Constructing a New Weighted Maxwell Probability Distribution with Comparing of its two Parameters

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Abstract

This paper discuss the Maxwell distribution with scale parameter (θ), and how to expand this family to find weighted Maxwell p.d.f which is necessary for sampling from population of different sizes, so we apply some transformation on $[f_T(t;\theta)]$ to obtain $[g_w(t)]$, then its C.D.F obtained. Also the formula for r^{th} moments about origin also derived to help in finding moment estimators of (β).

Keywords: Rayleigh distribution, Weighted Maxwell, Gamma Function, Cumulative distribution function, length biased moments, L – Moment, MLE

1. Introduction

In electromagnetism, one of the fundamental fields of physics, the introduction of Maxwell's equations (mainly in "A Dynamical Theory of the Electromagnetic Field") was one of the most important aggregations of empirical facts in the history of physics. It took place in the nineteenth century, starting from basic experimental observations, and leading to the formulations of numerous mathematical equations, notably by Charles-Augustin de Coulomb, Hans Christian Ørsted, Carl Friedrich Gauss, Jean-Baptiste Biot, Félix Savart, André-Marie Ampère, and Michael Faraday. The apparently disparate laws and phenomena of electricity and magnetism were integrated by James Clerk Maxwell, who published an early form of the equations, which modify Ampère's circuital law by introducing a displacement current term. He showed that these equations imply that light propagates as electromagnetic waves. His laws were reformulated by Oliver Heaviside in the more modern and compact vector calculus formalism he independently developed. Increasingly powerful mathematical descriptions of the electromagnetic field were developed, continuing into the twentieth century, enabling the equations to take on simpler forms by advancing more sophisticated mathematics.

If T_1 and T_2 are $i.i.d$ from$[N(0,\sigma^2)]$, then the radical distance $[T=\sqrt{T_1^2+T_2^2}]$ have Rayleigh distribution, but when $[T_1, T_2, T_3]$ are $i.i.d$ from $[N(0,\sigma^2)]$, then $[T=\sqrt{T_1^2+T_2^2+T_3^2}]$ is $r.v$ have Maxwell distribution p.d.f defined by;

$$f_T(t;\theta)=\frac{4}{\sqrt{\pi}\,\theta^3}\,t^2\,e^{-\left(\frac{t}{\theta}\right)^2} \qquad t,\theta>0$$

(1)

With mean and variance;

$$E(T)=\frac{2\,\theta}{\sqrt{\pi}}$$

(2)

$$v(T)=\frac{5\,\theta^2}{22}$$

(3)

Maxwell distribution represents a special case of general Gamma distribution given by;

$$f_T(t;\theta,p,\beta,\delta)=\frac{\beta}{\Gamma(\mathrm{p})\theta^{\beta p}}\,(t-\delta)^{\beta p-1}\,e^{-\left(\frac{t-\delta}{\theta}\right)^2} \qquad \theta,p,\beta,\delta>0 \qquad (4)$$

When $[p=\frac{3}{2},\beta=2,\delta=0]$, we have Maxwell.

Maxwell $p.d.f$ is a special case of general Gamma distribution which was introduced by [Diciccio J.J. (1987)].

When $[p = \frac{3}{2}, \beta = 2]$, this gives;

$$g(t) = \frac{4}{\sqrt{\pi}\,\theta^3}\, t^2\, e^{-\left(\frac{t}{\theta}\right)^2} \qquad t, \theta > 0 \tag{5}$$

The cumulative distribution function $[G(t) = pr(T \leq t)]$, is given by;

$$G(t) = \int_0^x \frac{4}{\sqrt{\pi}\,\theta^3}\, t^2\, e^{-\left(\frac{t}{\theta}\right)^2} dt$$

(6)

Let $u = \left(\frac{t}{\theta}\right)^2 \rightarrow \sqrt{u} = \frac{t}{\theta}, \quad t = \theta\sqrt{u} \quad dt = \frac{\theta}{2\sqrt{u}}\, du \quad if\ t = x \rightarrow u = \left(\frac{x}{\theta}\right)^2$

$$G(t) = \int_0^{\left(\frac{x}{\theta}\right)^2} \frac{4}{\sqrt{\pi}\,\theta^3}\, (\theta\sqrt{u})^2\, e^{-u}\, \frac{\theta}{2\sqrt{u}}\, du$$

$$G(t) = \frac{2}{\sqrt{\pi}} \int_0^{\left(\frac{x}{\theta}\right)^2} (u)^{\frac{1}{2}}\, e^{-u}\, du$$

(7)

Since;

$$\Gamma(\alpha) = \int_0^{\infty} (t)^{\alpha-1}\, e^{-t}\, dt$$

$$\gamma(\alpha; x) = \int_0^{x} (t)^{\alpha-1}\, e^{-t}\, dt$$

is incomplete Gamma function, then;

$$G(t) = \frac{2}{\sqrt{\pi}} \int_0^{\left(\frac{x}{\theta}\right)^2} (u)^{\frac{3}{2}-1}\, e^{-u}\, du$$

$$G(t) = \frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \left(\frac{x}{\theta}\right)^2\right)$$

(8)

When we sampling from population of different size we can transform the $p.d.f$ in equation (5) into weighted $p.d.f$ by applying;

$$g_w(t) = \frac{t^c\, g(t)}{E(t^c)}$$

(9)

Where;

$$E(t^c) = \int_0^\infty t^c\, g(t)\, dt$$

$$= \frac{4}{\pi}\int_0^\infty \frac{t^{c+2}}{\theta^3}\, e^{-\left(\frac{t}{\theta}\right)^2}\, dt$$

Let $z = \frac{t}{\theta} \rightarrow t = z\theta \quad dt = \theta dz$

$$E(t^c) = \frac{4}{\pi}\int_0^\infty \frac{(z\theta)^{c+2}}{\theta^3}\, e^{-z^2}\, \theta dz$$

$$= \frac{4}{\pi}\theta^c \int_0^\infty (z)^{c+2}\, e^{-z^2} dz$$

Let $w = z^2 \rightarrow z = \sqrt{w} \qquad dz = \frac{1}{2\sqrt{w}}\, dw$

$$E(t^c) = \frac{4}{\pi}\theta^c \int_0^\infty (\sqrt{w})^{c+2}\, e^{-w}\, \frac{1}{2\sqrt{w}}\, dw$$

$$E(t^c) = \frac{2}{\pi}\theta^c \int_0^\infty (\sqrt{w})^{c+1}\, e^{-w}\, dw$$

$$E(t^c) = \frac{2}{\pi}\theta^c \int_0^\infty w^{\frac{c+1}{2}}\, e^{-w}\, dw$$

$$E(t^c) = \frac{2}{\pi}\theta^c \Gamma\left(\frac{c+3}{2}\right)$$

(10)

Therefore the new weighted Maxwell distribution is;

$$g_w(t) = \frac{\frac{4}{\pi}\,\frac{t^{c+2}}{\theta^3}\;e^{-\left(\frac{t}{\theta}\right)^2}}{\frac{2}{\pi}\;\theta^c\Gamma\left(\frac{c+3}{2}\right)}$$

$$g_w(t) = \frac{2}{\Gamma\left(\frac{c+3}{2}\right)}\,\frac{t^{c+2}}{\theta^{c+3}}\;e^{-\left(\frac{t}{\theta}\right)^2} \quad t, c, \theta > 0 \qquad (11)$$

$$0 \qquad o/w$$

Cumulative distribution function (C.D.F) corresponding to $p.d.f$ in (11) is the WCDF obtained from;

$$G_w(t) = \int_0^u g_w(t)dt$$

$$G_w(t) = \frac{2}{\Gamma\left(\frac{c+3}{2}\right)\theta^{c+3}}\int_0^u t^{c+2}\;e^{-\left(\frac{t}{\theta}\right)^2}dt \qquad (12)$$

$$G_w(t) = \frac{2}{\Gamma\left(\frac{c+3}{2}\right)\theta^{c+3}}\int_0^x t^{c+2}\;e^{-\left(\frac{t}{\theta}\right)^2}dt$$

Let $z = \left(\frac{t}{\theta}\right)^2 \rightarrow \sqrt{z} = \frac{t}{\theta},\quad t = \theta\sqrt{z} \qquad dt = \frac{\theta}{2\sqrt{z}}\,dz \qquad if\ t = x \rightarrow z = \left(\frac{x}{\theta}\right)^2$

$$G_w(t) = \frac{2}{\Gamma\left(\frac{c+3}{2}\right)\theta^{c+3}}\int_0^{\left(\frac{x}{\theta}\right)^2} (\theta\sqrt{z})^{c+2}\;e^{-z}\,\frac{\theta}{2\sqrt{z}}\;dz$$

$$G_w(t) = \frac{1}{\Gamma\left(\frac{c+3}{2}\right)}\int_0^{\left(\frac{x}{\theta}\right)^2} (\sqrt{z})^{c+1}\;e^{-z}\;dz$$

$$G_w(t) = \frac{1}{\Gamma\left(\frac{c+3}{2}\right)}\int_0^{\left(\frac{x}{\theta}\right)^2} z^{\frac{c+3}{2}-1}\;e^{-z}\;dz$$

$$G_w(t) = \frac{1}{\Gamma\left(\frac{c+3}{2}\right)}\gamma\left(\frac{c+3}{2}, \left(\frac{t}{\theta}\right)^2\right)$$

(13)

Therefore the $p.d.f$ of weighted Maxwell distribution and C.D.F also of weighted Maxwell distribution are derived.

2. Method of Estimations

We introduce the methods of moments, maximum likelihood and L – moments to estimate the two parameters (c,θ) of new weighted Maxwell distribution.

2.1 Maximum Likelihood Method

Let $(t_1, t_2, \dots, t_n)$ be a $r.v$ from $[g_w(t)]$ then;

$$L = \prod_{i=1}^{n} g_w(t_i) = \frac{2^n \prod_{i=1}^{n}(t_i)^{c+2}}{\Gamma\left(\frac{c+3}{2}\right)\theta^{n(c+3)}} \quad e^{-\sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^2} \tag{14}$$

Then

$$\log L = n\log 2 + (c+2)\sum_{i=1}^{n}\log(t_i) - n\log\Gamma\left(\frac{c+3}{2}\right) - n(c+3)\log\theta - \sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^2 \tag{15}$$

$$\frac{\partial \log L}{\partial\theta} = \frac{-n(c+3)}{\theta} + \frac{2}{\theta^3}\sum_{i=1}^{n} t_i^2 = 0$$

$$\frac{2}{\theta^3}\sum_{i=1}^{n} t_i^2 = \frac{n(c+3)}{\theta}$$

$$\frac{2}{\theta^2}\sum_{i=1}^{n} t_i^2 = n(c+3)$$

$$\hat{\theta}^2 = \frac{2\sum_{i=1}^{n} t_i^2}{n(c+3)}$$

$$\hat{\theta}_{MLE} = \sqrt{\frac{2\sum_{i=1}^{n} t_i^2}{n(c+3)}} \tag{16}$$

$$\frac{\partial \log L}{\partial c}=\sum_{i=1}^{n}\log(t_i)-\frac{\frac{n}{2}\Gamma'\left(\frac{c+3}{2}\right)}{\Gamma\left(\frac{c+3}{2}\right)}-n\log\theta=0$$

$$\sum_{i=1}^{n}\log(t_i)-n\log\hat{\theta}=\frac{n}{2}\psi\left(\frac{c+3}{2}\right) \qquad (17)$$

Solving $\psi\left(\frac{c+3}{2}\right)=\frac{2(\sum_{i=1}^{n}\log(t_i)-n\log\hat{\theta})}{n}$

2.2 Moment Estimator

The moment estimator obtained from solving;

$\mu_r'=E(t^r)$ with $m_r=\frac{\sum_{i=1}^{n}t_i^r}{n}$

Where;

$$E(t^r)=\int_0^\infty t^r\, g_w(t)dt$$

$$=\frac{2}{\Gamma\left(\frac{c+3}{2}\right)\theta^{(c+3)}}\int_0^\infty t^{r+c+2}\, e^{-\left(\frac{t}{\theta}\right)^2}dt$$

Let $z=\left(\frac{t}{\theta}\right)^2\rightarrow\sqrt{z}=\frac{t}{\theta}\qquad t=\theta\sqrt{z}\quad dt=\frac{\theta}{2\sqrt{z}}\,dz$

$$E(t^r)=\frac{2}{\Gamma\left(\frac{c+3}{2}\right)\theta^{(c+3)}}\int_0^\infty(\theta\sqrt{z})^{r+c+2}\, e^{-z}\frac{\theta}{2\sqrt{z}}\,dz$$

$$E(t^r)=\frac{\theta^r}{\Gamma\left(\frac{c+3}{2}\right)}\int_0^\infty(z)^{\frac{r+c+2}{2}-\frac{1}{2}}e^{-z}\,dz$$

$$E(t^r)=\frac{\theta^r}{\Gamma\left(\frac{c+3}{2}\right)}\Gamma\left(\frac{r+c+3}{2}\right) \qquad (17)$$

Equating $E(t^r)$ with $m_r=\frac{\sum_{i=1}^{n}t_i^r}{n}$ $for\ r=1,2\ we\ obtain;$

$$\frac{\hat{\theta}}{\Gamma\left(\frac{c+3}{2}\right)}\Gamma\left(\frac{c+4}{2}\right)=\bar{t}\qquad for\quad r=1$$

$$\frac{\theta^2}{\Gamma\left(\frac{c+3}{2}\right)}\Gamma\left(\frac{c+5}{2}\right)=\frac{\sum_{i=1}^{2}t_i^2}{n} \qquad for \;\; r=2$$

Therefore from solving;

$$\frac{\hat{\theta}_{MOM}}{\Gamma\left(\frac{c+3}{2}\right)}\Gamma\left(\frac{c+4}{2}\right)=\bar{t} \tag{18}$$

$$\frac{\hat{\theta}_{MOM}^2}{\Gamma\left(\frac{c+3}{2}\right)}\Gamma\left(\frac{c+5}{2}\right)=\frac{\sum_{i=1}^{2}t_i^2}{n} \tag{19}$$

To obtain $\hat{\theta}_{MOM}$ and $\hat{c}_{MOM}$

2.3 L – Moment Estimation

The estimation by this method depends on ordered sample as indicated by Hosking (1990), and David and Nagaraja (2003). These estimator of parameters are obtained from equation the first and second sample L – moments, where;

$$L_1=\frac{\sum_{i=1}^{n}x_{(i)}}{n}$$

$$L_2=\frac{2\sum_{i=1}^{n}(i-1)x_{(i)}}{n(n-1)}-L_1$$

With $E(x)=E(t)$ and $E(x^2)=E(t^2)$

i.e

$$L_1=\frac{\theta}{\Gamma\left(\frac{c+3}{2}\right)}\Gamma\left(\frac{c+4}{2}\right) \quad \text{and} \quad L_2=\frac{\theta^2}{\Gamma\left(\frac{c+3}{2}\right)}\Gamma\left(\frac{c+5}{2}\right)$$

2.4 Estimation by method of length biased moments

Since our research consider the weighted distribution which is necessary when sampling from sample of different size, so we depend on the proposal introduced by Naeerat and Winai (2014), to apply the moments;

$$E_L(x^r) = \frac{E(x^{r+1})}{E(x)}$$

After deriving the formula of [$E_L(x^r)$], the best fitting model obtained at comparing the statistical measures like mean, variance, skeweness, kurtosis at different set given values of (θ, c), and then choose the set of estimated parameters which satisfy the smallest value of coefficient of skweness and kurtosis.

$$E(t^{r+1}) = \int_0^\infty t^{r+1}\, g_w(t) dt$$

$$= \frac{2}{\Gamma\left(\frac{c+3}{2}\right)\theta^{(c+3)}} \int_0^\infty t^{r+c+3}\, e^{-\left(\frac{t}{\theta}\right)^2} dt$$

Let $z = \left(\frac{t}{\theta}\right)^2 \rightarrow \sqrt{z} = \frac{t}{\theta} \qquad t = \theta\sqrt{z} \quad dt = \frac{\theta}{2\sqrt{z}}\, dz$

$$E(t^{r+1}) = k \int_0^\infty (\theta\sqrt{z})^{r+c+3}\, e^{-z} \frac{\theta}{2\sqrt{z}}\, dz$$

$$E(t^{r+1}) = \frac{k\theta^{r+c+4}}{2} \int_0^\infty (\sqrt{z})^{\frac{r+c+2}{2}}\, e^{-z}\, dz$$

$$E(t^{r+1}) = \frac{k\theta^{r+c+4}}{2} \Gamma\left(\frac{r+c+4}{2}\right)$$

$$E(t^{r+1}) = \frac{2\theta^{r+c+4}}{\frac{\Gamma\left(\frac{c+3}{2}\right)\theta^{c+3}}{2}} \Gamma\left(\frac{r+c+4}{2}\right)$$

$$E(t^{r+1}) = \frac{\theta^{r+1}}{\Gamma\left(\frac{c+3}{2}\right)} \Gamma\left(\frac{r+c+4}{2}\right)$$

(20)

Then;

$$E_L(x^r) = \frac{E(x^{r+1})}{E(x)} = \frac{\theta^r}{\Gamma\left(\frac{c+4}{2}\right)} \Gamma\left(\frac{r+c+4}{2}\right)$$

(21)

3. Simulation

Table (1)

Weighted Maxwell distribution two parameters estimators

n	Method	Experiment 1		Experiment 2		Experiment 3	
		$C = 3$	$th = 0.9$	$C = 2$	$th = 1$	$C = 1$	$th = 1$
25	MLE	3.7666	1.9329	2.6125	0.9721	1.4947	0.9710
	MOM	3.7674	1.9336	2.6286	0.9719	1.4957	0.9722
	PERCE	3.7870	1.9666	2.6299	0.9967	1.5781	0.9962
50	MLE	3.3293	1.9684	2.2822	0.9845	1.2156	0.9841
	MOM	3.3236	1.9701	2.2834	0.9847	1.2144	0.9849
	PERCE	3.3446	1.9815	2.2505	0.9967	1.2462	0.9918
100	MLE	3.1883	1.9814	2.1303	0.9940	1.1358	0.9896
	MOM	3.1909	1.9819	2.1287	0.9944	1.1392	0.9897
	PERCE	3.1720	1.9930	2.1215	0.9990	1.1468	0.9940
150	MLE	3.1425	1.9863	2.1040	0.9947	1.0918	1.9913
	MOM	3.1457	1.9863	2.1009	0.9952	1.0956	1.9998
	PERCE	3.1434	1.9926	2.1118	0.9969	1.0935	1.9987

Table (2)

Mean Square Error (MSE)

n	Method	Experiment 1		Experiment 2		Experiment 3	
		$C = 3$	$th = 0.9$	$C = 2$	$th = 1$	$C = 1$	$th = 1$
25	MLE	0.5876	0.0045	0.3751	0.7782 e-003	0.2447	0.8381e-003
	MOM	0.5890	0.0044	0.3951	0.7894e-003	0.2457	0.7703e-003
	PERCE	0.6194	0.0011	0.3968	0.0109e-003	0.3342	0.0146e-003
		perce	mle	perce	Mle	perce	mle
50	MLE	0.1084	0.9989e-003	0.0796	0.2413e-003	0.0465	0.2518e-003
	MOM	0.1047	0.8944e-003	0.0803	0.2331e-003	0.0460	0.2288e-003

	PERCE	0.1187	0.3432e−003	0.0628	0.0109e−003	0.0606	0.0677e−003
		perce	mom	perce	perce	perce	mom
100	MLE	0.0354	0.3447e−003	0.0170	0.3578e−004	0.0184	0.1075e−003
	MOM	0.0364	0.3281e−003	0.0166	0.3090e−004	0.0194	0.1055e−003
	PERCE	0.0296	0.0494e−003	0.0148	0.0103e−004	0.0216	0.0354e−003
		perce	mle	perce	perce	mom	perce
150	MLE	0.0203	0.1866e−003	0.0108	0.2840e−004	0.0084	0.4360e−004
	MOM	0.0212	0.1880e−003	0.0102	0.2318e−004	0.0091	0.4512e−004
	PERCE	0.0206	0.0550e−003	0.0125	0.0936e−004	0.0087	0.1201e−004
		perce	mle	mom	mom	perce	perce

Algorithm

1. Generation the variable $z = \left(\frac{t}{\sigma}\right)^2$ through inverse diversion method.
2. Calculating the variable (t), which distributed weighted Rayleigh by the equation $t = \sigma \times \sqrt{z}$
3. Estimating the two parameters by;
   a. Maximum Likelihood Method.
   b. Moment Method.
   c. Divisional Estimators Method.

Calculating mean square error [$\sum(\hat{\theta} - \theta)^2$] for comparison between the best estimators.

The main program:

```
clc

clear

c=1;

th=1;
```

```
n=150;
a=((c+3)/2);
for i=1:1000
   u=rand(n,1);
z=gammaincinv(u,a,'lower');
t=(sqrt(z))*th;
t=sort(t);
mle=fminsearch(@(d) wemax1(t,d),[c th],optimset('tolx',1e-15));
cmle(i)=mle(1);
thmle(i)=mle(2);
mom=fsolve(@(d) wemax2(t,d),[c th],optimset('tolx',1e-15));
cmom(i)=mom(1);
thmom(i)=mom(2);
per=fminsearch(@(d) wemax3(t,d),[c th],optimset('tolx',1e-15));
cper(i)=abs(per(1));
thper(i)=abs(per(2));
end
chad=[mean(cmle) mean(cmom) mean(cper)]
thhad=[mean(thmle) mean(thmom) mean(thper)]
msec=[mean(cmle-c).^2  mean(cmom-c).^2 mean(cper-c).^2 ]
mseth=[mean(thmle-th).^2 mean(thmom-th).^2 mean(thper-th).^2]
```

sub program

```
function F =wemax1(t,d)

n=length(t);

c=d(1);

th=d(2);

F=-(n*log(2)+((c+2)*(sum(log(t))))-(n*log(gamma((c+3)/2)))-(n*(c+3)*log(th))-

sum((t/th).^2));
```

```
function F=wemax2(t,d)

n=length(t);

c=d(1);

th=d(2);

m1=(sum(t))/n;

m2=(sum(t.^2))/n;

F=[((th*gamma((c+4)/2))/(gamma((c+3)/2)))-m1

   (((th.^2)*gamma((c+5)/2))/(gamma((c+3)/2)))-m2];
```

```
function F=wemax3(t,d)

n=length(t);

t=sort(t);

c=d(1);

th=d(2);

x=(t/th).^2;

a=((c+3)/2);
```

```
for i=1:n
    p(i)=(i-(3/8))/(n+(1/4));
 end
Fx=gammainc(x,a);
F=sum((p'-Fx).^2);
```

Conclusion

1. The weighted probability distribution are necessary when the samples taken are of different sample size so we need to modify the probability through weighting
2. For n=25 and for different values of c and θ we find estimators of(C andθ) is the best for percentiles estimators, this is clear from table (2) of mean square error.
3. The values of observations are generated using the algorithm of inverse gamma, method.
4. Also for (n=50,100,150) we find the best estimators are percentile's, which work on Minimizing the differences between cumulative distribution function and its non parametric estimators (x_i).

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