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A New Spectral Mittag-Leffler Weighted Method to Solve Variable Order Fractional Differential Equations

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ABSTRACT: ***Background:*** *In many studies, spectral methods based on one of the orthogonal polynomials and weighted residual methods (WRMs) have been used to convert the variable-order fractional differential equation (VO-FDEs) into a system of linear or nonlinear algebraic equations, and then solve this system to obtain the approximate solution.* ***Objective:*** *In this paper, a numerical method is presented for solving VO-FDEs. The proposed method is based on Chelyshkov polynomials (CPs).* ***Methods:*** *WRMs are used to obtain approximate solutions of the governing differential equations. In addition, a new weight function based on Mittag-Leffler functions is proposed. The proposed method is applied to a group of linear and non-linear examples with initial and boundary conditions.* ***Results:*** *Acceptable results are obtained in most of the examples. In addition, the effect of polynomials (such as Chebyshev, Jacobi, Legendre, Gegenbauer, Hermite, Taylor, Mittag-Leffler, and Bernstein polynomials) on the accuracy of the approximate solution is studied, and their effect was found to be minimal in most tests. The effect of the proposed weight function is also studied in comparison with the weight functions presented in WRMs, and it was found that it has a strong effect in most examples.* ***Conclusions:*** *The results obtained indicate that the proposed method is effective for solving equations of variable order.*

KEYWORDS: *Variable order derivative; Weighted residual method; Spectral method; Mittag-Leffler weight function; Chelyshkov polynomials*

INTRODUCTION

Variable order fractional differential equations (VO-FDEs) are one of the most important mathematical models used to describe real-world problems. Variable order (VO) is used in problems where materials change as a result of temperature, time, or any other variable. Examples include problems involving the change that occurs as a result of deformation in materials, or the transition from linear dynamics to nonlinear behavior [1].

The term variable order was first introduced in 1993 by Samko and Ross [2]. The theoretical framework was developed for the first time in 2011 by scientists Atanakovic and Pilipovic [3]. Many studies were conducted to determine the possibilities that can be utilized when applying variable order. It was found that the external forces remain constant, but an accurate description of the internal behavior of the system is obtained during the interactions. Therefore, the use of VO leads to a more accurate expression than the use of constant order (CO) describing a steady state [4].

Mathematically, the VO fractional derivative has several definitions. Since the initial and boundary conditions have integer orders, Caputo's definition is the appropriate definition of VO [5]. Despite the important practical utility of these terms, they still require significant computational efforts compared to CO terms. In addition, analytical solutions may be difficult to find in many cases. Therefore, numerical methods are considered a suitable alternative to find approximate solutions to the required problems [6].

Spectral methods are considered one of the numerical methods that have proven their efficiency in finding numerical solutions to VO-FDEs. In most applications, only the collocation method (COM) is relied upon to find approximate solutions. The reason of using COM may be to reduce the large computational efforts during the implementation of numerical algorithms. So, it is possible to use another Methods of weighted residual, developing new numerical methods, and proposing a weight function that leads to accurate results in most tests.

A spectral collocation method based on Bernstein, Bernoulli, Chebyshev, Genocchi, Legendre, Jacobi, shifted Legendre, shifted Jacobi , shifted fractional Jacobi, Bessel, Haar functions and Prabhakar polynomials is derived to solve linear and non-linear VO-FDEs [7]–[18]. In addition, the non-orthogonal Bernstein polynomial was created without utilizing the COM to solve VO-FDEs [19]. A COM method based on Chebyshev polynomials of the first and third types was applied to solve multi-term Delay and present VO-FDEs [20], [21]. The modified generalized Laguerre and Chebyshev polynomials with COM were used to solve the variable-order Bagley-Torvik equation [22], [23]. Haar wavelet and Legendre wavelet polynomials with COM were provided to solve VO-FDEs [24], [25].

Weighted residual methods (WRMs) convert differential equations into algebraic systems and aim to reduce error to a minimum. The most famous of these methods are Galerkin method, subdomain method, least squares method, and collocation method. Although these methods have succeeded in solving many problems, it is possible to propose new numerical schemes based on new weight functions, which are faster in convergence and more accurate than using the grouping method. In spectral methods, one of the orthogonal polynomials is often relied upon, through which we obtain the final form of the required solution.

In this work, we rely on Chelyshkov polynomials and compare the results with other commonly used functions such as Chebyshev polynomials of the first and second types, Legendre, Jacoby, Tayler, Bernstein, Gegenbauer, Hermite, Laguerre and Mittag-Leffler polynomials. Furthermore, WRMs are applied to solve linear and nonlinear VO-FDEs with initial and boundary conditions. In addition, the Mittag-Leffler function is proposed as a new weight function, and the obtained results are compared with other weight functions in WRMs. The following section includes the basic concepts, followed by the methodology, then the numerical examples, and finally the conclusions.

FUNDEMANTAL CONCEPT

Definition 1 Suppose $f(t)$ is an analytic function in the interval $I=[0,b]$, the Caputo fractional derivative of variable order (CFDVO) with $n-1<\alpha(t)<n$ of a function f is defined by [26]:

$$ {}_0^cD_t^{\alpha(t)}f(t)=\frac{1}{\Gamma\big(n-\alpha(t)\big)}\int_0^t (t-u)^{n-1-\alpha(t)}f^{(n)}(u)du, t>0. \tag{1}$$

For the special case where $f(t)=t^k, 0<\alpha(t)<1$. The CFDVO can be obtained as follows:

$$ {}_0^CD_t^{\alpha(t)}t^k=\begin{cases}\dfrac{\Gamma(k+1)}{\Gamma\big(k+1-\alpha(t)\big)}\,t^{k-\alpha(t)}, & k\geq 1, k\in N\\ \\ 0 & otherwise\end{cases}. \tag{2}$$

Definition 2 The Chelyshkov polynomials (CPs) are defined in the interval [0,b] as follows [27], [28]:

$$ C_{Nn}(t)=\sum_{k=n}^{N}\frac{(-1)^{k-n}}{b^k}\binom{N-n}{k-n}\binom{N+k+1}{N-n}t^k\,, \tag{3}$$

where, $t\in[0,b]$ and $b>0, n\in\{0,1,...,N\}$.

By applying the CFDVO to Eq. (3) , we obtain

$$ D^{\alpha(t)}C_{Nn}(t)=\sum_{k=n}^{N}\frac{(-1)^{k-n}}{b^k}\binom{N-n}{k-n}\binom{N+k+1}{N-n}\frac{\Gamma(k+1)}{\Gamma(k-\alpha(t)+1)}t^{k-\alpha(t)}, \tag{4}$$

where $k\geq 1, n\in\{1,...,N\}$ and $0<\alpha(t)<1$.

The CPs have the orthogonality properties with weights $w(t)=1$. This means that

$$ \int_0^b C_{Nn}(t)C_{Nk}(t)\,dt=\begin{cases}\frac{b}{2n+1}, & \text{if } n=k,\\ 0, & \text{otherwise.}\end{cases}$$

Definition 3 The general form of VO-FDEs is considered as follows [21]:

$$D^{\alpha(t)}u(t) = F\big(t, u, D^{\alpha_1(t)}u, ..., D^{\alpha_n(t)}u, D^{\beta_1}u, ..., D^{\beta_n}u\big), t \in [0, \text{b}], \tag{5}$$

where, $0 < \alpha_1(t) < \cdots < \alpha_n(t) < \alpha(t) < 1$, and $\beta_1, ..., \beta_n$ are constants.

If the operator F is linear then Eq.(5) is called linear VO-FDE, otherwise it is called non-linear VO-FDE. If Eq. (5) satisfies the following initial conditions:

$$u(0) = \gamma_1 \,, u'(0) = \gamma_2, ..., u^{(n-1)}(0) = \gamma_n, \tag{6}$$

where $\gamma_0, ..., \gamma_n$ are constants, and $n = \max_i\{\lceil\beta_i\rceil\}$, then the problem (5) along with (6) is called initial value problem (IVP). While if Eq. (5) satisfies the following conditions:

$$u(0) = \gamma_1, u'(0) = \gamma_2, ..., u^{(m)}(0) = \gamma_m, u(b) = \rho_0, u'(b) = \rho_1, ..., u^{(k)}(\text{b}) = \rho_k, \quad m + k = n, \tag{7}$$

then it is called a boundary value problem (BVP).

Definition 4 The spectral approximate solution to the problems defined in Eq.(5) can be written as follows:

$$u(t) \approx u_N(t) = \sum_{k=0}^{N} \sigma_k C_{Nk}(t), \tag{8}$$

where $C_{Nk}(t)$ are the Chelyshkov polynomials.

Multiplying both sides of Eq. (8) by $C_{Nn}(t)w(t)$ and then integrating from 0 to b, we obtain

$$\int_0^b u_N(t)C_{Nn}(t)w(t)dt = \sum_{k=0}^{N} \sigma_k \int_0^b C_{Nk}(t)C_{Nn}(t)w(t)\, dt\,.$$

The following result is obtained by applying the orthogonality property:

$$\sigma_n = \frac{(2n+1)}{b} \int_0^b u_N(t)C_{Nn}(t)w(t)dt\,, n = 0, 1, ..., N. \tag{9}$$

That is:

$$\sigma_n = \frac{\langle u, C_{Nn}\rangle_w}{\langle C_{Nn}, C_{Nn}\rangle_w}. \tag{10}$$

Weighted Residual Method (WRMs)

Suppose $U(t)$ is an approximate solution to the VO-FDE which is defined in Eq.(5), substituting $U(t)$ into this equation, we obtain the residual error $R(t)$ as follows [29]:

$$R(t) = D^{\alpha(t)}U(t) - F\big(t, U, D^{\alpha_1(t)}U, ..., D^{\alpha_n(t)}U, D^{\beta_1}U, ..., D^{\beta_n}U\big). \tag{11}$$

Clearly, $R(t) \neq 0$. If the solution to the given differential equation exists and is unique, then any function that makes $R(t)$ approach zero is an appropriate approximate solution to Eq.(5)

Theorem (1) suppose $R(t)$ is a function defined on $H(a, b)$, and let the value of the following integral be verified for any given positive function $w(t)$ defined on $H(a, b)$ [29], i.e.,

$$\int_a^b R(t)w(t)dt = 0, \tag{12}$$

then, $R(t) = 0, \forall\ t \in (a, b)$.

From Theorem 1, several methods can be deduced based on the weight function. Among the most famous methods that rely on this theorem are the Galerkin method (GLM), the subdomain method (SDM), the least squares method (LSM), the collocation method (COM), and the momentum method (MOM) [30].

MATERIALS AND METHODS

New Mittag-Leffler Weight Method (MLWM)

The Mittag- Leffler function of two parameters α and β is defined by [31]

$$E_{\alpha,\beta}(t)=\sum_{j=0}^{\infty}\frac{t^j}{\Gamma(\alpha j+\beta)}. \tag{13}$$

In this research, the truncated Mittag-Leffler function is proposed to be the weight function, which is defined by

$$M_k(t)=\sum_{j=0}^{k}\frac{t^j}{\Gamma(j+1)}. \tag{14}$$

Convergence Analysis

Proposition (1) if the weight function is defined as $w_{\mathrm{k}}(\mathrm{t})=M_k(t)$ in the integral represented by Eq.(12), then the residual $R(t)$ is vanished.
Proof: For all $k=1,2,...,N$, the functions $M_k(t)$ are analytic on $H(0,b)$. Suppose $R(t)\neq 0$, then either $R(t)>0$, or $R(t)<0$. If it is positive for some subinterval $[a_1,b_1]$ in $[0,b]$, since $M_k(t)>0,\forall k$, and $M_k(t)=0$ only if $t=0$, the value of the following integral is positive and not equal to zero:

$$\int_0^b R(t)M_k(t)dt=\int_{a_1}^{b_1}R(t)\sum_{j=0}^{k}\frac{t^j}{\Gamma(j+1)}dt>0, \tag{15}$$

which is contradicts the hypotheses. The same approach is followed when $R(t)<0$. Therefore $R(t)=0$.

The Proposed Spectral Mittag-Leffler Residual Method (SMLRM-CPs)

Suppose the approximate solution of Eq.(5) can be written as spectral Chelyshkov polynomials as follows:

$$u_N(t)=\sum_{j=0}^{N}q_jC_{Nj}(t). \tag{16}$$

Substituting (16) into (5), we get

$$\sum_{j=0}^{N}q_jD^{\alpha(t)}C_{Nj}(t)=F\left(t,\sum_{j=0}^{N}q_jC_{Nj}(t),D^{\alpha_1(t)}\sum_{j=0}^{N}q_jC_{Nj}(t),\ldots,\right.$$
$$\left.D^{\alpha_n(t)}\sum_{j=0}^{N}q_jC_{Nj}(t),D^{\beta_1}\sum_{j=0}^{N}q_jC_{Nj}(t),\ldots,D^{\beta_n}\sum_{j=0}^{N}q_jC_{Nj}(t)\right). \tag{17}$$

Multiplying Eq.(17) by the Mittag-Leffler weights $M_i(t),i=1,2....,N-r$, where r is the number of initial or boundary conditions, and integrating from 0 to $b,b>0$, yields

$$\int_0^b\sum_{j=0}^{N}q_jD^{\alpha(t)}C_{Nj}(t)M_i(t)dt$$
$$=\int_0^b F(t,\sum_{j=0}^{N}q_jC_{Nj}(t),D^{\alpha_1(t)}\sum_{j=0}^{N}q_jC_{Nj}(t),...,D^{\beta_n}\sum_{j=0}^{N}q_jC_{Nj}(t))M_i(t)dt. \tag{18}$$

which leads to a system of $N-r$ nonlinear equations.
The fully discrete scheme of $N-r$ nonlinear algebraic equations is represented as following:

$$\sum_{h=1}^{N_h} v_h \Bigg[\sum_{j=0}^{N} q_j D^{\alpha(t_h)} C_{Nj}(t_h) - F\left(t_h, \sum_{j=0}^{N} q_j C_{Nj}(t_h), D^{\alpha_1(t_h)} \sum_{j=0}^{N} q_j C_{Nj}(t_h), ..., D^{\beta_n} \sum_{j=0}^{N} q_j C_{Nj}(t_h) \right) \Bigg] M_i(t_h) = 0, \tag{19}$$

where $N_h = \frac{b}{\hbar}$, and v_h are the weights of the Simpson's quadrature integration. By applying boundary conditions, we obtain

$$\sum_{j=0}^{N} q_j C_{Nj}(0) = \gamma_1 , ..., \sum_{j=0}^{N} q_j C_{Nj}^{(r-1)}(0) = \gamma_r, \tag{20}$$

$$\sum_{j=0}^{N} q_j C_{Nj}(0) = \gamma_1, ..., \sum_{j=0}^{N} q_j \mathrm{C}_{Nj}^{(m)}(0) = \gamma_m,$$

$$\sum_{j=0}^{N} q_j C_{Nj}(\mathrm{b}) = \rho_0, ..., \sum_{j=0}^{N} q_j \mathrm{C}_{Nj}^{(k)}(\mathrm{b}) = \rho_k, \quad m+k=r. \tag{21}$$

The approximate solution is obtained by solving the previous system of N nonlinear algebraic equations. If F is a linear operator, then the system of N linear algebraic equations is computed. The linear algebraic system is directly solved, but the nonlinear system is solved by using newton's method.

Error Analysis

Suppose $D^n u(t) \in C[0,1], n = 0, 1, ..., N+1$. Let $u_N(t)$ be the best approximate solution to the problem generated by the proposed MLWM-CPs, and let the exact solution $u(t)$ be expanded as a generalized Taylor's formula

$$u(t) \approx U(t) = \sum_{k=0}^{N} \frac{t^k}{\Gamma(k+1)} D^{kv} u(0^+) , t \in (0,1]. \tag{22}$$

From Taylor series, the bound of error is given by:

$$\begin{aligned} |u(t) - U(t)| &= \left| \frac{t^{(N+1)}}{\Gamma\Big((N+1)+1\Big)} D^{(N+1)} u(\xi) \right|, \xi \in (0,1] \\ &\le \frac{Q_v t^{(N+1)}}{\Gamma\big((N+1)+1\big)}, \end{aligned} \tag{23}$$

where, $Q_v = \sup_{0<t\le 1} \left\{ \left| D^{(N+1)} u(t) \right| \right\}$

Since $u(t)$ and u_N belong on $S_N = \{C_{N0}, C_{N1}, ..., C_{NN}\}$, then

$$\begin{aligned}\|u - u_N\|_w^2 \le \|u - U\|_w^2 &= \int_0^1 \Big(u(t) - U(t)\Big)^2 w(t)dt \\ &\le \int_0^1 \left(\frac{Q_v t^{(N+1)}}{\Gamma\Big((N+1)+1\Big)}\right)^2 w(t)dt \\ &= \left(\frac{Q_v}{\Gamma\Big((N+1)+1\Big)}\right)^2 \int_0^1 \Big(t^{(N+1)}\Big)^2 dx \\ &= \left(\frac{Q_v}{\Gamma\Big((N+1)+1\Big)}\right)^2 \frac{1}{(2N+3)}. \qquad (24)\end{aligned}$$

Hence,

$$\|u - u_N\|_w \le \frac{Q_v}{\Gamma\Big((N+1)+1\Big)} \sqrt{\frac{1}{(2N+3)}}. \qquad (25)$$

NUMERICAL EXAMPLES

In this section, the proposed method (SMLRM-CPs) is applied to a set of examples using the MATLAB 2020a, and the results are compared using the following criterion:

$$RMS = \sqrt{\sum_{i=1}^{M} \frac{(u(t_i) - u_N(t_i))^2}{M}}, \qquad (26)$$

where $t_i \in [0, b], \forall i$, and u and u_N are the exact and approximate solutions, respectively.

Example 1 Consider the following linear VO- fractional Bagley-Torvik differential equation [9]

$$u''(t) + {}_0^c D_t^{\alpha(t)} u(t) + u(t) = 6\frac{t^{3-\alpha(t)}}{\Gamma\Big(4 - \alpha(t)\Big)} + t^3 + 7t + 1, t \in [0, \pi].$$

with the initial conditions $u(0) = u'(0) = 1, \alpha(t) = 1 + 0.5|\sin(t)|$. The exact solution is $u(t) = t^3 + t + 1$. Applying the proposed method (SMLRM-CPs) when $N = 3$, we obtain the following approximate solution:

$$u_N(t) = t^3 + t + 1 - 1.5941 \times 10^{-18} t^2,$$

with $RMS = 1.6913 \times 10^{-16}$.

Example 2 Consider the following nonlinear IVP of VO-FDE

$${}_0^C D_t^{\alpha(t)} u(t) + \sin(t) u^2(t) = 6\frac{t^{3-\alpha(t)}}{\Gamma(4 - \alpha(t))} + t^6 \sin(t), t \in [0, 1].$$

with the initial condition $u(0) = 0, \alpha(t) = 1 - 0.5\mathrm{e}^{-\mathrm{t}}$ and the exact solution $u(t) = t^3$. Applying the proposed method (SMLRM-CPs) when $N = 3$, we obtain the following approximate solution $u_N(t) = t^3 - 3.16 \times 10^{-10} t^2 + 1.21 \times 10^{-10} t$, with $RMS = 2.2696 \times 10^{-12}$.

Example 3 Consider the following linear functional BVP of VO-FDE [9]

$${}_0^c D_t^{\alpha(t)} u(t) + \cos(t)\, u'(t) + 4u(t) + 5u(t^2) = h(t), t \in [0, 4].$$

with boundary conditions $u(0) = 0, u(4) = 16, \alpha(t) = 0.25(5 + \sin(t))$. The exact solution is $u(t) = t^2$, and the non-homogeneous part is

$$h(t) = 2tcos(t) + 5t^4 + 4t^2 + 2\,\frac{t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))}.$$

Applying the proposed method (SMLRM-CPs) when $N = 2$, we obtain the following approximate solution $u_N(t) = t^2 - 2.34 \times 10^{-12}t$, with $RMS = 4.908 \times 10^{-13}$.

Example 4 Suppose the following delay BVP of VO-FDE [15]

$${}_{0}^{c}D_t^{\alpha(t)}u(t) + e^t u'(t) + 2u(t) + 8u(t-1) = g(t), t \in [0, 1]$$

with boundary conditions $u(0) = 4, u(1) = 9, \alpha(t) = 0.25(6 + \cos(t))$. The exact solution is $u(t) = t^2 + 4t + 4$, and the non-homogeneous part is $g(t) = 2te^t + 4e^t + 32e^{t-1} + 8e^{2(t-1)} + 32 + 2t^2 + 8t + 8 + 2\frac{t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))}$. Applying the proposed method (SMLRM-CPs) when $N = 2$, we obtain the following approximate solution $u_N(t) \approx t^2 + 4t + 4$, with $RMS = 1.3257 \times 10^{-13}$.

Example 5 Consider the following linear VO-FDE [32]

$${}_{0}^{C}D_t^{\alpha(t)}u(t) + u(t) = \sum_{k=1}^{\infty} \frac{t^{k-\alpha(t)}}{\Gamma\big(k + 1 - \alpha(t)\big)} + e^t,\ t \in [0, 1],$$

with initial conditions $u(0) = 1, \alpha(t) = 0.5|\sin(t)|$ and the exact solution $u(t) = e^t$. The following approximate solution is obtained using MLWM-CPs with $N = 10, u_{10}(t) = 1 + t + 0.5t^2 + 0.1667t^3 + 0.0417t^4 + 0.0083t^5 + 0.0014t^6 + 1.9777 \times 10^{-4}t^7 + 2.5509 \times 10^{-5}t^8 + 2.2689 \times 10^{-6}t^9 + 4.5955 \times 10^{-7}t^{10}$. Figure 1 shows the approximate solution with errors in examples 1 to 5 according to the criteria mentioned in front of each example. It is noted that most of the results have an acceptable error rate of less than 10^{-10}.

In this example, the effect of N on the solutions is explained as shown in Table 1. In addition, a numerical comparison between the WRMs is presented, as shown in Table 2. Furthermore, Table 3 shows the effect of the basic functions on the approximate solution. Finally, Table 4 illustrate the effect of variable orders on the approximate solution.

Table 1. The effect of the value of N of the approximate solution using MLWM-CPs when $v = 1$

N	RMS	Time (s)
2	0.0023	2.42
4	6.7897×10^{-5}	4.64
6	8.8947×10^{-9}	9.43
8	6.9442×10^{-12}	12.72
10	1.0431×10^{-14}	20.23

Table 2. The effect of weighted residual methods on the approximate solution when $N = 7$

Weighted residual method (WRM)	**RMS**	**Time (s)**
GLM -CPs	2.5043×10^{-9}	10.10
COM -CPs	7.2615×10^{-9}	5.08
SDM -CPs	1.4983×10^{-9}	10.21
MNM -CPs	2.4908×10^{-10}	10.11
MLWM -CPs	2.4658×10^{-10}	10.92
LSM -CPs	2.5456×10^{-10}	9.98

Table 3. The effect of the expansion function on the approximate solution using MLWM, with $N = 7$

Polynomials	**RMS**	**Time (s)**
Chelyshkov	2.4658×10^{-10}	10.9
Chebyshev of first kind [33]	2.4658×10^{-10}	19.62
Chebyshev of second kind [34]	2.4658×10^{-10}	19.62
Bernstein [35]	$2.4656 \times 10 - 10$	25.53
Legendre [36]	2.4658×10^{-10}	19.08
Jacobi [37]	2.4658×10^{-10}	19.26
Gegenbauer [38]	2.4658×10^{-10}	19.17
Hermite [39]	2.4658×10^{-10}	19.39
Laguerre [40]	2.4658×10^{-10}	18.45
Taylor	2.4658×10^{-10}	16.12
Mittag-Leffler	2.4658×10^{-10}	20.26

Table 4. A numerical comparison showing the effect of VO on the accuracy of the approximate solution using MLWM-CPs with $N = 7$

$\alpha(t)$	$0.5\lvert\sin(t)\rvert$	$0.5\lvert\cos(t)\rvert$	$0.1 + 0.5t$
RMS	2.4658×10^{-10}	3.4629×10^{-10}	2.9034×10^{-10}
Time (s)	10.92	11.38	7.13

Through the comparison shown in the Tables 1-4, it is clear that the effect of the basic functions is small, while the weighted residual methods have a strong effect on the accuracy of the solutions. In addition, the appropriate choice of the number of functions is directly proportional to the accuracy of the approximate solution. Table 1 confirms the results obtained in [4]. Thus, the proposed method is capable of solving this type of linear and non-linear examples, subject to initial or boundary conditions of variable order.

CONCLUSION

In this research, a spectral numerical method based on the Mittag-Leffler weight function is proposed. The approximate solution is based on Chelyshkov polynomials. Satisfactory results are obtained in many tests. Convergence and error are analyzed. In Example 5, a numerical comparison is made between polynomials, which are considered the basis of many studies. It was found that their effect is minimal in most tests. The weighted residual methods were compared and it was found that the proposed method was faster in obtaining the required solution and more accurate. We recommend applying the proposed method to solve these types of equations.

SUPPLEMENTARY MATERIAL

None.

AUTHOR CONTRIBUTIONS

Abdulrazzaq T. Abed: Conceptualization, Methodology, Investigation, Formal analysis, Software, Visualization and Writing. Ekhlass S. Al-Rawi: Investigation, Review, Editing and Supervision.

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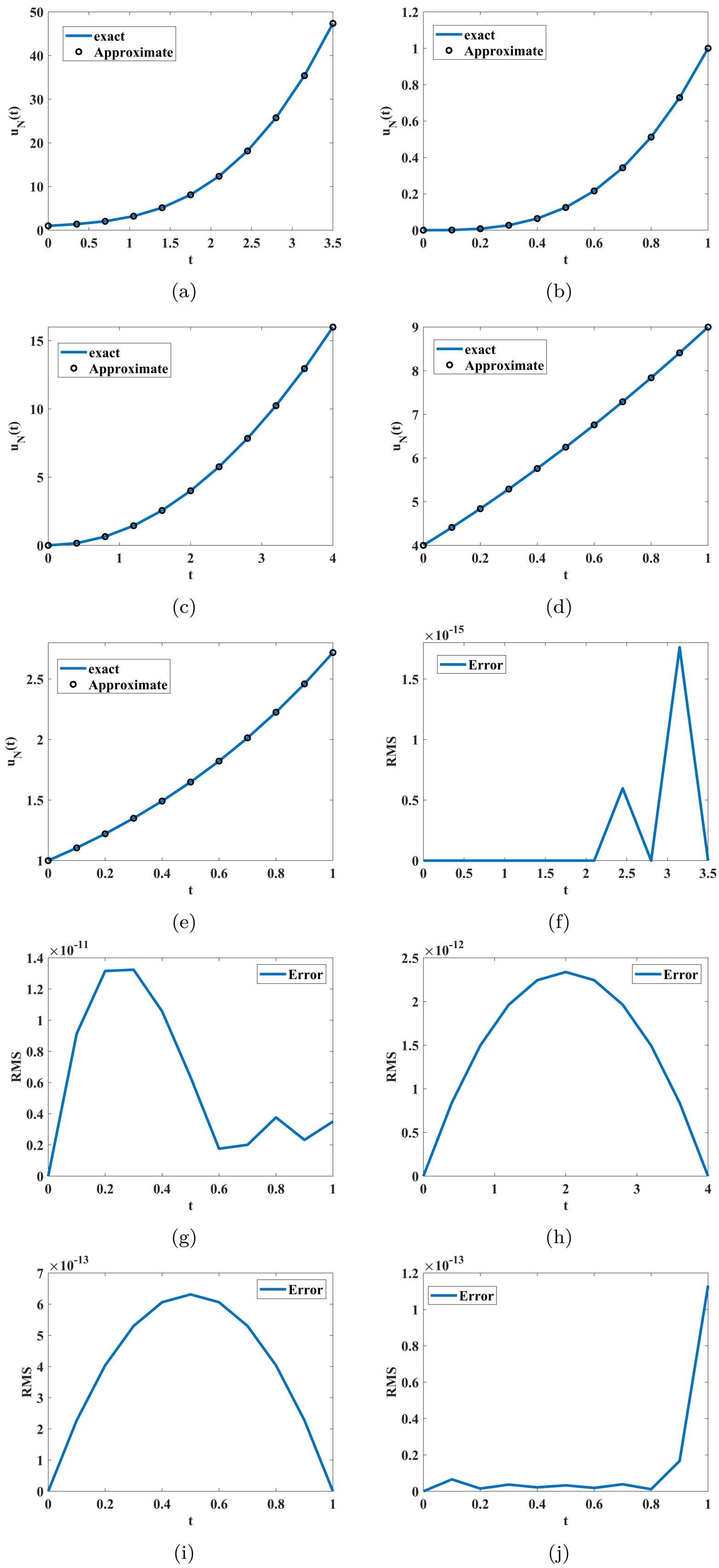

Figure 1. (a)-(e) Comparison between the exact and numerical solutions for Examples 1 to 5, respectively; (f)-(j) Absolute errors of the approximate solutions for Examples Examples 1 to 5, respectively

DATA AVAILABILITY STATEMENT

None.

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CONFLICTS OF INTEREST

The authors declare no conflicts of interest.

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