

Diameter and width via Rotating Calipers of a Set of Two-dimensional Convex Hull Vertices using Graham scan algorithm

Asia Mahdi Naser Alzubaidi

Computer Science Department / College of Science / Karbala university

Abstract

A convex hull is an important shape in graphic geometry that can be used to construct many other geometric structures. In this paper we compute the Convex hull polygon by eliminating all the interior points and bounding box the exterior points using Graham scan algorithm then we compute the Diameter and width of a convex hull polygon using a powerful, easy and well-designed method that can solve several computational geometric difficulties professionally in practice called Rotating Caliper method. The proposed idea of a well-known rotating caliper algorithm includes rotate pairs of parallel straight lines called antipodal pairs around a given polygon's vertices, the first one located in maximum Y-coordinate and the second one is in minimum Y-coordinate of Convex hull vertices. If we continuously sweep the tangent of the antipodal pairs through 360 degrees, then the width and diameter of the convex hull polygon are simply detected by minimum and maximum distance between the antipodal pairs during this sweep process. The experimental results reveal that The rotating calipers algorithm is fast, powerful, easy and general tool for solving geometric problems.

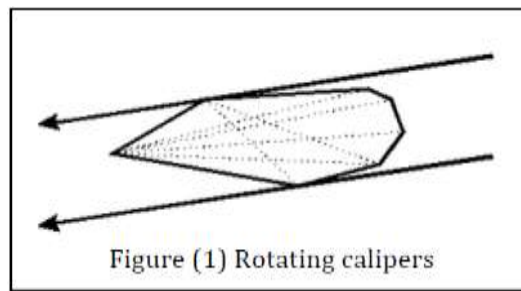
Keywords: Rotating Calipers; Computational Geometric; Graham Convex Hull.

المستخلص

التل المحدث هو شكل مهم في الرسم الهندسي يمكن استخدامه لبناء العديد من الاشكال الهندسية الأخرى. في هذه البحث تم حساب متعدد الاضلاع المحدث من خلال التخلص من جميع النقاط الداخلية واحاطة النقاط الخارجية فقط باستخدام مفهوم bounding box استخدام خوارزمية مسح غراهام ثم حساب القطر والعرض لمتعدد الاضلاع المحدث. تعد خوارزمية الفرجار الدوار احد الخوارزميات القوية ، السهلة و المصممة بشكل جيد لحل العديد من المشاكل الهندسية الحسابية بكفاءة في الجوانب العملية. فكرة الخوارزمية المقترحة والمعروفة بالفرجار الدوار تشمل تدوير أزواج من الخطوط المستقيمة المتوازية تسمى أزواج تقابلية حول القيم لمتعدد الاضلاع المعطى ، الخط الأول يقع في أقصى Y- تنسيق ثنائي الابعاد والاخر يقع في الحد الأدنى Y- تنسيق ثنائي الابعاد للتل المحدث. في كل مرة نحصل فيها على أزواج تقابلية من خلال التدوير بزوايا حتى الوصول الى زاوية 360 درجة ، يتم الكشف ببساطة عن العرض والقطر لمتعدد الاضلاع المحدث من خلال حساب المسافة بينا لحد الأدنى والحد الأقصى للأزواج التقابلية خلال عملية التدوير. النتائج التجريبية تكشف عن أن الخوارزمية لفرجار الدورية هي أداة سريعة وقوية وسهلة وعامة من أجل حل معظم المشاكل الهندسي.

1. Introduction

Computational Geometry is the study of geometric problems and the algorithms to solve them. Problems range from the direct to the very multifaceted in their details [2]. Let $P = (P_0, P_1, \dots, P_n)$ be a sequence of two dimension points order in Cartesian coordinate plane, we present a familiar algorithm for solving many problems concerning about building a convex hull polygon from a given two-Dimension set points in plane. Then, a more instinctive of rotating calipers algorithm used to computing the width and diameter by computing minimum and maximum distance between two parallel lines named antipodal pair around the convex polygon vertices or boundary [5]. Calculation the width or diameter of convex polygon has many applications in collision prevention difficulties and in approximating polygonal shape curves. In this research algorithm to find the diameter or width of a convex hull polygon will be offered, depended on Shamos' "rotating calipers" algorithm for calculating the diameter or width. The algorithms run in $O(n)$ time and $O(n)$ space complexity [1]. Figure (1) shows the concept of Rotating calipers. In the current paper, we first consider the convex hull of the set of points using Graham scan algorithm. Second, scan the hull and find antipodal pairs. Then, Compute the distances between antipodal pairs, Report the largest and smallest such distance. Finally, the conclusions and references.



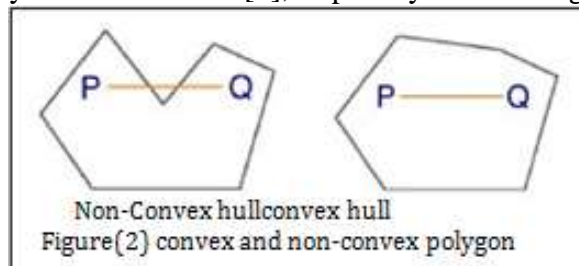
2. Related Work

Related work on finding a maximum and minimum distance of convex hull polygon. The Research done in 1978 by M.I. Shamos started the field of Computational Geometry within theoretical Computer Science, his Ph.D. thesis carrying the same name. In his work, he discusses the problem of finding the diameter of a convex polygon and gives a fast and well-organized algorithm as a solution. This algorithm is the basis for the Rotating Calipers procedure [1]. The term "rotating calipers" later coined in 1983 by G.T. Toussaint to solve Computational geometry problems by computing distances such as finding diameter of a convex hull polygon or computing Enclosing bounding box for instance the minimum area of enclosing rectangle [4]. Another piece of work in 1999 by H. Pirzadeh, he use Rotating Calipers as a method to solve a number of problems in the field of Computational Geometry. The resulting algorithms are often optimal in their time and space complexity, and always efficient and powerful to implement [3]. Since then, the Rotating Calipers paradigm was also been generalized to applications of three dimension points [6], although we limit the scope of this paper to solve some of computational geometric problems in two dimensional problems.

3. Computing a 2-D Convex Hull: Graham's Algorithm

3-1 Graham scan algorithm

Computing a convex hull (or just "hull") is one of the first sophisticated computational geometry algorithms, and there are many variations of it [8]. The most public form of this algorithm involves determining the smallest convex set (called the "convex hull") containing a discrete set of two dimensional points in the plane [7]. A convex hull CH means that a subset of space is called convex if and only if for every pair of points P and Q in the set of CH, the line segment from P point to Q point is enclosed completely in the set of CH [9], as portrayed on the figure (2) below.



The most popular algorithms for computing convex hull CH are the "Graham scan" algorithm of a set P of n points in the plane [11]. The algorithm is fascinating for a number of reasons. First of all, it is simple to compute and is very intuitive. Moreover, for a Data Structures class it is quite convincing. This algorithm describes below [9].

Graham scan algorithm

Input: A set $P = (P_0, P_1, \dots, P_n)$ of two- dimension points in the plane.

Output: Convex Hull $CH(S)$ contains S vertices.

Start

Step1: Find an extreme point p_0 of P that is a vertex of CH and denote it as the anchor point,

We can for example, pick as our anchor point the point in P with minimum y-coordinate value (in screen coordinate p_0 has maximum y-coordinate) .

Step2: Translate the anchor point of P and all the other point, so the extreme point is at the origin by subtract each point of P from anchor or extreme point [11].

Step3:Find the angle between the line connecting P0 to each of the points and the positive X- axis using the following steps[17].

- Find differences between the X-axis of two points. $\Delta X = P_x - P_{0x}$
- Find differences between the Y-axis of two points. $\Delta Y = P_y - P_{0y}$
- if ($\Delta X = 0$)and ($\Delta Y = 0$)then Angle = 0 else
- Compute angle in radian degree.

$$\text{angle} = \text{ArcTan2}(\Delta Y, \Delta X) * 57.295779513082 \text{ ----- (1)}$$

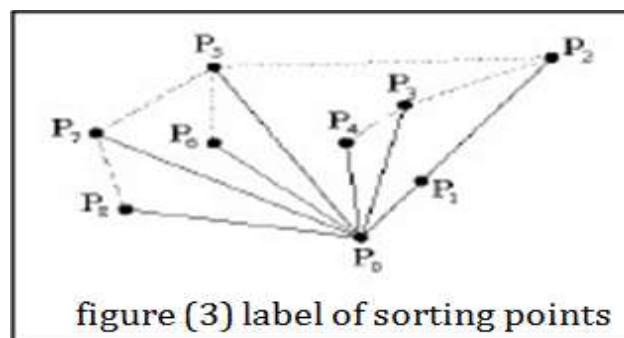
if ($\text{angle} < 0$)then add 360 to angle.

Step4: Sort the points in order of increasing angle about anchor point using Mergesort algorithm. The Resulting sorted list of points order in counterclockwise (CCW)with respect to the extreme point [12].

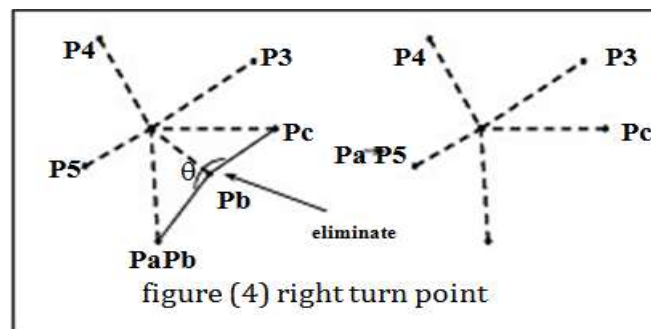
Step5:If two points have the same value of angle, delete the point with the smaller amplitude or distance to anchor point using Euclidean distance measure[10], as in equation(1).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \text{ ----- (2)}$$

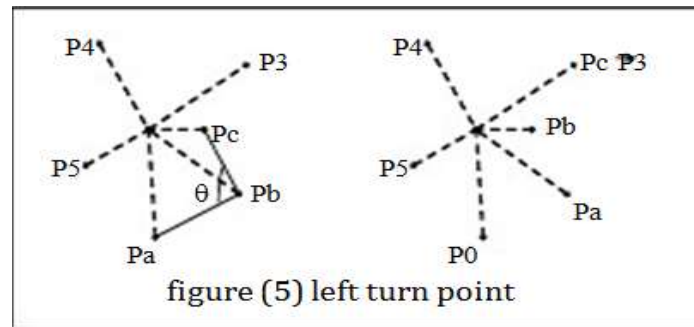
Step6:Starting from the anchor point p0and in CCW manner, labelthe points (P0, P1, P2, ...) It looks like a fan with anchor at the point P0, as in figure (3).



Step7: Remove right turn point If the inner angle formed by Pa, Pb,Pc is greater than or equal to 180° then we eliminate the point labeled with Pb. Setpoint Pb to be point Pa. Set point Pa to the previous point in the sequence (in this case P5) [13]. As showing in figure(4).



Step8: Add to CH the left turn points. If the inner angle formed by Pa, Pb, Pcfrom before is less than 180° then No points are discarding. Each of Pa,Pb and Pc are advances forward one point [14] as shown in figure (5).



Step9: The Algorithm continues by repeating step 7 until $P_b = P_0$. At this case, the algorithm stops and only the points of the convex hull CH remain as output [15].

Finish

3-2 Mergesort algorithm

Merge sort depends on divide and conquer strategy to sort the points of list or matrix. This means Merge sort recursively divides the list of points into smaller sub lists and sorts them individually and then merges these sub lists to form a complete sorted list [16]. Conceptually, merge sort works as follows:

Mergesort algorithm

Input: Set of angles (list) between P_0 and other points P .

Output: The Set of angles sorted in counter clockwise order.

Start

Step1: Finding the median (mid) of list .

$Mid = (first + last) \div 2$ ----- (3)

Where:- mid : index of median point. first, last: index of first and last point respectively.

Step2: Divide the unsorted list into two sub lists of about half the size.

Mergesort (list, first, Mid)----- (4)

Mergesort (list, Mid+1, last)

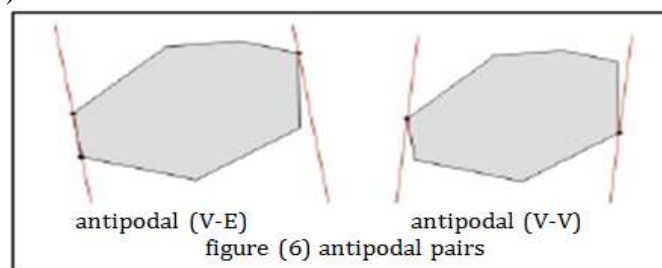
Step3: Divide each of the two sub lists recursively until we have list sizes of length one point.

Step4: Merge the two sorted sub lists back into one sorted list.

Finish

4. Diameter-finding and width-finding: Rotating calipers

Before explaining the Diameter-finding algorithm it is beneficial to describe what is the Diameter and width of a set of two dimension points, and current terminology [6]. We call any two edges of the convex polygon CH of a set of two dimension points S an antipodal vertex-vertex (V-V) pair if parallel lines of convex polygon of S contain these vertices. Similarly, we can define vertex-edge (V-E) pairs. Note that these are the only probable combinations to find width or diameter of convex hull polygon, since a line of support of S must cross CH either at only vertex, or along one of CH edges [3], as in figure(6).



4.1 Diameter-finding algorithm

The diameter of a convex polygon is defined as the maximum distance between parallel lines of the polygon. Note that the pair of points determining the diameter of a convex polygon CH does not belong to the interior of CH, [2]. The search should focus on the external. In fact, only antipodal pairs of points should be checked, for the diameter is the highest distance between two parallel lines of polygon [4].

Diameter-finding algorithm

Input: Set points CH= (S0 , S1 , S2 ,...) of convex polygon vertices.

Output: Antipodal pair has maximum distance.

Start

Step1: Compute the polygon's finishing points in the y direction. Call them ymin and ymax.

Step2: Construct two horizontal lines of support through ymin and ymax. Since this is already an antipodal pair, we compute the area of triangle of ymin, ymax and S vertices, then keep as maximum [17].

Given the coordinates of the three vertices of a triangle ABC, the area is given by

$$area = \left| \frac{Ax(By - Cy) + Bx(Cy - Ay) + Cx(Ay - By)}{2} \right|$$

where Ax and Ay are the x and y coordinates of the point A etc..

----- (5)

Step3: Use Rotating calipers to generate antipodal pairs with an edge of the convex Polygon.

Step4: Compute the new area and compare to old maximum, then update if necessary.

Step5: Repeat steps 3 and 4 until the antipodal pair reached is (ymin, ymax) again.

Step6: Report or output the antipodal pair with maximum distance as the diameter pair.

Finish

4.2 Width-finding algorithm

The width of a convex hull polygon distinct as the smallest distance between two parallel lines of polygon [1]. Note not all directions of polygon need to emphasis. Presume a polygon vertices were obtain, alongside with two parallel lines of convex polygon. If neither of the parallel lines coincides with an edge, then it is continuously possible to rotate them to minimize the distance between them. Therefore, two parallel lines of CH do not measure the width except one of them concurs with an edge [3].

Width-finding algorithm

Input: Set points CH= (S0 , S1 , S2 ,...) of convex polygon vertices.

Output: Antipodal pair has minimum distance.

Start

Step1: compute the finishing point of polygon in the y orientation, denoted as ymin and ymax.

Step2: Construct two parallel lines of CH through ymin and ymax. If one (or both) of the Lines tangent with an edge, then a antipodal vertex-edge or edge-edge pair is detected, compute the area of triangle (ymin, ymax , S) vertices, and save as the minimum distance.

Step3: Rotate the lines until one is attached with an edge of the polygon. A new antipodal pair is detected. Compute the new distance between the lines, compare to old minimum, and update the distance if it necessary.

Step4: Repeat steps 3 until the lines of calipers have reached their original horizontal position.

Step5: Report or output the antipodal pair with minimum distance as the width of polygon.

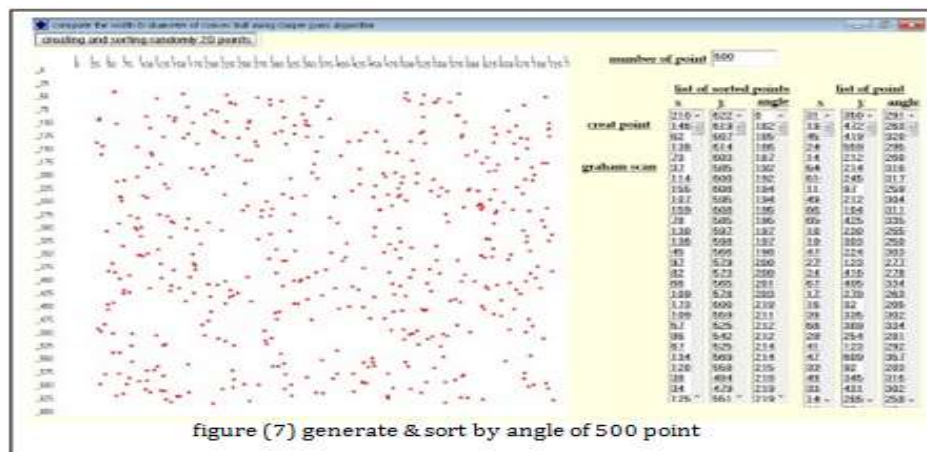
Finish

5. implementation of programming system and results

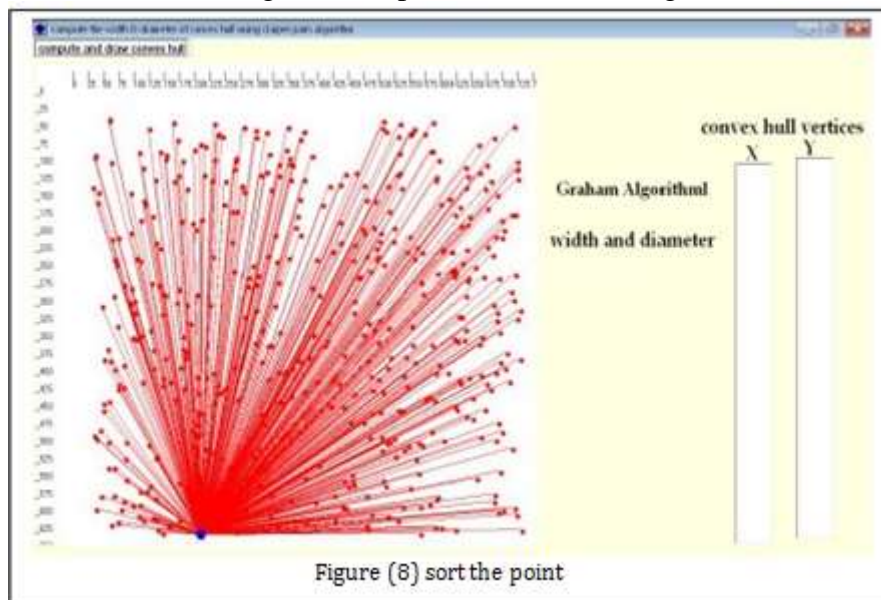
In this paper, the rotating calipers algorithm to find Diameter and Width of convex polygon starts with an unordered set of points defined by Cartesian coordinates. Each point has a position on the X-axis and Y-axis. To illustrate the algorithm we'll start with example of generate 500 points.

5-1 Implementation of programming system

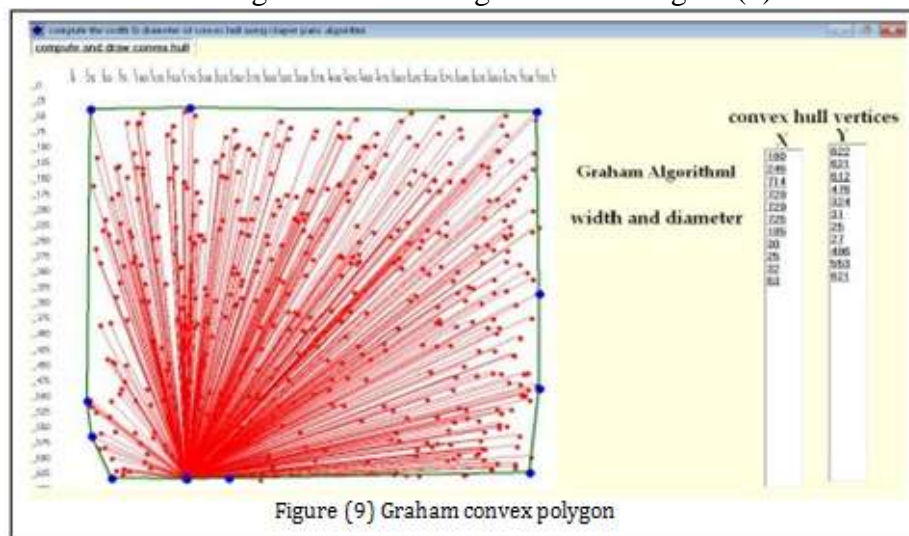
First: Generate the two dimensional points with maximum number not exceeding 10000 points and draw them using point drawing instruction according to programming language. Then the algorithm compute or take starting point to be anchor point which represent the maximum Y- Coordinate, after that compute the angles between the anchor point to all other points from the horizontal line using atan2 function in the equation (1) as shown in figure (7).



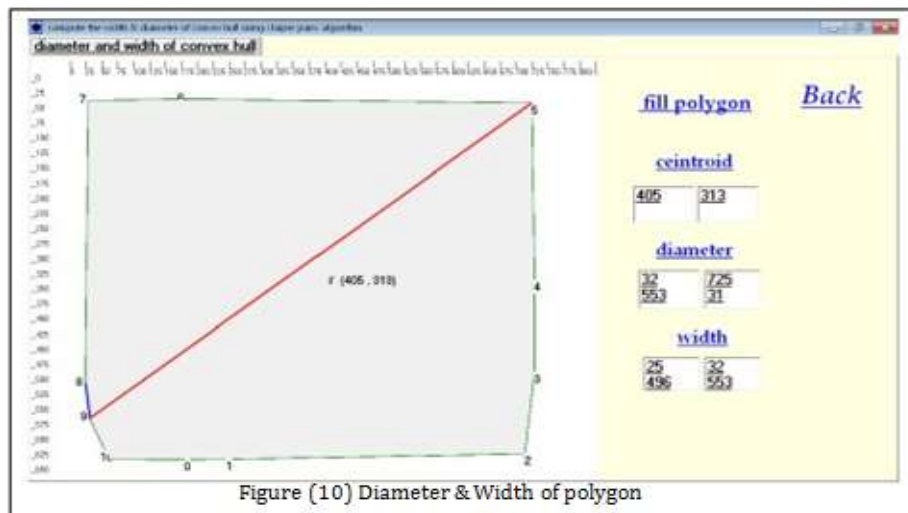
Second: Sort the points by angles in counter clockwise order using equation number(3,4) represents the mergesort algorithm steps, in the same time the two dimension points are also rearrange according to value of angle belongs to each point. Finally we draw line between anchor point and all othersorted points to show the arrangement of points as shown in figure (8)



Third: Build the convex hull using Graham scan algorithm as in figure (9).



Forth: Compute the Width & Diameter of convex polygon using rotatingcalipers algorithm as in figure (10). Diameter in red and width in blue color.



5-2 Results

From the practical implementation of proposed software system, we find the number of convex hull polygon vertices for variety of two dimension data size and puts the resulting in one table as bellow

#OF POINTS	#HULL POINTS
5	4
10	7
50	10
100	11
300	14
500	15
600	15
800	16
1000	15
1500	17
10000	20

6. Conclusion

1. The experimental results reveal that convex hull points are only a small subset of all the randomly two dimension points, so we recommends computing the hull and then doing all antipodal pairs distances computation.
2. The Random function use in generating two dimension points is a speedy function in having random two dimension points.
3. The rotating calipers algorithm is fast, easy and common tool for solving computational geometric problems.
4. Mergesort algorithm used to sort floating points represent angles of points is fast algorithm and influences the runtime of Graham scanning procedure.
5. Rotating Calipers a highly useful and adaptable tool that deserves further research.
6. The Graham Scan algorithm has the optimal worst case complexity when not taken account output sensitivity. The basic concept is that we take an infinity point, sort all the other points angularly in $O(n \log n)$, with a pass in linear time to compute the convex hull polygon. Since we sort in this algorithm, we are bounding by the output insensitive complexity.
7. The convex hull is one of the most basic constructions of computational geometry, used as a stepping stone to solving much firmer and more complex problems.
8. There is a fact that optimal algorithms for building convex hull polygon already exist for both n-space with 2D or 3D point sets.

References

- [1] M.I. Shamos, "Computational geometry", Ph.D. thesis, Yale University, 1978.
- [2] M. De Berg, O. Cheong, M. van Kreveld, and Mark Overmars. "Computational geometry: algorithms and applications", Springer, 2008.
- [3] H.Pirzadeh,"Computational geometry with the rotating calipers",MSc.thesis,McGill University,1999.
- [4] G.T.Toussaint,"Solving geometric problems with the rotating calipers",Proceedings of IEEE MELECON'83, Athens, Greece, May 1983.
- [5] M. E. Houle and G. T. Toussaint, "Computing the width of a set," Tech. Rep. SOCS84.22,1984.
- [6] online document [accessed on November 26th, 1999] available at <http://cgm.cs.mcgill.ca/~orm/rotcal.frame.html>.
- [7] CSEE vova,v144,"Convex Hulls in Two Dimensions", University of South,Florida,2004
- [8] Mark Nelson, "Building the Convex Hull", August, 22nd, 2007.
- [9] D.Sunday,"Convex Hull of a 2D Point Set or Polygon",softsurfer.com, 2009.
- [10] S.Suri ,"Computational Geometry ",UC Santa Barbara,CS-235,2002.
- [11] M.T.Goodrich,R. Tamassia,"Algorithm Design", John Wiley & Sons Inc,2002.
- [12] Jeff Erickson,online document [accessed on January 2011] available at <http://www.cs.uiuc.edu/~jeffe/teaching/algorithms>.
- [13] rec24" Intro to Algorithms" ,Recitation 24 ,May 6, 2011.
- [14] E.Eilberg," Convex Hull Algorithms", Denison University, available at <http://drc.denison.edu/bitstream/handle/2374.DEN/5092/eilberg.pdf?sequence=1>
- [15] NengxiongXu, Gang Mei, John C.Tipper,"An Algorithm for Finding Convex Hulls of Planar Point Sets" Grant Numbers 40602037 and 40872183,2012
- [16]Cormen,Thomas,H.Leiserson,Charles,E., Rivest, Ronald L., Stein,"Clifford Introduction to Algorithms", MIT Press and McGraw-Hill. ISBN 0-262-03384-4,2001.
- [17] online document [accessed on 2013] available at <http://www.mathwarehouse.com/geometry/triangles/area>