds of arbitrary point $x \in X$.
let y be any point of X different from x . since y does not belong to the intersection,Theremust existb-closed nhd Nof x such that $y \notin N$ since N is b-nhd of x, There existsb-open set G such that $x \in G \subset N$.Thus G and X-N are b-open sets such that $x \in G$,$y \in X-N$ and $G \cap (X-N) = \varnothing$.Itfollows that the space is b-T_2 space.

Definition(4.8):

Let (X, τ) and (Y,v) be two b-topological spaces and let f be a mapping ofXintoYthen f is said to be b–open mapping ifff[G] is b–open in Ywhenever G isb–open in X .

Theorem(4.9):

The property of a space being a b-T_2 space is preserved by one-to-one ,onto,and b–open mapping.

Proof:

Let (X, τ) be b- T_2 space and let*f*be one-to-one,b–open mapping of (X, τ) onto another space(Y,v). we shall show that (Y,v) is b- T_2 space,let y_1,y_2 be twodistinct points of Y. since f is one-to-one onto map ,there exist distinct point $x_1,x_2$of X such that $f(x_1)=y_1$, $f(x_2)=y_2$,since (X, τ) isb- T_2 space ,there exist b–opensets G and Hsuch that $x_1 \in G$, $x_2 \in H$ and $G \cap H = \varnothing$. Again since f is an b–open mapping$f[G]$ and $f[H]$areb–open in Y such that
$y_1 = f(x_1) \in f[G]$,
$y_2 = f(x_2) \in f[H]$ and $f[G] \cap f[H] = f[G \cap H] = f[\varnothing] = \varnothing$
Since *f*is one-to-one ,we have $f[G] \cap f[H] = f[G \cap H]$ it follows that (Y,v)is also b-T_2 space.

5. b-regular space

Definition (5.1):

Let (X, τ) be a topological space, then X is called **b-regular space**, if for each $x \in X$ and F b-closed subset of X such that $x \notin F$ there exist two b-open sets say U and V such that $x \in U$, $F \subseteq V$ and $U \cap V = \varnothing$.

Remark (5.2):

Every regular space is a b-regular.

Proof :

Let (X, τ) be aregular space, Then for each $x \in X$ and F closed subset of X such that $x \notin F$ there exist two open sets say U and V such that $x \in U$, $F \subseteq V$ and $U \cap V = \varnothing$. and because every open set is b-open set ,Then every regular is b-regular .
But the converse of remark (5.2) is not true in general as the following example shows:

Example (5.3):

Let X ={a, b, c}, τ= {X, ∅, {a, b}}
PO(X) = {X, ∅, {a, b}, {a}, {b}, {a, c}, {b, c}}
PC(X) = {X, ∅, {c}, {b, c}, {a, c}, {b}, {a}}
(X, τ) is a b-regular space, but (X, τ) is not a regular space.

Theorem (5.4):

A topological space X is b-regular iff for every point $x \in X$ and every b-nhdNof x, there exists a b-nhd Mof x such thatbcl(M) ⊂ N .In other words, a topological space isb-regulariff the collection of all b-closed nhds of x from a local base at x .

Proof :

It is enough to prove the theorem for b-open nhds .
Let N be any b-nhd of x . Then there exist b-open set G such that $x \in G \subset N$. Since G′ is

b-closed and $x \notin G'$, by definition there exist b-open sets L and M such that $G' \subset L$,$x \in M$ and $L \cap M = \emptyset$ so that $M \subset L'$.
It follows that
$bcl(M) \subset bcl(L') = L'$
Also $G' \subset L \rightarrow L' \subset G \subset N$.
From (1) and (2), we get $bcl(M) \subset N$.
Let the condition hold .Let F be any b-closed set and let $x \notin F$ .Then $x \in F'$. Since F' is a b-open set containing x , by hypothesis there exist a b-open set M such that $x \in M$ and $bcl(M) \subset F' \rightarrow F \subset [bcl(M)]'$. Then $[bcl(M)]'$ is a b-open set containing F . Also

$$M \cap M' = \emptyset \rightarrow M \cap [bcl(M)]' = \emptyset$$

Hence the space is b-regular .

Definition (5.5):

A topological space (X, τ) is called **b-T_3space**, if X is a b-T_1 and b-regular space.

Remark (5.6):

1. Every b-T_3 space is a b-T_2 space.
2. Every T_3 space is a b-T_3 space.

Proof :

1-Let (X,τ) be a b-T_3 space and let x,y be two distinct points of X . Now by definition, X is also a b-T_1 space and so $\{x\}$ is a b-closed set .Also $y \notin \{x\}$. Since (X,τ) is a b-regular space, There exist b-open sets G and H such that

$$\{x\} \subset G,\ y \in H \text{ and } G \cap H = \emptyset .$$

Also $\{x\}\} \subset G \rightarrow x \in G$. Thus x,y belong respectively to disjoint b-open sets G and H . Accordingly (X,τ) is a b-T_2space .

2- Let (X,τ) be a T_3space , Then X is a T_1 and regular space. And from remark(3.4) every T_1 space is b-$T_1$and remark(5.2) every regular space is b-regular ,Then every T_3 space is b-T_3 .

Corollary (5.7):

Every T_3-space is a b-T_2space.

Proof :

From remark(5.6-2) every T_3 space is b-T_3 and from remark(5.6-1) every b-T_3 space is a b-T_2 space , Then every T_3-space is a b-T_2 space .

Remark (5.8):

Every b-T_3 space is a b-regular space.

Proof :

From definition(5.5) let (X, τ) is a b-T_3 space, Then X is a b-T_1 and b-regular space . Hence X is b-regular space , Hence everyb-T_3 space is a b-regular space .

6. b-normal space

Definition (6.1):

Let (X, τ) be a topological space is called **b-normal**, if for each $F_1$and F_2 two b-closed sets such that $F_1 \cap F_2 = \emptyset$, there exist two b-open sets say U and V such that $F_1 \subseteq U$, $F_2 \subseteq V$ and $U \cap V = \emptyset$.

Example (6.3):

Let $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}$, Then (X, τ) is a b-normal space .

Remark (6.2):

Every normal space is a b-normal space.

Proof :

Let (X, τ) is a normal space,Thenfor each F_1 and F_2 two closed sets such that $F_1 \cap F_2 = \varnothing$, there exist two open sets say U and V such that $F_1 \subseteq U$, $F_2 \subseteq V$ and $U \cap V = \varnothing$. And since every open set is a b-open ,Hence every normal space is a b-normal space .

Theorem(6.3):

(X, τ) is a b-normal space, iff for each b-closed set F and for each b-open set Esuch that $F \subset E$, there exists a b-open set V such that $F \subset V \subset bcl(V) \subset E$.

Proof :

Let X be a b-normal space and let F be any b-closed set and E a b-open set such that $F \subset E$. Then E' is a b-closed set such that $F \cap E' = \varnothing$

Thus E' and F are disjoint b-closed subsets of X. Since the space is b-normal, there exist b-open sets U and V such that $E' \subset U$, $F \subset V$ and $U \cap V = \varnothing$ so that $V \subset U'$

But $V \subset U' \rightarrow bcl(V) \subset bcl(U') = U$ [since U' is b-closed] …….(1)

Also $E' \subset U \rightarrow U' \subset E$ ...…..(2)

From (1) and (2) , we get $bcl(V) \subset E$. Thus there exists a b-open set V such that $F \subset V$ and $bcl(V) \subset E$.

Let the condition hold .Let N and M be b-closed subsets of X such that $N \cap M = \varnothing$ so that $N \subset M'$

Thus the b-closed set N is contained in the b-open set M' .By hypothesis there exists a b-open set V such that

$N \subset V$ and$bcl(V) \subset M'$ which implies $M \subset [bcl(V)]'$

Also $V \cap [bcl(V)]' = \varnothing$

Thus V and $[bcl(V)]'$ are two disjoint b-open sets such that

$N \subset V$ and $M \subset [bcl(V)]'$

Implies that the space is b-normal .

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