Semi-Small Submodules

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Abstract

Let R be a commutative ring with unity and let M be an unitary R-module. In this work we present semi-small sub-module concept as a generalization of small submodule. Also we generalize some properties of small submodules to semi-small. And we study the relation between small submodules and semi-small submodules.

1 Semi-Small submodules :

In this section, we introduce a generalization for small submodule concept namely semi-small submodule.

Recall that a proper submodule B of an R-module M is called primary if whenever $rm \in B$ for $r \in R$ and $m \in M$, either $m \in$ B or $r^n \in (B:M)$ for some positive integer n, where $(B:M) = \{ r \in R : rM \subseteq B\}$, [5]. And an ideal E in a ring R is called a primary ideal in R if $xy \in E$, where $\boldsymbol{x}, y \in R$, then either $x^n \in E$ or $y^k \in E$ for some positive integers n and k [10].

Definition 1.1 :

An R-submodule A of an R-module M is called a semi-small submodule of M if and only if $A+B \neq M$, for each primary R-submodule B of M. And an ideal I in a ring R is called a semi-small ideal in R if and only if $I + E \neq R$, for each primary ideal E of R.

Remarks and Examples 1.2 :

Recall that a non-zero R-submodule A of an R-module M is called a small submodule of M if only and only if $A + C \neq M$, for each proper submodule C of M. And a non-zero ideal I in a ring R is called a small ideal if and only if $I + E \neq R$, for each proper ideal E of R[4].

1. Each small submodule is semi-small submodule. We do not know the converse is true or not in general .
2. For each module M , we have (0) is a semi-small submodule of M.
3. If M is a semi-simple module , then (0) is the only semi-small submodule of M.
4. Each finitely generated submodule of Q_Z is semi-small submodule in Q_Z.

Recall that a module M , which satisfies the following conditions:

1. M has a basis.
2. $M = \oplus_{i \in I} A_i$ and $\forall i \in I$ [$R_R \approx A_i$]

is called a free module, [4].

5. In a free Z-module , only the trivial submodule (0) is a semi-small submodule.

The following proposition shows that the two concepts semi-small and small submodule are equivalent in the class of finitely generated modules.

Proposition 1.3 :

Let M be a finitely generated R-module , and let A be an R-submodule of M. Then A is semi-small if and only if A is small.

Proof:

Suppose that A is a semi-small submodule of M and B is a proper R-submodule of M such that $A + B = M$. Since M is finitely generated R-module , so there exists a maximal ideal L such that $B \subseteq L$ [6]. Thus $A + L = M$. But L is a primary R-submodule of M and N is semi-small so $L = M$, a contradiction. Thus $A + B \neq M$ for each proper R-submodule B of M. Hence A is a small R-submodule of M. The converse is clear.

Proposition 1.4 :

Let M be an R-module such that every proper R-submodule of M is primary , and let A be an R-submodule of M. Then A is semi-small if and only if A is small.

Proposition 1.5 :

Let M be an R-module and let N be a semi-small submodule of M. If A is an R-submodule of N , then A is a semi-small submodule of M.

Proof:

Let $A + B = M$, for some primary R-submodule B of M. Since A is an R-submodule of N. Thus $N + B = M$, and because N is semi-small R-submodule of M , $B = M$ a contradiction. Therefore A is semi-small submodule of M.

The converse of prop.1.5 is not true in general as the following example shows.

Example 1.6

Consider Z_{12} as a Z-module , the only primary submodules of Z_{12} are $(\bar{2})$, $(\bar{3})$ and $(\bar{4})$. It's clear that $(\bar{6})$ is submodule of Z_{12} , so $(\bar{6})$ is a submodule of $(\bar{3})$. And $(\bar{6})$ is a semi-small submodule of Z_{12} , but $(\bar{3})$ is not semi-small submodule of Z_{12} , since $(\bar{3}) + (\bar{4}) = Z_{12}$.

Now , we have the following result .

Corollary 1.7 :

The intersection of any two semi-small submodules is also semi-small submodule.

In general , we have the following result

Corollary 1.8 :

Let N be an R-submodule of an R-module M. If A is a semi-small R-submodule of M , then $A \cap N$ is a semi-small R-submodule of M.

Corollary 1.9 :

Let A_i be a semi-small R-submodule of an R-module M for each $i = 1,2,\ldots, n$. Then $\bigcap_{i=1}^{n} A_i$ is a semi-small submodule of M.

Corollary 1.10 :

Let A_i be a submodule of an *R*-module *M* for each *i* = 1,2,…, *n*. If there exists *j* such that A_j is a semi-small submodule of *M* , where $1 \le j \le n$, then $\bigcap_{i=1}^{n} A_i$ is a semi-small submodule of ***M.***

Proposition 1.11 :

Suppose that *A* and *N* are *R*-submodules of an *R*-module *M* such that *N* is small. Then *N* + *A* is a semi-small *R*-submodule of *M* if and only if *A* is a semi-small *R*-submodule of *M*.

Proof :

Suppose *N* + *A* is semi-small and *B* is a primary *R*-submodule of *M*. Thus $(N + A) + B \neq M$, and hence $N + (A + B) \neq M$. From this we get $A + B \neq M$ and consequently , *A* is semi-small.

Conversely ; Assume that *A* is semi-small and *B* is a primary *R*-submodule of *M*. Then $A + B \neq M$. Because *N* is small , $N + (A + B) \neq M$. Therefore $(N + A) + B \neq M$, and hence *N* + *A* is semi-small.

Furthermore , we introduce in this section a generalization for small homomorphism concept namely semi-small homomor- phism.

Recall that an *R*-homomorphism $f : M \to N$, where *M* and *N* are *R*-modules is called a small homomorphism if $f (M)$ is a small submodule of N , [4].

We introduced the following definition for the semi-small homomorphism.

Definition 1.12 :

An *R*-homomorphism $\theta : M \to N$, where *M* and *N* are *R*-modules, is called a semi-small homomorphism , if $\theta(M)$ is a semi-small submodule of *N*.

Before we give the following proposition we will need the following lemma , which appeared in [11].

Lemma 1.13 :

Let *M* and *N* be *R*-modules with $\theta : M \to N$ is an *R*-epimorphism. If *B* is a primary submodule of *M* , such that $\ker \theta \subseteq B$, then $\theta (B)$ is a primary submodule of *N*.

Proposition 1.14 :

Let *M* and *N* be R-modules with $\theta : M \to N$ is an *R*-epimorphism such that $\ker \theta \subseteq B$, for each primary sub-module *B* of *M* , if A is a semi-small submodule of N, then $\theta^{-1} (A)$ is a semi-small submodule of *M*.

Proof :

Suppose $\theta^{-1} (A) + B = M$ for some primary submodule *B* of *M* , then $A + \theta(B) = \theta(M)$, and since θ is an epimorphism , so $\theta(M) = N$. And since $\ker \theta \subseteq B$, therefore $\theta(B)$ is a primary submodule of *N* , (Lemma 1.15). Since *A* is a semi-small submodule of *N* , thus $\theta (B) = N$, this is a contradiction. Therefore $\theta^{-1} (A) + B + \neq M$ for each primary submodule *B* of *M*. Thus $\theta^{-1} (A)$ is a semi-small submodule of *M*.

The converse of prop.1.14 is not true in general unless we put some conditions:

Next we need the following lemma , which appeared in [11].

Lemma 1.15 :

Let *M* and *N* be *R*-modules with $\theta : M \to N$ is an *R*-homomorphism. If *A* is a primary submodule of N , such that $A \subseteq Im\ \theta$, then $\theta^{-1} (A)$ is a primary submodule of *M*.

Proposition 1.16 :

Let *M* and *N* be R-modules with $\theta : M \to N$ is an *R*-epiomorphism . If *A* is a semi-small submodule of *M* such that $\ker \theta \subseteq A$, then $\theta (A)$ is semi-small submodule of *N*.

Proof :

Let $\theta (A) + B = N$ for some primary submodule *B* of *N*. Since θ is an epimorphism , then $\theta^{-1} (N) = M$. Thus $\theta^{-1}\theta (A) + \theta^{-1} (B) = M$. Then $\theta^{-1} (B)$ is a primary submodule of *M* (lemma 1.17). But $\ker\theta \subseteq A$, so $\theta^{-1} \theta (A) = A + \ker \theta = A$. Therefore $A + \theta^{-1} (B) = M$, a contradiction. Thus $\theta(A) + B \neq M$, for each primary submodule *B* of *N* . Hence $\theta(A)$ is a semi-small submodule of *M*.

Proposition 1.17 :

Let *M* be an R-module , and let be *A* is an *R*-submodule of *M*. Then *A* is a semi-small submodule of *M* if and only if the inclusion function $i : A \to M$ is a semi-small monomorphism.

Finally , we give the following proposition.

Proposition 1.18 :

Let *A* and *N* are submodules of an *R*-module *M* such that $A \subseteq N$ and $A \subseteq B$ for each primary submodule *B* of *M* , if *A* is semi-small in *M* , then *N* / *A* is semi-small in *M* / *A* if and only if N is semi-small in *M*.

Proof :

Suppose that *N* / *A* is semi-small in *M* / *A* and suppose that $N + B = M$ for some primary submodule *B* of *M*. Then $(N + B) / A = M / A$ and $N / A + B / A = M / A$. This is a contradiction , since *B* / *A* is a primary submodule of *M* / *A* and *N* / *A* is semi-small in *M*/ *A* , thus $N + B \neq M$. Therefore *N* is semi-small submodule of *M*.

Conversely ; Let *N* be a semi-small *R*-submodule of *M* and $\pi : M \to M / A$ is a natural projective function. Let $N / A + B = M / A$ for some primary submodule *B* of *M* / *A*. Then there exists a primary *R*-submodule *P* in *M* such that $P = \pi^{-1} (B)$. Thus $\pi (P) = B = P / A$ is a primary submodule of *M* / *A* , hence $N / A + P / A =$ M / A , so $(N + P) / A = M / A$ and consequently , $N + P = M$ a contradiction , since *N* is a semi-small *R*-submodule in *M* .So $N / A + B \neq M / A$. Therefore *N* / *A* is a semi-small *R*-submodule of *M* / *A*.

2: Semi-Small Submodules and Multiplication R-modules

An *R*-module *M* is called a multiplication *R*-module if every submodule *N* of *M* is of the form *BM* , for some ideal *B* of *R*[2]. And an *R*-module *M* is called a faithful *R*-module if *ann*(*M)* = 0 , [l].

In the following theorem , we give a condition under which a submodule of a faithful multiplication R-module is semi-small. First , we will need the following lemma , which appeared in [9].

Lemma 2.1 :

If M is a multiplication R-module , and L is a proper submodule of M. Then the following statements are equivalent :

1 - L is a primary submodule of M.

2 - $(L : M)$ is a primary ideal in the ring R.

3- $L = PM$ for some primary ideal P in the ring R , such that $ann(M) \subseteq P$.

In the following theorem , we put a condition on finitely generated faithful multiplication R-modules to find a relation between semi-small submodule and semi-small ideal.

Theorem 2.2 :

Let M be a finitely generated faithful multiplication R-module and let $A = IM$ be a proper R-submodule of M. Then I is a semi-small ideal in R if and only if A is a semi-small submodule of M.

Proof :

Assume I is a semi-small ideal in R , and let $A + B = M$ for some primary submodule B of M , since M is a multiplication R-module , then there exists a primary ideal E in R , such that $B = EM$ (Lemma 2.1). Thus $A + B = IM + EM = M = RM$, and so $(I + E) M) = RM$. But M is a multiplication R-module , so $I + E = R$. This is a contradiction. Therefore A is a semi-small submodule of M.

Conversely ; assume A is a semi-small submodule of M , and let $I + E = R$ for some primary ideal E in R. Since M is a multiplication R-module , then $IM + EM = RM$, and since $ann(M) \subseteq E$ for each primary ideal E in R , then EM is a primary submodule of M (Lemma 2.1). Thus $A + EM = RM =$ M. This is a contradiction , since A is a semi-small. Therefore I is a semi-small ideal in R.

The following corollary follows directly from the previous theorem.

Corollary 2.3 :

Let M be a finitely generated faithful multiplication R-module and let A be a proper submodule of M. Then A is a semi-small R-submodule of M if and only $(A : M)$ is a semi-small ideal in R.

The following proposition shows that the semi-small concept and small concept are equivalent in the class of multiplication module.

Proposition 2.4 :

Let M be a multiplication R-module and let A be an R-submodule of M. Then A is semi-small if and only if A is small.

Proof :

The proof is similar to the proof of proposition 1.3 , hence is omitted.

Furthermore , we study in this section the ascending chain condition (Acc) M is said to be satisfy the ascending chain if each ascending chain of submodule of M terminate. Moreover , M is called Noetherian module if and only if M satisfies Acc. And M is said to be satisfy the descending chain condition (Dcc) if each descending chain of submodules of M terminates [7].

We start by the following definition:

Definition 2.5 :

An R-module M is said to satisfy the ascending chain condition (Acc) on semi-small submodules if each ascending chain of semi-small submodules $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$ is terminates.

We preface the section by the following proposition.

Proposition 2.6 :

Let M be a finitely generated faithful multiplication R-module , then R satisfies Acc on semi-small ideal if and only if M satisfies Acc on semi-small submodules.

Proof :

Let R satisfies *Acc* on semi-small ideal , and let $A_1 \subseteq A_2 , \subseteq \ldots . \subseteq A_k \subseteq \ldots$ be ascending chain of semi-small submodules of M. Since M is a multiplication R-module ,then $A_i = B_i M$ for some semi-small ideal B_i of R, for each i (Th.2.2). Hence $B_1 M \subseteq B_2 M \subseteq \ldots \subseteq B_k M \subseteq \ldots$. But M is a finitely generated faithful module , then $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_k \subseteq \ldots$ is ascending chain of ideals in R [7]. Since R satisfies Acc on semi-small ideals , then there exists $k \in N$ such that $B_k = B_{k+j}$, $\forall j \geq 1$,and hence $A_k=A_{k+j}$, $\forall j \geq 1$. Thus M satisfies Acc on semi-small submodules.

Conversely ; let M satisfies Acc on semi-small submodules and let $B_1 \subseteq B_2 \subseteq \ldots \subseteq \ldots$ be ascending chain of semi-small ideals in R. Then $B_1 M \subseteq B_2 M \subseteq \ldots \subseteq B_R M \subseteq \ldots$ is ascending chain of semi-small submodules of M , (Th.2.2). Since M satisfies Acc on semi-small submodules, there exists k such that $B_k M= B_{k+j} M$, $\forall j \geq 1$. But M is a finitely generated faithful multiplication module , then $B_k = B_{k+j}$, $\forall j \geq 1$ [3]. Thus R satisfies Acc on semi-small ideals.

Now , we give the main theorem of this section.

Theorem 2.7 :

Let M be a finitely generated faithful multiplication R-module , then the following are equivalent :

1. M satisfies Acc on semi-small R-submodules of M.
2. R satisfies Acc on semi-small ideals.
3. $S = End_{\boldsymbol{R}} (M)$ satisfies Acc on semi-small ideals.
4. M satisfies Acc on semi-small submodules as an S-modules.

Proof :

(1) $\Rightarrow$ (2) by Prop.2.5

(2) $\Rightarrow$ (3) since M is a finitely generated faithful multiplication R-module , then $R \approx S$ [8]. But R satisfies Acc on semi-small ideals , thus S satisfies Acc on semi-small ideals.

(3) $\Rightarrow$ (4) by Prop.2.5

(4) $\Rightarrow$ (1) by Prop.2.5 , S satisfies Acc on semi-small ideals , and $R \approx S$ [8] , hence R satisfies Acc on semi-small submodules as on R-module.

Definition 2.8 :

An R-module M is said to satisfy the descending chain condition (Dcc) on semi-small submodules if each descending chain of semi-small submodules $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n \supseteq \dots$ terminates.

Proposition 2.9 :

Let M be a finitely generated faithful multiplication R-module . Then M satisfies Dcc on semi-small submodules if and only if R satisfies Dcc on semi-small ideals.

Finally , we get the following result.

Theorem 2.10 :

Let M be a finitely generated faithful multiplication R-modules . Then the following statements are equivalent :

1. M satisfies Dcc on semi-small submodules as an R-modules.
2. R satisfies Dcc on semi-small ideals.
3. $S = End_R (M)$ satisfies Dc**c** on semi-small ideals.
4. M satisfies Dcc on semi-small submodules as an S-modules.

Proof :

The proof is similar to the proof of Th.2.6 , hence is omitted.

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الملخص

لتكن R حلقة تبادلية ذات عنصر محايد و M مقاس أحادي على R. قدمنا في هذا البحث مفهوم المقاس الجزئي شبه- الصغير بصفته أعماما إلى مفهوم المقاس الجزئي الصغير.و عممنا بعض خصيصات المقاسات الجزئية الصغيرة إلى المقاسات الجزئية شبه- الصغيرة وكذلك درسنا العلاقة بينهما.