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Orthonormal Vector Polynomials in a Unit Annular Pupil

Sundus Y. Hasan* and Thill A. Kadhum **

***Physics Department, Education College for Girls, Kufa University, Najaf 54001, Iraq**
****Physics Department, Science college, Al-Muthanna university, Iraq**
Corresponding author: sunds2004@yahoo.com

Abstract

A set of orthonormal polynomials, which represent scalar functions, are given in this paper that represent balanced basis over an area of annular pupil. These functions are used to represent the aberration of imaging systems with annular pupils. Another orthonormal set of polynomials related to those, which represent vector quantities, can be generated from the gradients of annular Zernike polynomials (ZP), and the orthogonality is made by MATLAB code for Gram Schmidt orthogonalization method (GSOM). A relation of these polynomials to the circular ZP and circular ZP gradients are represented also in this work.

1. Introduction

Annular pupils were taking the interest of several researchers for their use in optical systems.

In (1981) Mahajan discussed the aberrations of imaging systems with uniformly illuminated annular pupils in terms of complete set of polynomials that are orthogonal over an annular region. These polynomials, which were called annular ZP, are similar to the circular ZP and reduce to them as the annulus approaches the full circle, and he described how the higher order classical aberration of a power-series expansion is balanced with one or more lower-order classical aberrations to minimize its variance [1]. In this paper he introduced six polynomials of annular Zernike polynomials, while in (1994) he introduced 19 polynomials [2].

In (1987), Liu Ying and Moh Edward compared the first (16) terms of annular ZP with circular ZP [3]. While in(1996), Dai discussed the wavefront reconstruction by the first (15) Zernike annular polynomials with obscuration ratio (ϵ =0.8), and compared the results with circular ZP, he noted that the amount of total error in the wavefront reconstruction became smaller by using annular pupil instead of circular pupil [4].

In (2006), Dai and Mahajan discussed the imaging through atmospheric turbulence by systems with annular pupils using the annular ZP[5].

In (2007) Dai and Mahajan analyzed the error that arises if circular ZP are used for annular pupil [6]. And in the next year they used Zernike polynomials for focal system with annular aperture, with value (ϵ ≤0.2) to analyze wavefronts through the numerical results that have found that up to this value, the aberration of astigmatism be much smaller than both coma and spherical aberration, but as the value of the (ϵ) approaches (1), the aberration of astigmatism will be much larger than spherical aberration, and noted that spherical aberration does not appear significantly if approached the value of the (ϵ) than the one [7].

In year (2010), both Mahajan and Aftab, compared the use of the first (11) terms of annular ZP and the corresponding circular ZP to analyze annular wavefront aberration, and they found that annular ZP more efficient to analyze the aberrations of wavefront instead of circular ZP [8].

In (2013) Mahajan and Diaz used the first (11) terms of annular ZP with value of (ϵ =0.5), to study wavefront aberrations, and found that the occur deviations in the wavefronts and calculated by annular pupils have the same symmetry properties of wavefronts which calculated by circular pupils [9].

The vector Zernike polynomial or Zernike gradients are also important to study. These new vector functions have immediate application for fitting data from a Shack-Hartmann wavefront sensor or for fitting mapping distortion for optical testing .These polynomials are studied for circular aperture in (1976) by Robert J. Noll, who gave the rules for computing the derivatives of Zernike polynomials as a linear combination of the polynomials themselves. Derivatives of Zernike polynomials can be useful whenever the gradient of a wavefront is required. Wave-front gradients occur in some geometrical optics problems as well as direct measurements in an electronic Hartmann Test [10].

We developed a complete and orthonormal set of vector polynomials defined over a unit circle. One application of these vector polynomials is for fitting the mapping distortions in an interferometric null test.

This paper discusses the source of the mapping distortions and the approach of fitting the mapping relations, and in (2007), Zhao and Burge provide a set of complete basis for representing vector fields that can be defined as a gradient of some scalar functions across circular pupil. These polynomials can be transformed to the scalar circular ZP[11]. Then in the next year, in(2008), they gave an additional set of vector functions consisting only of rotational terms with zero divergence [12].These two sets together provide a complete basis that can represent all vector distributions in a circular domain. In (2009), they justified, with examples, why the set of vector polynomials is the appropriate choice for describing mapping distortions, and they showed the excellent fitting results with the polynomials [13].

In this work, after finding the scalar annular ZP, a set of vector polynomials are presented, which are orthonormal in a unit annular aperture with obscuration ratio equal to (ϵ¿ .These polynomials are perfect for fitting slope data, and the fitted slope map can be easily converted to the wavefront map expressed in terms of Zernike polynomials.

In the next section, a procedure for deriving the orthonormal annular ZP using (GSOM) and the circular Zernike polynomials were produced. Then the gradients of the annular ZP are calculated, and by using, for the second time, the (GSOM) the orthonormal vector annular ZPs were found. Then the relation of them with both the circular ZPs and circular gradient ZPs have been found, and finally several equations representing these relations were concluded.

2. Results

2.1-Annular Zernike polynomials

There are different numbering schemes for circular ZP, and in this work, Noll's notation has been adopted which is the same as what (C).Zhao and J. Burge considered, (see appendix I, which shows the first (21) polynomials in polar coordinates) [11].

Circular ZP are not suitable for annular pupils. So, these polynomials must be converted to annular ZP, and this could be done with (GSOM), which can be illustrated by the following equation:

$$A'_2(\rho,\theta;\epsilon) = A_2(\rho,\theta;\epsilon) - \frac{\int_0^{2\pi}\int_0^{1} A_1(\rho,\theta;\epsilon)A_2(\rho,\theta;\epsilon)\rho\, d\rho\, d\theta - \int_0^{2\pi}\int_0^{\epsilon} A_1(\rho,\theta;\epsilon)A_2(\rho,\theta;\epsilon)\rho\, d\rho\, d\theta}{\int_0^{2\pi}\int_0^{1} A_1^2(\rho,\theta;\epsilon)\rho\, d\rho\, d\theta - \int_0^{2\pi}\int_0^{\epsilon} A_1^2(\rho,\theta;\epsilon)\rho\, d\rho\, d\theta} * A_1(\rho,\theta;\epsilon)$$

(1)

And to normalize these polynomials, the normalization rule must be submitted:

$$C^2 \frac{\int_0^{2\pi}\int_0^{1} A_1^2(\rho,\theta;\epsilon)\rho\, d\rho\, d\theta - \int_0^{2\pi}\int_0^{\epsilon} A_1^2(\rho,\theta;\epsilon)\rho\, d\rho\, d\theta}{\int_0^{2\pi}\int_0^{1} \rho\, d\rho\, d\theta - \int_0^{2\pi}\int_0^{\epsilon} \rho\, d\rho\, d\theta} = 1$$

(2)

Where (A_1) and (A_2) are two non orthogonal functions, while (A'_2) is the function orthogonal with (A_1).

The above two equations were programmed in MATLAB, and the results were illustrated in Table (1), which show the first (21) orthonormal annular ZP in polar coordinates, for annular pupil with obscuration ratio (ϵ¿ . These polynomials can be reduced to circular ZPs by putting (ϵ =0), (see appendix I). And also the low orders of annular ZP were compared with that mentioned in reference, by Mahajan and Dais [9], and they found to be the same.

Table (2), shows the annular Zernike polynomials in Cartesian coordinates, the conversion is made also using a MATLAB code using these equations:

$$x = r\cos(\theta) \quad , \quad y = r\sin(\theta) \quad , \quad r^2 = x^2 + y^2 \quad , \quad \theta = \tan^{-1} ¿ \; ¿ \; ¿ \; y/¿ \; x \qquad (3)$$

Then the relationship between the annular ZP and the circular ZP were shown in table (3). Where (Z_i) here represents circular ZP.

2.2-The Gradients of annular Zernike polynomials

To find the gradient of annular ZP, a MATLAB code was written for this purpose. So, either the results in table (4), which were written in Cartesian coordinates, are used to compute gradients with the equation:

$$\nabla A = \frac{\partial A}{\partial x}\hat{i} + \frac{\partial A}{\partial y}\hat{j}$$

(4)

Or the results of table (1) were used, which were written in polar coordinates,

$$\nabla A = \frac{\partial A}{\partial \rho}\hat{a}_\rho + \frac{\partial A}{\rho \partial \theta}\hat{a}_\theta$$

(5)

Where $(\hat{a}_\rho \wedge \hat{a}_\theta)$ represent the unit vectors in polar coordinates. Here a transformation of the coordinates to the Cartesian coordinates must be done to get the components of ∇A in($\hat{i} \wedge \hat{j}$).

Table (4) represents the gradients of the first (21) annular polynomials. These functions are not easy to work with, because they are not orthogonal to each other over an annular aperture. So to convert them to an orthonormal functions, equations (1) and (2) must be worked but for the annular gradient polynomials (vector polynomials). Note that the first polynomial is zero, so it is not included in the process of MATHLAB code and the process begin from the second polynomial, because if not, an error of dividing by zero would appear. Note also that the dot product of the different unit vector is equal to (1) and that of the same is equal to(0).

The process now became sort of complicated, and the computer became slow in finding the functions. So this process can be done in another way by applying (GSOM) for gradients of Zernike circular aperture but over the annular aperture, i.e. the limits of integration will be the limits of annular. The results, table (5), show the orthonormal vector annular polynomials. And Table (6), represents the orthonormal vector annular ZP as Function of Circular ZPZ(x,y), while Table (7) represents them in terms of gradient of circular Zernike Polynomials ∇ Z($x,y¿$ (see appendix II).

By having a glance at table (8), it can be concluded that:

1)For all (j) with (n=m),

$$\vec{S}_j=\frac{1}{\sqrt{2n(n+1)(\epsilon^{2n-2}+\epsilon^{2n-4}+..+1)}}\nabla Z_j$$

(6)

2)And for all j with (n $\neq$ m), and (n-m=2)

$$\vec{S}_j=C_j\left[k\nabla Z_j-\sqrt{\frac{n+1}{n-1}}\nabla Z_{j'(n'=n-2,m'=m)}\left(n\epsilon^{2(n-2)}+\epsilon^{2(n-2)-2}+\ldots+1\right)\right](7)$$

Where (C) represent the normalization constant for the orthogonal vector annular ZP (table 7), and k= $\left(\epsilon^{2(n-3)}+\epsilon^{2(n-3)-2}+\ldots+1\right)$ except for (n<3), k=1.

3)For (n $\neq$ m), (n $-¿$ m=4)

$\vec{S}$ j = $\frac{k_1\nabla Z_j}{C_j¿}$ – k_2 ∇Z_j (n-2,m) – k_3 ∇Z_j (n-4,m))

(8)

k_1= $(2\epsilon^4-\epsilon^2+2)$

k_2= $\sqrt{\frac{3}{2}}(10\epsilon^6+2\epsilon^4-\epsilon^2+12)$, $k_3=10\sqrt{3}(\epsilon^8+3\epsilon^6-3\epsilon^4+2\epsilon^2+3)$ Where (j-j') is even when (m $\neq$ 0).

In general, the vector polynomial ($\vec{S}¿$ is equal to:

$\vec{S} = \hat{i} S_{jx} + \hat{j} S_{jy}$

(9)

Figure (1) shows the plots of first (21) vector annular polynomials. The arrows represent the amount and direction of the displacement of a particular point.

2.3-The relation between the vector annular ZP and the scalar circular ZP

The space of vector distribution over the unit annular can be written as a linear combination of a set of S polynomials:

$\vec{V} = \sum_j \alpha_j S_j$

(10)

and it can be written as a gradient of scalar function:

$\vec{V} = \nabla \Phi$

(11)

but $\vec{S}_j = \nabla \phi_j$

so, from equations (6, 7, and 8), we get:

1)For all j with (n=m),

$\phi_j = \frac{1}{\sqrt{2n(n+1)(\epsilon^{2n-2} + \epsilon^{2n-4} + .. + 1)}} Z_j$

(12)

2)For all (j) with (n $\neq$ m), and (n-m=2)

$$\phi_j = C_j \left[k Z_j - \sqrt{\frac{n+1}{n-1}} Z_{j'(n'=n-2, m'=m)} \left(n\epsilon^{2(n-2)} + \epsilon^{2(n-2)-2} + \ldots + 1 \right) \right] \quad (13)$$

3)For (n $\neq$ m), (n $-¿$ m=4)

ϕ_j = $k_1 Z_j$ $C_j ¿$ – k_2 Z_j (n-2,m) – k3 Z_j (n-4,m))

(14)

Where(j-j') is even when (m $\neq$ 0).

That means, the scalar polynomials ($\phi ¿$ can be found form the vector polynomials ($\vec{S} ¿$, for example:

$$\vec{S}_7 = \left[\nabla Z_7 - \sqrt{2}(3\epsilon^2 + 1) \nabla Z_3 \right] / \sqrt{48\epsilon^4 - 24\epsilon^2 + 48}$$

leads to:

$\phi_7 = \left[Z_7 - \sqrt{2}(3\epsilon^2 + 1) Z_3 \right] / \sqrt{48\epsilon^4 - 24\epsilon^2 + 48}$

$$\vec{V}=\sum_j \alpha_j S_j=\nabla \Phi$$

(15)

Then the scalar function can be written as a linear combination of standard circular ZP.

$$\Phi=\sum_j \alpha_j \phi_j=\sum_j \gamma_j Z_j$$

(16)

Where(γ_j¿ is:

1)For (j) with (n=m),

$$\gamma_j = \frac{\alpha_{j(n,m)}}{\sqrt{2n(n+1)(\epsilon^{2n-2}+\epsilon^{2n-4}+..+1)}} - \alpha_{j(\acute{n}=n+2,\acute{m}=m)} C_{j(n',m)} \sqrt{\frac{n'+1}{n'-1}}\left(n'\epsilon^{2(n'-2)}+\epsilon^{2(n'-2)-2}+\ldots+1\right) - \alpha_{j(\acute{n}=n+4,\acute{m}=m)} C_{j(n',m)} k_3$$

(17)

2)For all (j) with ($n \neq m$) and (n-m=2),

$$\gamma_j = \alpha_{j\,(n,m)} C_{j(n,m)}\, k - \alpha_{j(\acute{n}=n+2,\acute{m}=m)} C_{j'(n'+m)} \sqrt{\frac{n'+1}{n'-1}}\left(n'\epsilon^{2(n'-2)}+\epsilon^{2(n'-2)-2}+\ldots+1\right)$$

(18)

3)For (j) with ($n \neq m$) and (n-m=4),

$$\gamma_j = C_j\, \alpha_{j\,(n,m)}$$

(19)

3. Discussion:

By looking at the forms of annular ZPs, It can be seen that these polynomials are having the same properties of that of circular ZP, they have axial symmetry (because they can be written in one form of triangular function (sin or cos), and circular symmetry (because they were separable in r and ϵ¿).

When these polynomials were written in terms of circular ZPs, it can be concluded that they are a linear combination of at most three circular ZP, and when (n=m), Annular ZPs equal to the circular ZP multiplied by a constant depends on obscuration ratio (ϵ¿ , while when $n \neq m$ then annular ZP is a linear combination of circular ZP ($Z_{(n,m)}$, $Z_{(n-2)}$, and $Z_{(n-4)}$).

As like as when the annular ZP with (ϵ=0) gives the circular ZP, the vector annular ZP are returned to circular vector polynomials when (ϵ=0), and, like vector circular ZP, the annular vector ZP can be represented as linear combination of at most four circular ZP gradient. And again when (n=m) orthonormal annular vector ZPs equal to the circular ZP gradient multiplied by a constant depends on obscuration ratio (ϵ¿ , while when (n $\neq$ m) the S is a linear combination of circular ZP gradient ($\nabla Z_{(n,m)}$, $\nabla Z_{(n-2)}$, and $\nabla Z_{(n-4)}$).

From table (6), which represent the (S) as a function of (Z), the annular vector ZP is a linear combination of at most of three circular ZP, unlike that of circular vector which contain at most of two circular ZP.

From table (5) of (S), equations (6, 7, and 8) were concluded, which were used to conclude equations (12,13, and 14) that illustrate the relation between the vector polynomials and the scalar circular ZP.

Because the (S) polynomials are representing the divergence of a scalar functions f, it can be known from books of electromagnetism that the curl of these polynomials is equal to zero, so this makes these polynomials are not complete, and another work must be done to get the other polynomials with zero divergence and nonzero curl.

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Table (1): Zernike Annular Polynomials A(ρ , $\theta;\epsilon$¿ in polar coordinates

No.	A
1	1
2	$2\rho\cos(\theta)/\sqrt{\epsilon^2+1}$
3	$2\rho\sin(\theta)/\sqrt{\epsilon^2+1}$
4	$\sqrt{3}(2\rho^2-1-\epsilon^2)/(\epsilon^2-1)$
5	$\sqrt{6}\rho^2\sin(2\theta)/\sqrt{\epsilon^4+\epsilon^2+1}$

6	$\sqrt{6}\rho^2\cos(2\theta)/\sqrt{\epsilon^4+\epsilon^2+1}$
7	$-2\sqrt{2}\rho\sin(\theta)\left[2(\epsilon^4+\epsilon^2+1)-3(\epsilon^2-1)\rho^2\right]/[(\epsilon^2-1)\sqrt{\epsilon^6+5\epsilon^4+5\epsilon^2+1}]$
8	$-2\sqrt{2}\rho\cos(\theta)\left[2(\epsilon^4+\epsilon^2+1)-3(\epsilon^2-1)\rho^2\right]/[(\epsilon^2-1)\sqrt{\epsilon^6+5\epsilon^4+5\epsilon^2+1}]$
9	$2\sqrt{2}\rho^3\sin(3\theta)/\sqrt{\epsilon^6+\epsilon^4+\epsilon^2+1}$
10	$2\sqrt{2}\rho^3\cos(3\theta)/\sqrt{\epsilon^6+\epsilon^4+\epsilon^2+1}$
11	$\sqrt{5}\left[6\rho^4-6(\epsilon^2+1)\rho^2+(\epsilon^4+4\epsilon^2+1)\right]/(\epsilon^4-2\epsilon^2+1)$
12	4 $\sqrt{10}\rho^2\cos\frac{(2\theta)[\text{¿}(\epsilon^4+\epsilon^2+1)\rho^2-3(\epsilon^6+\epsilon^4+\epsilon^2+1)]}{[(\epsilon^2-1)\sqrt{\epsilon^{12}+5\epsilon^{10}+15\epsilon^8+18\epsilon^6+15\epsilon^4+5\epsilon^2+1}]}$ ¿
13	$\frac{\sqrt{10}\rho^2\sin(2\theta)\left[4(\epsilon^4+\epsilon^2+1)\rho^2-3(\epsilon^6+\epsilon^4+\epsilon^2+1)\right]}{[(\epsilon^2-1)\sqrt{\epsilon^{12}+5\epsilon^{10}+15\epsilon^8+18\epsilon^6+15\epsilon^4+5\epsilon^2+1}]}$
14	$\sqrt{10}\rho^4\cos(4\theta)/\sqrt{\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$
15	$\sqrt{10}\rho^4\sin(4\theta)/\sqrt{\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$
16	$2\sqrt{3}\rho\cos(\theta)\left[3(\epsilon^8+4\epsilon^6+10\epsilon^4+4\epsilon^2+1)+10(\epsilon^4+4\epsilon^2+1)\rho^4-12(\epsilon^6+4\epsilon^4+4\epsilon^2+1)\rho^2\right]$ $\sqrt{\epsilon^{10}+13\epsilon^8+46\epsilon^6+46\epsilon^4+13\epsilon^2+1}/(\epsilon^4-2\epsilon^2+1)$
17	$2\sqrt{3}\rho\sin(\theta)\left[3(\epsilon^8+4\epsilon^6+10\epsilon^4+4\epsilon^2+1)+10(\epsilon^4+4\epsilon^2+1)\rho^4-12(\epsilon^6+4\epsilon^4+4\epsilon^2+1)\rho^2\right]$ $\sqrt{\epsilon^{10}+13\epsilon^8+46\epsilon^6+46\epsilon^4+13\epsilon^2+1}/(\epsilon^4-2\epsilon^2+1)$
18	ϵ $(\text{¿¿}8+\epsilon^6+\epsilon^4+\epsilon^2+1)$ $5\rho^2(\epsilon^6+\epsilon^4+\epsilon^2+1)-4\text{¿}$ ¿ $2\sqrt{3}\rho^3\cos(3\theta)\text{¿}$
19	$\epsilon^6+\epsilon^4+\epsilon^2+1$ $2\sqrt{3}\rho^3\sin(3\theta)\left[(5\rho^2-4)+4(\epsilon^{10}-\epsilon^8)/(1-\epsilon^8)\right]/\text{¿}$)

20	$2\sqrt{3}\rho^5\cos(5\theta)/\sqrt{\epsilon^{10}+\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$
21	$2\sqrt{3}\rho^5\sin(5\theta)/\sqrt{\epsilon^{10}+\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$

Table (2): Zernike Annular Polynomials A(x , $y;\epsilon$¿ in Cartesian coordinates

No.	A
1	1
2	$2x/\sqrt{\epsilon^2+1}$
3	$2y/\sqrt{\epsilon^2+1}$
4	$\sqrt{3}(2x^2+2y^2-1-\epsilon^2)/(\epsilon^2-1)$
5	$2\sqrt{6}xy/\sqrt{\epsilon^4+\epsilon^2+1}$
6	$\sqrt{6}(x^2-y^2)/\sqrt{\epsilon^4+\epsilon^2+1}$
7	ϵ^2+1 ¿ $\epsilon^4+\epsilon^2+1$ $(x^2y+y^3)-2($¿$)y$ 3¿ $2\sqrt{2}$¿
8	ϵ^2+1 ¿ $\epsilon^4+\epsilon^2+1$ $(x^3+xy^2)-2($¿$)x$ 3¿ $2\sqrt{2}$¿
9	$2\sqrt{2}(3yx^2-y^3)/\sqrt{\epsilon^6+\epsilon^4+\epsilon^2+1}$
10	$2\sqrt{2}(x^3-3xy^2)/\sqrt{\epsilon^6+\epsilon^4+\epsilon^2+1}$

11	$x^4+2x^2y^2+y^4$ ¿ $6\left(\epsilon^4+4\epsilon^2+1-6\left(\epsilon^2+1\right)\left(x^2+y^2\right)+(\text{¿})\right]$ ¿ $\epsilon^4-2\epsilon^2+1$) ¿ $\sqrt{5}$¿ ¿
12	$\epsilon^6+\epsilon^4+\epsilon^2+1$ ¿ $4\left(\epsilon^4+\epsilon^2+1\right)\left(x^4-y^4\right)-3\left(x^2-y^2(\text{¿})\right]$ ¿ $\sqrt{10}$¿ ¿
13	ϵ $(\text{¿¿}6+\epsilon^4+\epsilon^2+1)xy$ $4(\epsilon^4+\epsilon^2+1)(yx^3+xy^3)-3$¿ ¿ $2\sqrt{10}$¿ ¿
14	$\sqrt{10}(x^4-6x^2y^2+y^4)/\sqrt{\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$
15	$4\sqrt{10}(yx^3-xy^3)/\sqrt{\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$
16	$2\sqrt{3}\left[3\left(\epsilon^8+4\epsilon^6+10\epsilon^4+4\epsilon^2+1\right)x+10\left(\epsilon^4+4\epsilon^2+1\right)\left(x^5+2x^3y^2+xy^4\right)-12\left(\epsilon^6+4\epsilon^4+4\epsilon^2+1\right)\left(x^3+\right.\right.$
17	$2\sqrt{3}\left[3\left(\epsilon^8+4\epsilon^6+10\epsilon^4+4\epsilon^2+1\right)y+10\left(\epsilon^4+4\epsilon^2+1\right)\left(yx^4+2x^2y^3+y^5\right)-12\left(\epsilon^6+4\epsilon^4+4\epsilon^2+1\right)\left(yx\right.\right.$
18	$2\sqrt{3}\left[5\left(\epsilon^6+\epsilon^4+\epsilon^2+1\right)\left(x^5-2x^3y^2-3y^4x\right)-4\left(\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1\right)\left(x^3-3xy^2\right)\right]$ $\left(\epsilon^{18}+5\epsilon^{16}+15\epsilon^{14}+35\epsilon^{12}+44\epsilon^{10}+44\epsilon^8+35\epsilon^6+15\epsilon^4+5\epsilon^2+1\right)^{\frac{1}{2}}/(\epsilon^2-1)$
19	$\dfrac{2\sqrt{3}\left[5\left(\epsilon^6+\epsilon^4+\epsilon^2+1\right)\left(3yx^4+2x^2y^3-y^5\right)-4\left(\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1\right)\left(3yx^2-y^3\right)\right]}{\left(\epsilon^6+\epsilon^4+\epsilon^2+1\right)}$
20	$2\sqrt{3}(x^5-10x^3y^2+5xy^4)/\sqrt{\epsilon^{10}+\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$
21	$2\sqrt{3}(5yx^4-10x^2y^3+y^5)/\sqrt{\epsilon^{10}+\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$

Table (3): Annular Zernike Polynomials A(x , $y;\epsilon$¿ as function of circular ZP, Z(x , y¿

No.	A
1	Z_1
2	Z_2 ¿$\sqrt{\epsilon^2+1}$
3	Z_3 ¿$\sqrt{\epsilon^2+1}$
4	$(Z_4-\sqrt{3}Z_1\epsilon^2)/(\epsilon^2-1)$
5	$Z_5/\sqrt{\epsilon^4+\epsilon^2+1}$
6	$Z_6/\sqrt{\epsilon^4+\epsilon^2+1}$
7	$\left[Z_7(\epsilon^2+1)-2\sqrt{2}Z_3\epsilon^4\right]/\left[(\epsilon^2-1)\sqrt{\epsilon^6+5\epsilon^4+5\epsilon^2+1}\right]$
8	$\left[Z_8(\epsilon^2+1)-2\sqrt{2}Z_2\epsilon^4\right]/\left[(\epsilon^2-1)\sqrt{\epsilon^6+5\epsilon^4+5\epsilon^2+1}\right]$
9	$Z_9/\sqrt{\epsilon^6+\epsilon^4+\epsilon^2+1}$
10	$Z_{10}/\sqrt{\epsilon^6+\epsilon^4+\epsilon^2+1}$
11	$\left[Z_{11}-\sqrt{15}Z_4\epsilon^2+\sqrt{5}Z_1(\epsilon^4-\epsilon^2)\right]/\sqrt{\epsilon^4-2\epsilon^2+1}$
12	$\left[Z_{12}(\epsilon^4+\epsilon^2+1)-\sqrt{15}Z_6\epsilon^6\right]/\left[(\epsilon^2-1)\sqrt{\epsilon^{12}+5\epsilon^{10}+15\epsilon^8+18\epsilon^6+15\epsilon^4+5\epsilon^2+1}\right]$
13	$\left[Z_{13}(\epsilon^4+\epsilon^2+1)-\sqrt{15}Z_5\epsilon^6\right]/\left[(\epsilon^2-1)\sqrt{\epsilon^{12}+5\epsilon^{10}+15\epsilon^8+18\epsilon^6+15\epsilon^4+5\epsilon^2+1}\right]$
14	Z_{14} ¿$\sqrt{\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$
15	Z_{15} $\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1$ ¿√¿¿
16	$\left[Z_{16}(\epsilon^4+4\epsilon^2+1)-2\sqrt{6}\epsilon^4Z_8(\epsilon^2-3)+\sqrt{3}\epsilon^4Z_2(3\epsilon^4+4\epsilon^2+3)\right]\sqrt{\epsilon^{10}+13\epsilon^8+46\epsilon^6+46\epsilon^4+13\epsilon}$
17	$\left[Z_{17}(\epsilon^4+4\epsilon^2+1)-2\sqrt{6}\epsilon^4Z_7(\epsilon^2-3)+\sqrt{3}\epsilon^4Z_3(3\epsilon^4+4\epsilon^2+3)\right]\sqrt{\epsilon^{10}+13\epsilon^8+46\epsilon^6+46\epsilon^4+13\epsilon}$
18	$\left[Z_{18}(\epsilon^6+\epsilon^4+\epsilon^2+1)-2\sqrt{6}Z_{10}\epsilon^8\right](\epsilon^{18}+5\epsilon^{16}+15\epsilon^{14}+35\epsilon^{12}+44\epsilon^{10}+44\epsilon^8+35\epsilon^6+15\epsilon^4+5\epsilon^2$

19	$\epsilon^6+\epsilon^4+\epsilon^2+1$ $\left[Z_{19}(\epsilon^6+\epsilon^4+\epsilon^2+1)-2\sqrt{6}Z_9\epsilon^8\right]/¿$)
20	Z_{20} $¿\sqrt{\epsilon^{10}+\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$
21	Z_{21} $¿\sqrt{\epsilon^{10}+\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$

Table (4): Gradient of Zernike Annular Polynomials ∇ A(x , $y;\epsilon¿$ in Cartesian coordinate

No.	∇ A(x , $y;\epsilon¿$
1	0
2	$2\hat{i}/\sqrt{\epsilon^2+1}$
3	$2\hat{j}/\sqrt{\epsilon^2+1}$
4	$4\sqrt{3}(x\hat{i}+y\hat{j})/(\epsilon^2-1)$
5	$2\sqrt{6}(y\hat{i}+x\hat{j})/\sqrt{\epsilon^4+\epsilon^2+1}$
6	$2\sqrt{6}(x\hat{i}-y\hat{j})/\sqrt{\epsilon^4+\epsilon^2+1}$
7	ϵ^2+1 ¿ $6(\epsilon^2+1)xy\hat{i}+\left(3(\epsilon^4+\epsilon^2+1(x^2+3y^2)-2(¿))\right)\hat{j}\Big]$ ¿ $2\sqrt{2}¿$ ¿
8	ϵ^2+1 ¿ $\left(3(\epsilon^4+\epsilon^2+1(3x^2+y^2)-2(¿))\right)\hat{i}+6(\epsilon^2+1)xy\hat{j}\Big]$ ¿ $2\sqrt{2}¿$ ¿
9	$6\sqrt{2}[2xy\hat{i}+(x^2-y^2)\hat{j}]/\sqrt{\epsilon^6+\epsilon^4+\epsilon^2+1}$

10 $6\sqrt{2}[(x^2-y^2)\hat{i}-2xy\,\hat{j}]/\sqrt{\epsilon^6+\epsilon^4+\epsilon^2+1}$

$2(x^3+x\,y^2)-(\epsilon^2+1)\,x$
¿
y
11 $((¿¿3+y\,x^2)-(\epsilon^2+1)\,y\,\hat{i}+(2¿)\,\hat{j}]/(\epsilon^4-2\epsilon^2+1)$
$12\sqrt{5}¿$

$\epsilon^4+4\epsilon^2+1$
¿
$\epsilon^4+4\epsilon^2+1$
¿
$\epsilon^6+\epsilon^4+\epsilon^2+1$
12 $y^3-3(¿)\,y$
$8¿$
$8(\epsilon^6+\epsilon^4+\epsilon^2+1x^3-3(¿)x)\hat{i}-¿$
¿
$2\sqrt{10}¿$

$¿[(1-\epsilon^2)\sqrt{\epsilon^{12}+5\epsilon^{10}+15\epsilon^8+18\epsilon^6+15\epsilon^4+5\epsilon^2+1}]$

ϵ
¿
$(¿6+\epsilon^4+\epsilon^2+1¿)\,y$
13 $4(\epsilon^4+\epsilon^2+1)(3\,yx^2+y^3)-3¿$
¿
¿
$2\sqrt{10}¿$

ϵ
¿
$(¿6+\epsilon^4+\epsilon^2+1¿)\,x$
$4(\epsilon^4+\epsilon^2+1)(x^3+3\,xy^2)-3¿\,\hat{j}$
$+¿$

$¿[(1-\epsilon^2)\sqrt{\epsilon^{12}+5\epsilon^{10}+15\epsilon^8+18\epsilon^6+15\epsilon^4+5\epsilon^2+1}]$

y
¿
$(¿3-3\,y\,x^2¿)\,\hat{j}$
14 $(x^3-3\,x\,y^2)\,\hat{i}+¿/\sqrt{\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$
$4\sqrt{10}¿$

x
¿
$(¿3-3x\,y^2¿)\,\hat{j}$
15 $(3\,y\,x^2-y^3)\,\hat{i}+¿/\sqrt{\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$
$4\sqrt{10}¿$

16	$2\sqrt{3}\left[\left(3\left(\epsilon^8+4\epsilon^6+10\epsilon^4+4\epsilon^2+1\right)+10\left(\epsilon^4+4\epsilon^2+1\right)\left(5x^4+6x^2y^2+y^4\right)-12\left(\epsilon^6+4\epsilon^4+4\epsilon^2+1\right)\left(3\right.\right.\right.$
17	$2\sqrt{3}\left[\left(40\left(\epsilon^4+4\epsilon^2+1\right)\left(yx^3+xy^3\right)-24\left(\epsilon^6+4\epsilon^4+4\epsilon^2+1\right)xy\right)\hat{i}+\left(3\left(\epsilon^8+4\epsilon^6+10\epsilon^4+4\epsilon^2+1\right)+10\right.\right.$
18	$2\sqrt{3}\left[\left(5\left(\epsilon^6+\epsilon^4+\epsilon^2+1\right)\left(5x^4-6x^2y^2-3y^4\right)-12\left(\epsilon^4+\epsilon^2+1\right)\left(x^2-y^2\right)\right)\hat{i}-\left(20\left(\epsilon^6+\epsilon^4+\epsilon^2+1\right)\left(yx^3\right.\right.\right.$
19	$\epsilon^6+\epsilon^4+\epsilon^2+1$ $2\sqrt{3}\left[\left(20\left(\epsilon^6+\epsilon^4+\epsilon^2+1\right)\left(3yx^3+xy^3\right)-24\left(\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1\right)xy\right)\hat{i}+\left(5\left(\epsilon^6+\epsilon^4+\epsilon^2+1\right)\left(3x^4+6x\right.\right.\right.$)
20	x ¿ $(¿4-6x^2y^2+y^4¿)\hat{i}-4\left(yx^3-xy^3\right)\hat{j}$ ¿ $10\sqrt{3}$¿ $¿\sqrt{\epsilon^{10}+\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$
21	x ¿ $4\left(yx^3-xy^3\right)\hat{i}+(¿4-6x^2y^2+y^4¿)\hat{j}$ ¿ $10\sqrt{3}$¿ $¿\sqrt{\epsilon^{10}+\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$

Table (5): Orthonormal Vector of Annular ZP $(\acute{S})$ in Cartesian coordinates.

No.	$\acute{S}$
1	0
2	$\hat{i}$
3	$\hat{j}$
4	$\sqrt{2}(x\hat{i}+y\hat{j})/\sqrt{\epsilon^2+1}$
5	$\sqrt{2}(y\hat{i}+x\hat{j})/\sqrt{\epsilon^2+1}$
6	$\sqrt{2}(x\hat{i}-y\hat{j})/\sqrt{\epsilon^2+1}$
7	$\sqrt{3}\left[2xy\hat{i}+\left(\left(x^2+3y^2\right)-\left(\epsilon^2+1\right)\right)\hat{j}\right]/\sqrt{2\epsilon^4-\epsilon^2+2}$

8	$\sqrt{3}\left[\left(\left(3x^2+y^2\right)-\left(\epsilon^2+1\right)\right)\hat{i}+2xy\,\hat{j}\right]/\sqrt{2\epsilon^4-\epsilon^2+2}$
9	$\sqrt{3}\left[2xy\,\hat{i}+\left(x^2-y^2\right)\hat{j}\right]/\sqrt{\epsilon^4+\epsilon^2+1}$
10	$\sqrt{3}\left[\left(x^2-y^2\right)\hat{i}-2xy\,\hat{j}\right]/\sqrt{\epsilon^4+\epsilon^2+1}$
11	ϵ (¿¿$4+\epsilon^2+1)x$ $3\left(\epsilon^2+1\right)\left(x^3+xy^2\right)-2$¿ ¿ ϵ (¿¿ $4+\epsilon^2+1)y$ $3\left(\epsilon^2+1\right)\left(yx^2+y^3\right)-2$¿ $\hat{i}+($¿$\hat{j}]$ ¿ 2¿ ¿
12	ϵ (¿¿ $4+\epsilon^2+1)x$ $2x^3\left(\epsilon^2+1\right)-$¿ ¿ ϵ (¿¿$4+\epsilon^2+1)y-2y^3\left(\epsilon^2+1\right)$ ¿ $\hat{i}+($¿$\hat{j}]$ ¿ $2\sqrt{2}$¿ ¿
13	ϵ (¿¿$4+\epsilon^2+1)y$ $\left(\epsilon^2+1\right)\left(3yx^2+y^3\right)-$¿ ¿ ϵ (¿¿ $4+\epsilon^2+1)x$ $\left(\epsilon^2+1\right)\left(3xy^2+x^3\right)-$¿ $\hat{i}+($¿$\hat{j}]$ ¿ $2\sqrt{2}$¿ ¿
14	$2\left[\left(x^3-3xy^2\right)\hat{i}-\left(3yx^2-y^3\right)\hat{j}\right]/\sqrt{\epsilon^6+\epsilon^4+\epsilon^2+1}$
15	$2\left[\left(3yx^2-y^3\right)\hat{i}+\left(x^3-3xy^2\right)\hat{j}\right]/\sqrt{\epsilon^6+\epsilon^4+\epsilon^2+1}$
16	$2\sqrt{5}\left[\left(\left(\epsilon^8+2\epsilon^6-3\epsilon^4+2\epsilon^2+1\right)+(2\epsilon^4-\epsilon^2+2)(5x^4+6y^2x^2+y^4)-3(\epsilon^6+1)\left(3x^2+y^2\right)\right)\hat{i}+(4(2\epsilon\right.$

17	$2\sqrt{5}[¿\,\frac{4(2\epsilon^4-\epsilon^2+2)(yx^3+xy^3)-6(\epsilon^6+1)yx}{}\,\hat{i}+\left(\left(\epsilon^8+2\epsilon^6-3\epsilon^4+2\epsilon^2+1\right)+(2\epsilon^4-\epsilon^2+2)(x^4+6y^2x^2+5y^4)-3(\epsilon^6+1)\left(x^2+3y^2\right)\right)\hat{j}]/M$
18	$\sqrt{5}\left[\left(\left(\epsilon^4+\epsilon^2+1\right)\left(5x^4-6x^2y^2-3y^4\right)-3\left(\epsilon^6+\epsilon^4+\epsilon^2+1\right)\left(x^2-y^2\right)\right)\hat{i}+\left(6\left(\epsilon^6+\epsilon^4+\epsilon^2+1\right)xy-4\left(\epsilon^4+\right.\right.\right.$
19	$\sqrt{5}\left[\left(4\left(\epsilon^4+\epsilon^2+1\right)\left(3yx^3+xy^3\right)-6\left(\epsilon^6+\epsilon^4+\epsilon^2+1\right)xy\right)\hat{i}+\left(\left(\epsilon^4+\epsilon^2+1\right)\left(3x^4+6x^2y^2-5y^4\right)-3\left(\epsilon^6+\right.\right.\right.$
20	$\sqrt{5}\left[\left(x^4-6x^2y^2+y^4\right)\hat{i}-4\left(yx^3-xy^3\right)\hat{j}\right]/\sqrt{\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$
21	$\sqrt{5}\left[4\left(yx^3-xy^3\right)\hat{i}+\left(x^4-6x^2y^2+y^4\right)\hat{j}\right]/\sqrt{\epsilon^8+\epsilon^6+\epsilon^4+\epsilon^2+1}$

Table (6): Orthonormal vector annular ZP $(\acute{S})$ as a function of circular ZP Z(x , y¿

No.	$\acute{S}$
1	0
2	$Z_1\hat{i}$
3	$Z_1\hat{j}$
4	$(Z_2\hat{i}+Z_3\hat{j})/\sqrt{2\epsilon^2+2}$
5	$(Z_3\hat{i}+Z_2\hat{j})/\sqrt{2\epsilon^2+2}$
6	$(Z_2\hat{i}-Z_3\hat{j})/\sqrt{2\epsilon^2+2}$
7	$\left[2\sqrt{3}Z_5\hat{i}+2\left(-\sqrt{3}Z_6+\sqrt{6}Z_4-3\sqrt{2}\epsilon^2Z_1\right)\hat{j}\right]/\sqrt{48\epsilon^4-24\epsilon^2+48}$
8	$\left[2\left(-\sqrt{3}Z_6+\sqrt{6}Z_4-3\sqrt{2}\epsilon^2Z_1\right)\hat{i}+2\sqrt{3}Z_5\hat{j}\right]/\sqrt{48\epsilon^4-24\epsilon^2+48}$
9	$\left[Z_5\hat{i}+Z_6\hat{j}\right]/\sqrt{2\epsilon^4+2\epsilon^2+2}$

10	$\left[Z_6\hat{i}-Z_5\hat{j}\right]/\sqrt{2\epsilon^4+2\epsilon^2+2}$
11	Z Z $(¿¿7(\epsilon^2+1)-2\sqrt{2}\epsilon^4Z_3)\hat{j}$ $(¿¿8(\epsilon^2+1)-2\sqrt{2}\epsilon^4Z_2)\hat{i}+¿$ ¿ ¿ ¿ ¿
12	$Z_{10}(\epsilon^2+1)$ $Z_9(\epsilon^2+1)$ $(¿¿2-Z_7(\epsilon^2+1)/2+\sqrt{2}Z_3\epsilon^4)\hat{j}$ $(¿¿2+Z_8(\epsilon^2+1)/2-\sqrt{2}Z_2\epsilon^4)\hat{i}+¿$ ¿ ¿ ¿ ¿
13	$Z_9(\epsilon^2+1)$ $-Z_{10}(\epsilon^2+1)$ $(¿¿2+Z_8(\epsilon^2+1)/2-\sqrt{2}Z_2\epsilon^4)\hat{j}$ $(¿¿2+Z_7(\epsilon^2+1)/2-\sqrt{2}Z_3\epsilon^4)\hat{i}+¿$ ¿ ¿ ¿ ¿
14	$\left[Z_{10}\hat{i}-Z_9\hat{j}\right]/\sqrt{2\epsilon^6+2\epsilon^4+2\epsilon^2+2}$
15	$\left[Z_9\hat{i}+Z_{10}\hat{j}\right]/\sqrt{2\epsilon^6+2\epsilon^4+2\epsilon^2+2}$
16	$\left[\left(Z_{12}M_{13}^2+\sqrt{2}Z_{11}M_{13}^2-\sqrt{15}Z_6(2\epsilon^6-2\epsilon^4+\epsilon^2)-\sqrt{30}Z_4(2\epsilon^6-2\epsilon^4+\epsilon^2)+2\sqrt{10}Z_1(\epsilon^8-\epsilon^6-\epsilon^4\right.\right.$
17	$\left[\left(Z_{13}M_{13}^2-\sqrt{15}Z_5(2\epsilon^6-2\epsilon^4+\epsilon^2)\right)\hat{i}+\left(-Z_{12}M_{13}^2+\sqrt{2}Z_{11}M_{13}^2+\sqrt{15}Z_6(2\epsilon^6-2\epsilon^4+\epsilon^2)-\sqrt{30}Z_4\right.\right.$

18	ϵ ¿ $Z_{14}(¿4+\epsilon^2+1¿)+Z_{12}(\epsilon^4+\epsilon^2+1)-\sqrt{15}\epsilon^6 Z_6$ ¿ ϵ ¿ $Z_{15}(¿4+\epsilon^2+1¿)-Z_{13}(\epsilon^4+\epsilon^2+1)+\sqrt{15}\epsilon^6 Z_5$ $(¿\hat{j})¿\sqrt{2}M_4$ ¿ ¿ ¿
19	ϵ ¿ $Z_{15}(¿4+\epsilon^2+1¿)+Z_{13}(\epsilon^4+\epsilon^2+1)-\sqrt{15}\epsilon^6 Z_5$ ¿ ϵ ¿ $-Z_{14}(¿4+\epsilon^2+1¿)+Z_{12}(\epsilon^4+\epsilon^2+1)-\sqrt{15}\epsilon^6 Z_6$ $(¿\hat{j})¿\sqrt{2}M_4$ ¿ ¿ ¿
20	$[Z_{14}\hat{i}-Z_{15}\hat{j}]/\sqrt{2\epsilon^8+2\epsilon^6+2\epsilon^4+2\epsilon^2+2}$
21	$[Z_{15}\hat{i}+Z_{14}\hat{j}]/\sqrt{2\epsilon^8+2\epsilon^6+2\epsilon^4+2\epsilon^2+2}$

Table (7): Orthonormal vector annular ZP $(\acute{S})$ as a function of gradient of circular ZP ∇ Z(x , y¿

No.	$\acute{S}$
1	0
2	$\nabla Z_2/\sqrt{4}$
3	$\nabla Z_3/\sqrt{4}$
4	$\nabla Z_4/\sqrt{24\epsilon^2+24}$

5	$\nabla Z_5/\sqrt{12\epsilon^2+12}$
6	$\nabla Z_6/\sqrt{12\epsilon^2+12}$
7	$\left[\nabla Z_7-\sqrt{2}(3\epsilon^2+1)\nabla Z_3\right]/\sqrt{48\epsilon^4-24\epsilon^2+48}$
8	$\left[\nabla Z_8-\sqrt{2}(3\epsilon^2+1)\nabla Z_2\right]/\sqrt{48\epsilon^4-24\epsilon^2+48}$
9	$\left[Z_5\hat{i}+Z_6\hat{j}\right]/\sqrt{2\epsilon^4+2\epsilon^2+2}$
10	$\left[Z_6\hat{i}-Z_5\hat{j}\right]/\sqrt{2\epsilon^4+2\epsilon^2+2}$
11	$\dfrac{\left[\nabla Z_{11}(\epsilon^2+1)-\sqrt{\frac{5}{3}}(4\epsilon^4+\epsilon^2+1)\nabla Z_4\right]}{\sqrt{80\epsilon^{10}+240\epsilon^8+240\epsilon^2+80}}$
12	$\dfrac{\left[\nabla Z_{12}(\epsilon^2+1)-\sqrt{\frac{5}{3}}(4\epsilon^4+\epsilon^2+1)\nabla Z_6\right]}{\sqrt{80\epsilon^{10}+240\epsilon^8+240\epsilon^2+80}}$
13	$\dfrac{\left[\nabla Z_{13}(\epsilon^2+1)-\sqrt{\frac{5}{3}}(4\epsilon^4+\epsilon^2+1)\nabla Z_5\right]}{\sqrt{80\epsilon^{10}+240\epsilon^8+240\epsilon^2+80}}$
14	$\left[Z_{10}\hat{i}-Z_9\hat{j}\right]/\sqrt{2\epsilon^6+2\epsilon^4+2\epsilon^2+2}$
15	$\left[Z_9\hat{i}+Z_{10}\hat{j}\right]/\sqrt{2\epsilon^6+2\epsilon^4+2\epsilon^2+2}$
16	$\Big[\Big(Z_{12}M_{13}^2+\sqrt{2}Z_{11}M_{13}^2-\sqrt{15}Z_6(2\epsilon^6-2\epsilon^4+\epsilon^2)-\sqrt{30}Z_4(2\epsilon^6-2\epsilon^4+\epsilon^2)+2\sqrt{10}Z_1(\epsilon^8-\epsilon^6-\epsilon^4$
17	$\Big[\Big(Z_{13}M_{13}^2-\sqrt{15}Z_5(2\epsilon^6-2\epsilon^4+\epsilon^2)\Big)\hat{i}+\Big(-Z_{12}M_{13}^2+\sqrt{2}Z_{11}M_{13}^2+\sqrt{15}Z_6(2\epsilon^6-2\epsilon^4+\epsilon^2)-\sqrt{30}Z_4$
18	ϵ ¿ $Z_{14}($¿$4+\epsilon^2+1$¿$)+Z_{12}(\epsilon^4+\epsilon^2+1)-\sqrt{15}\epsilon^6Z_6$ ¿ ϵ ¿ $Z_{15}($¿$4+\epsilon^2+1$¿$)-Z_{13}(\epsilon^4+\epsilon^2+1)+\sqrt{15}\epsilon^6Z_5$ (¿$\hat{j}$) ¿ ¿ ¿ ¿$\sqrt{2}M_4$

19	ϵ ¿ $Z_{15}(¿4+\epsilon^2+1¿)+Z_{13}(\epsilon^4+\epsilon^2+1)-\sqrt{15}\epsilon^6 Z_5$ ¿ ϵ ¿ $-Z_{14}(¿4+\epsilon^2+1¿)+Z_{12}(\epsilon^4+\epsilon^2+1)-\sqrt{15}\epsilon^6 Z_6$ $(¿\hat{j})¿\sqrt{2}M_4$ ¿ ¿ ¿
20	$\left[Z_{14}\hat{i}-Z_{15}\hat{j}\right]/\sqrt{2\epsilon^8+2\epsilon^6+2\epsilon^4+2\epsilon^2+2}$
21	$\left[Z_{15}\hat{i}+Z_{14}\hat{j}\right]/\sqrt{2\epsilon^8+2\epsilon^6+2\epsilon^4+2\epsilon^2+2}$

Table (8): The normalization constants for orthogonal vector annular ZP.

j	C
1	0
2	$1/2$
3	$1/2$
4	$1/\sqrt{24\epsilon^2+24}$
5	$1/\sqrt{12\epsilon^2+12}$
6	$1/\sqrt{12\epsilon^2+12}$
7	$1/\sqrt{48\epsilon^4-24\epsilon^2+48}$
8	$1/\sqrt{48\epsilon^4-24\epsilon^2+48}$
9	$1/\sqrt{24\epsilon^4+24\epsilon^2+24}$
10	$1/\sqrt{24\epsilon^4+24\epsilon^2+24}$
11	$\frac{(\epsilon^4-1)/\sqrt{80}}{\sqrt{\epsilon^{10}+\epsilon^8-8\epsilon^6+8\epsilon^4-\epsilon^2-1}}$

12	$\frac{(\epsilon^4-1)/\sqrt{80}}{\sqrt{\epsilon^{10}+\epsilon^8-4\epsilon^6+4\epsilon^4-\epsilon^2-1}}$
13	$\frac{(\epsilon^4-1)/\sqrt{80}}{\sqrt{\epsilon^{10}+\epsilon^8-4\epsilon^6+4\epsilon^4-\epsilon^2-1}}$
14	$1/\sqrt{40\epsilon^6+40\epsilon^4+40\epsilon^2+40}$
15	1 ¿ $\sqrt{40\epsilon^6+40\epsilon^4+40\epsilon^2+40}$
16	$\frac{(2\epsilon^6-3\epsilon^4+3\epsilon^2-2)/\sqrt{240}}{\sqrt{\epsilon^{14}-3\epsilon^{12}+11\epsilon^{10}-25\epsilon^8+25\epsilon^6-11\epsilon^4+3\epsilon^2-1}}$
17	$\frac{(2\epsilon^6-3\epsilon^4+3\epsilon^2-2)/\sqrt{240}}{\sqrt{\epsilon^{14}-3\epsilon^{12}+11\epsilon^{10}-25\epsilon^8+25\epsilon^6-11\epsilon^4+3\epsilon^2-1}}$
18	$\frac{(\epsilon^6+2\epsilon^4+2\epsilon^2+1)/\sqrt{60}}{\sqrt{2\epsilon^{14}+2\epsilon^{12}+2\epsilon^{10}-15\epsilon^8+15\epsilon^6-2\epsilon^4-2\epsilon^2-2}}$
19	$\frac{(\epsilon^6+2\epsilon^4+2\epsilon^2+1)/\sqrt{60}}{\sqrt{2\epsilon^{14}+2\epsilon^{12}+2\epsilon^{10}-15\epsilon^8+15\epsilon^6-2\epsilon^4-2\epsilon^2-2}}$
20	1/ $\sqrt{60\epsilon^8+60\epsilon^6+60\epsilon^4+60\epsilon^2+60}$
21	1/ $\sqrt{60\epsilon^8+60\epsilon^6+60\epsilon^4+60\epsilon^2+60}$

Appendix I

Table (I) Zernike Circular Polynomials Z(x , y¿ in Cartesian coordinate

No.	n	m	Z(x , y¿
1	0	0	1
2	1	1	$2x$
3	1	-1	$2y$
4	2	0	$\sqrt{3}(2x^2+2y^2-1)$
5	2	-2	$2\sqrt{6}xy$
6	2	2	$\sqrt{6}(x^2-y^2)$
7	3	-1	$2\sqrt{2}\left(3(yx^2+y^3)-2y\right)$
8	3	1	$2\sqrt{2}\left(3(xy^2+x^3)-2x\right)$
9	3	-3	$2\sqrt{2}(3yx^2-y^3)$
10	3	3	$2\sqrt{2}(x^3-3xy^2)$
11	4	0	x (¿¿$4+2x^2y^2+y^4)-6\left(x^2+y^2\right)+1$ 6¿ $\sqrt{5}$¿
12	4	2	$\sqrt{10}\left[4\left(x^4-y^4\right)-3\left(x^2-y^2\right)\right]$
13	4	-2	$2\sqrt{10}[4(yx^3+xy^3)-3xy]$
14	4	4	$\sqrt{10}(x^4-6x^2y^2+y^4)$
15	4	-4	$4\sqrt{10}(yx^3-xy^3)$
16	5	1	$2\sqrt{3}\left[3x+10\left(x^5+2x^3y^2+xy^4\right)-12\left(x^3+xy^2\right)\right]$
17	5	-1	$2\sqrt{3}\left[3y+10\left(yx^4+2x^2y^3+y^5\right)-12\left(yx^2+y^3\right)\right]$
18	5	3	$2\sqrt{3}\left[5\left(x^5-2x^3y^2-3y^4x\right)-4\left(x^3-3xy^2\right)\right]$
19	5	-3	$2\sqrt{3}\left[5\left(3yx^4+2x^2y^3-y^5\right)-4\left(3yx^2-y^3\right)\right]$
20	5	5	$2\sqrt{3}(x^5-10x^3y^2+5xy^4)$

21	5 -5	$2\sqrt{3}\left(5yx^4-10x^2y^3+y^5\right)$

Appendix II

Table (II) Gradient of Circular Zernike Polynomials $\nabla Z(x,y)$ In Cartesian Coordinate

No.	$\nabla Z(x,y)$
1	0
2	$2\hat{i}$
3	$2\hat{j}$
4	$4\sqrt{3}\left(x\hat{i}+y\hat{j}\right)$
5	$2\sqrt{6}\left(y\hat{i}+x\hat{j}\right)$
6	$2\sqrt{6}\left(x\hat{i}-y\hat{j}\right)$
7	$2\sqrt{2}\left[6xy\hat{i}+\left(3\left(x^2+3y^2\right)-2\right)\hat{j}\right]$
8	$2\sqrt{2}\left[\left(3\left(x^2+3y^2\right)-2\right)\hat{i}+6xy\hat{j}\right]$
9	$6\sqrt{2}\left[2xy\hat{i}+\left(x^2-y^2\right)\hat{j}\right]$
10	$6\sqrt{2}\left[\left(x^2-y^2\right)\hat{i}-2xy\hat{j}\right]$

11

$2(x^3+x\,y^2)-x$
¿
y
$((¿¿3+y\,x^2)-y\,\hat{i}+(2¿)\hat{j}]$
$12\sqrt{5}¿$

12 $2\sqrt{10}[(8\,x^3-3\,x)\hat{i}-(8\,y^3-3\,y)\hat{j}]$

13

$4(y^3+3\,yx^2)-3\,y$
$(¿)\hat{i}+(4(x^3+3\,xy^2)-3\,x)\hat{j}$
$2\sqrt{10}¿$

14

y
¿
$(¿3-3\,y\,x^2¿)\hat{j}$
$(x^3-3\,x\,y^2)\hat{i}+¿$
$4\sqrt{10}¿$

15

x
¿
$(¿3-3\,x\,y^2¿)\hat{j}$
$(3\,y\,x^2-y^3)\hat{i}+¿$
$4\sqrt{10}¿$

16

$$2\sqrt{3}[(3+10(5\,x^4+6\,x^2\,y^2+y^4)-12(3\,x^2+y^2))\hat{i}+(40(y\,x^3+x\,y^3)-24\,xy)\hat{j}]$$

17

$$2\sqrt{3}[(40(yx^3+x\,y^3)-24\,xy)\hat{i}+(3+10(x^4+6\,x^2\,y^2+5\,y^4)-12(x^2+3\,y^2))\hat{j}]$$

18

$$2\sqrt{3}[(5(5\,x^4-6\,x^2\,y^2-3\,y^4)-12(x^2-y^2))\hat{i}-(20(yx^3+3\,xy^3)-24\,xy)\hat{j}]$$

19

$$2\sqrt{3}[(20(3\,y\,x^3+x\,y^3)-24\,xy)\hat{i}+(5(3\,x^4+6\,x^2\,y^2-5\,y^4)-12(x^2-y^2))\hat{j}]$$

20

x
¿
$(¿4-6\,x^2\,y^2+y^4¿)\hat{i}-4(y\,x^3-x\,y^3)\hat{j}$
¿
$10\sqrt{3}¿$

21	x ¿ $4\left(yx^3-xy^3\right)\hat{i}+(¿4-6x^2y^2+y^4¿)\hat{j}$ ¿ $10\sqrt{3}$¿

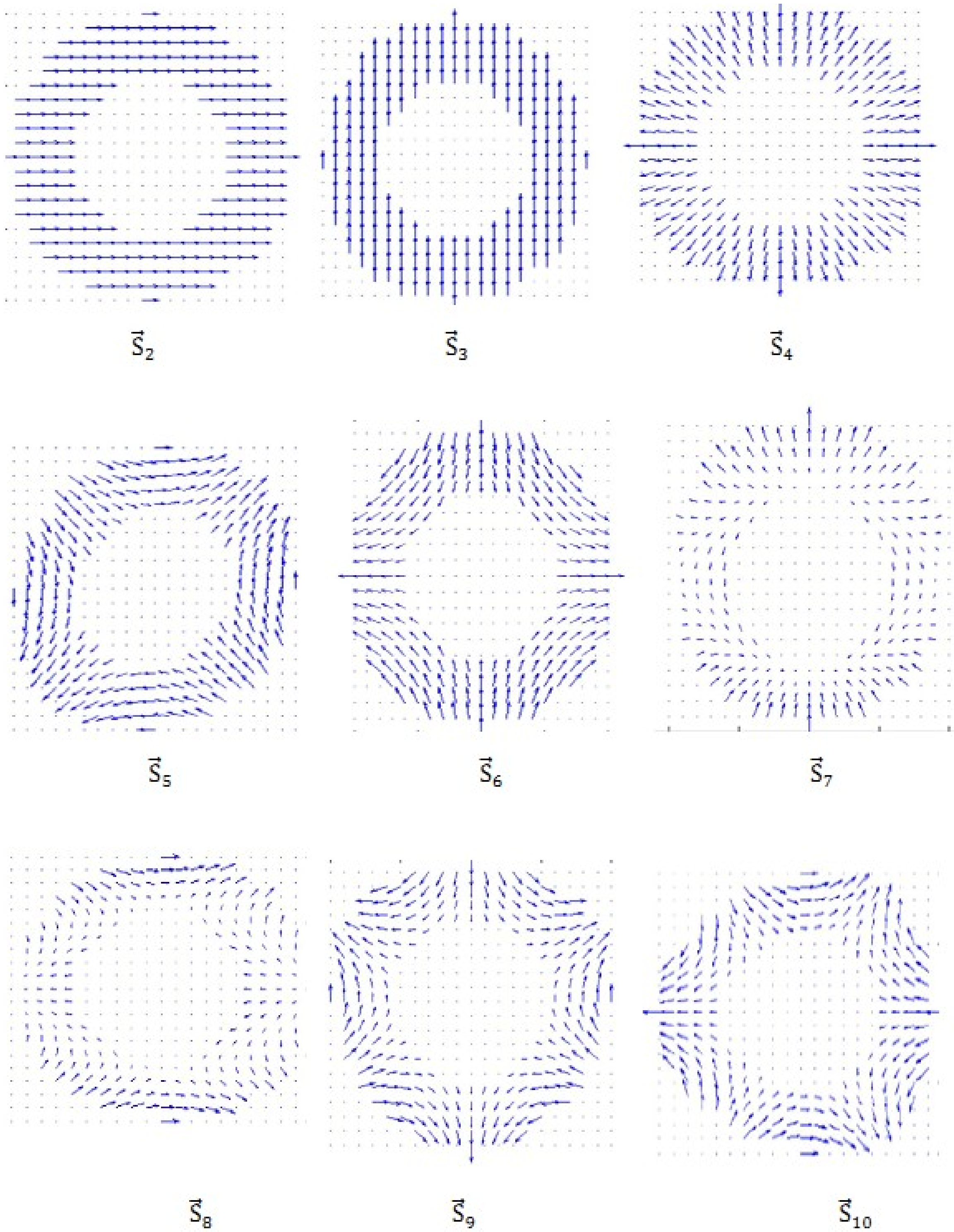

Fig.1: Plots of first (9) orthonormal annular vector Zernike polynomials

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متعددات حدود متـّجهة متعامدة ومعايرة على وحدة البؤبؤ الحلقي

سندس ياسين حسن الأسدي-قسم الفيزياء, كلية التربية للبنات, جامعة الكوفة, النجف
ظل عقيل كاظم الموسوي-قسم الفيزياء, كلية العلوم, جامعة المثنى, المثنى

الخلاصة:

مجموعة متعددات الحدود المتعامدة والمعايرة, والتي تمثّل دوالاً عددية, المشتقـّة في هذا البحث, تمثـّل أساسات متوازنة على كل مساحة البؤبؤ الحلقي, هذه المتعددات الحدود المتعامدة والمعايرة تـُستخدم في تمثيل زيوغ الأنظمة البصرية (الصورية), التي تتعامل مع البؤبؤ الحلقي.

هناك مجموعة أخرى من المتعددات الحدود المتعلّقة بالمجموعة الأولى, تمثّل كميات متـّجهة, والتي يمكن اشتقاقها من انحدارات متعددات زرنيك الحلقية, إذ اُستخدمت طريقة غرام-شمت التعامدية العيارية (GSOM) في توليدها. حـُسبت جميع هذه المتعددات باستخدام برنامج الماتلاب.

إن العلاقة بين متعددات الحدود المتـّجهة المتعامدة والمعايرة, ومتعددات زرنيك الدائرية (ZP), وكذا العلاقة بين هذه المتعددات وانحدارات متعددات زرنيك الدائرية (∇ ZP) معطاة في هذا البحث أيضاً.