J – Class composition operators

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Abstract

Let X be a separable Banach space. For every $x \in X$, $L(x)$, $J(x)$, $J^{mix}(x)$ are denoted to the limit set ,extended limit set , extended mixing limit set of x under the operator T respectively. In this paper we give several properties about these sets and we study these sets for the composition operators C_φ ,also we give some conditions on φ to make C_φ is J – class operator.

1. Introduction:

We see in the last years that the dynamics of linear operators on infinite dimensional spaces has been extensively studied [5], [6]. Recall that if X is separable Banach space and $T: X \to X$ is bounded linear operator then T is said to be hypercyclic provided there exists a vector $x \in X$ such that its orbit under T, $Orb(T,x) = \{T^n x : n = 0,1,2,...\}$ is dense in X, in this case the vector x is called hypercyclic vector [1] .On the other hand if there exists $x \in X$ such that $Orb(T,x)$ has dense linear span in X then T is said to be cyclic operator and the vector x is called cyclic vector . If X is Banach space (possibly non – separable) and $T: X \to X$ is a bounded linear operator then T is said to be topologically transitive (topologically mixing) if for every pair of non – empty open subsets U, V of X there exists a positive integer n such that $T^n U \cap V \neq \varnothing$ ($T^m U \cap V \neq \varnothing$ for every $m \geq n$ respectively) [2]. One can prove easily that if T is a bounded linear operator acting on separable Banach space X then T is hypercyclic if and only if T is topologically transitive [2]. Let us assume from now and so on that X is a separable complex Banach space. If x is a vector in X and $T: X \to X$ is bounded linear operator then define : $J(x) = \{y \in X :$ there exist a strictly increasing sequence of positive integers $\{k_n\}$ and a sequence $\{x_n\} \subset X$ such that $\{x_n\} \to x$ and $\{T^{k_n} x_n\} \to y$ $\}$, and $J^{mix}(x) = \{y \in X :$ there exists a sequence $\{x_n\} \subset X$ such that $\{x_n\} \to x$ and $\{T^n x_n\} \to y\}$ denote the extended limit sets and extended mixing limit set of x under T respectively, see [2] for more details . We study the dynamics of operators by replacing the orbit of a vector with its extended limit set and extended mixing limit set. George Costakis and Antonios Manoussos proved that the operator $T: X \to X$ is hypercyclic if and only if $J(X) = X$ for every $x \in X$ [2]. An operator $T: X \to X$ will be call a J – class (J^{mix} – class) operator provided there exists a non – zero vector $x \in X$ so that the extended limit set of x under T (the extended mixing limit set of x under T) is the whole space i.e. $J(x) = X$ ($J^{mix}(x) = X$). It is clear from the above discussion that every hypercyclic operator is J – class operator, but the converse is not necessarily true [2].

Suppose that U is the unit ball of the complex numbers and H (U) is the set of all holomorphic functions on U, it is well known that every function f belongs to H (U) can be written as $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n, (z \in U)$. If the sequence of the coefficients $\{\hat{f}(n)\}$ is a square - summable sequence, i.e. $\sum_{n=0}^{\infty} \left|\hat{f}(n)\right|^2 < \infty$, then we say that the function f belongs to H^2 or $H^2(U)$. Therefore $H^2 = \left\{ f \in H(U) : \sum_{n=0}^{\infty} \left|\hat{f}(n)\right|^2 < \infty \right\}$.

H^2 is called the Hardy space [3], [7]. If φ is a holomorphic self map of U (i.e. φ(U) ⊆U) then the operator $C_\varphi : H^2 \to H^2$ defined by $C_\varphi f = f o \varphi$ for every $f \in H^2$ is bounded linear operator on H^2[3], this operator is called the composition operator. The mapping φ is called linear fractional transformation if $\varphi(z) = \frac{az+b}{cz+d}$ for every $z \in U$,where a,b,c,d are complex numbers. We denote LFT(U) to the set of all linear fractional transformations that take U into itself i.e. LFT(U) $= \{\varphi : \varphi$ is linear fractional transformation and $\varphi(U) \subseteq U\}$.

If $\varphi \in$ LFT(U) then φ has either one or two fixed point [3]. If φ is holomarphic self map of U then

1- $\varphi_n = \underbrace{\varphi \circ \varphi \circ \ldots \circ \varphi}_{n-th\ time}$ where $\circ$ is the composition of functions
2- $\varphi_n \xrightarrow{k} p$ means $\varphi_n(z) \to p$ on every compact subset of U.

This paper organized as follows: In section 2 we discuss the definitions of J – sets and study some basic properties of these sets ,also we introduce some information of the composition operators that we needs in the next section. In section 3 we study the J – sets of the composition operators and we give some conditions to make C_φ is J – class operator.

2. Preliminaries:

In this section we introduce some definitions and basic information about the J – sets and J – class operators, Let us assume that X is a separable complex Banach space, we begin with the following definition

Definition (2.1):

Let $T : X \to X$ be a bounded linear operator. For every $x \in X$ the sets

$L(x) = \{y \in X :$ there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that $\{T^{K_n}(x)\} \to y\}$ and

$J(x) = \{y \in X :$ there exists a strictly increasing sequence of positive integers $\{k_n\}$ and a sequence $\{x_n\} \subset X$ such that $\{x_n\} \to x$ and $\{T^{k_n} x_n\} \to y$ $\}$

are denoting the limit set and the extended limit set of x under T respectively.

Remark (2.2) [2]:

It is clear that an equivalent definition of $J(x)$ is the following

$J(x) = \{y \in X :$ for every pair of neighborhoods U, V of x, y respectively, there exists a positive integer n such that $T^n U \bigcap V \neq \emptyset$ $\}$.

Observe now from the above remark and the definition of topologically transitive operator that T is topologically transitive if and only if $J(x) = X \quad \forall x \in X$.

Definition (2.3):

Let $T : X \to X$ be a bounded linear operator. For every $x \in X$ the set

$J^{mix}(x) = \{y \in X :$ there exists a sequence $\{x_n\} \subset X$ such that $\{x_n\} \to x$ and $\{T^n x_n\} \to y\}$ is denote the extended mixing limit set of x under T.

Remark (2.4) [2]:

An equivalent definition of $J^{mix}(x)$ is the following

$J^{mix}(x) = \{y \in X :$ for every pair of neighborhoods U, V of x, y respectively, there exists a positive integer N such that $T^n U \bigcap V \neq \emptyset$ for every $n \geq N\}$.

Observe from remark (2.4) and the definition of topologically mixing operator that T is topologically mixing if and only if $J^{mix}(x) = X \quad \forall x \in X$.

Remark (2.5):

We can prove easily that $L(x) \subseteq J(x)$ and $J^{mix}(x) \subseteq J(x)$ for every $x \in X$.

The following lemma appears in [2].

Lemma (2.6):

Let $T : X \to X$ be a bounded linear operator and $\{x_n\}, \{y_n\}$ are two sequences in X such that $\{x_n\} \to x$ and $\{y_n\} \to y$ for some $x, y \in X$.

(i) If $y_n \in J(x_n)$ for every n=1, 2,…, then $y \in J(x)$.

(ii) If $y_n \in J^{mix}(x_n)$ for every n=1,2,…,then $y \in J^{mix}(x)$.

The following proposition consequence immediate from the above lemma.

Proposition (2.7) [2]:

For every $x \in X$ the sets $L(x)$, $J(x)$ and $J^{mix}(x)$,are closed and T – invariant moreover, the set $J^{mix}(x)$ is convex and especially $J^{mix}(0)$ is a (closed) linear subspace of X.

Recall that the operator T is called power bounded if there exists a positive number M such that $\|T^n\| \leq M$ for every positive integers n.

Proposition (2.8) [2]:

Let $T : X \to X$ be a bounded linear operator. If T is power bounded then $J(x) = L(x) \quad \forall x \in X$.

We prove the following lemmas

Lemma (2.9):

If $T : X \to X$ is power bounded and $\{x_n\} \to x$, $\{y_n\} \to y$. If $y_n \in L(x_n)$ then $y \in L(x)$.

Proof:

It is clear from proposition (2.8) that $L(x_n) = J(x_n) \quad \forall n$, therefore $y_n \in J(x_n)\ \forall n$. Lemma (2.6) shows that $y \in J(x)$.Since $J(x) = L(x)\ \forall x \in X$ then $y \in L(x)$.

Lemma (2.10):

Let $T : X \to X$ be a bounded linear operator then for every $x \in X$, $L(x) = L(T^n x)$, n=0,1,2,…

Proof :

Suppose that $z \in L(x)$ and r is non – negative integer then there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that $\{T^{K_n}(x)\} \to z$, hence $\{T^{k_n - r}(T^r x)\} \to z$. Thus $z \in L(T^r x)$

Conversely: suppose that $z \in L(T^p x)$ for some non – negative integer p, then there exists a strictly increasing sequence of positive integers $\{k_n\}$ such that $\{T^{K_n}(T^p x)\} \to z$, that is $\{T^{K_n + p}(x)\} \to z$. Thus $z \in L(x)$.

We prove the following results .

Theorem (2.11):

Let $T: X \to X$ be a bounded linear operator and $x, y \in X$. If $y \in L(x)$ then $L(y) \subseteq L(x)$

Proof :

Since $y \in L(x)$ then there exist a strictly increasing sequence of positive integers $\{k_n\}$ such that $\{T^{K_n}(x)\} \to y$. If $z \in L(y)$ then there exist a strictly increasing sequence of positive integers $\{F_n\}$ such that $\{T^{F_n}(y)\} \to z$. Let $\varepsilon > 0$, then there exists a positive integer m such that $\left\| T^{F_n} y - z \right\| < \frac{\varepsilon}{2} \quad \forall n \geq m$.

Since $\{T^{K_n}(x)\} \to y$ then $\{T^{K_n + F_m}(x)\} \to T^{F_m}(y)$ as $n \to \infty$, therefore there exists a positive integer q such that $\left\| T^{K_n + F_m}(x) - T^{F_m}(y) \right\| < \frac{\varepsilon}{2} \quad \forall n > q$.

So that

$\left\| T^{K_n + F_m}(x) - z \right\| \leq \left\| T^{K_n + F_m}(x) - T^{F_m}(y) \right\| + \left\| T^{F_m}(y) - z \right\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \varepsilon \ \forall n > q$. This implies that $\{T^{K_n + F_m}(x)\} \to z$ as $n \to \infty$.

Thus $z \in L(x)$.

Corollary (2.12):

If $T: X \to X$ is power bounded and $y \in L(x)$ then $L(y) = L(x)$.

Proof :

We see in theorem (2.11) that $L(y) \subseteq L(x)$.

Since $y \in L(x)$ then there exist a strictly increasing sequence of positive integers $\{k_n\}$ such that $\{T^{K_n}(x)\} \to y$. If $z \in L(x)$ then $z \in L(T^{k_n} x) \ \forall n$ (lemma(2.10)), so that $z \in L(y)$ (lemma (2.9)), so that $L(x) \subseteq L(y)$. Thus $L(y) = L(x)$.

Corollary (2.13):

If $T: X \to X$ is power bounded then either $L(y) = L(x)$ or $L(y) \cap L(x) = \varnothing$ for every $x, y \in X$.

Proof:

If $L(y) \cap L(x) \neq \varnothing$ then there exists $z \in L(y) \cap L(x)$ that is $z \in L(y)$ and $z \in L(x)$, so that $L(y) = L(z) = L(x)$ (corollary (2.12)).

Definition (2.14) [4]:

Suppose that (X,d) , (Y,d') are metric spaces then the mapping $T: X \to Y$ is contraction mapping if there

exists $\theta, 0 \leq \theta < 1$ such that

$d(T(x), T(y)) \leq \theta d'(x, y)$ for every $x, y \in X$

Contraction mapping theorem (2.15) [4]:

Suppose that X is complete metric space and $T: X \to X$ is contraction mapping then there exist a unique point $x_0 \in X$ such that $T(x_0) = x_0$, x_0 is called fixed point for T .

Corollary (2.16):

If $T: X \to X$ is contraction linear operator then 0 is the only fixed point for T .

Proof:

Since T is linear operator then 0 is fixed point for T and hence from the contraction mapping theorem that 0 is the only fixed point for T .

We prove the following theorem

Theorem (2.17):

Suppose that X is Banach space and d is the metric on X induced by its norm i.e.

$d(x, y) = \|x - y\| \quad \forall x, y \in X$, then the operator $T: (X, d) \to (X, d)$ is contraction mapping if and only if $\|T\| < 1$.

Proof:

We know (see [8] p.95) that $\|T\| = \sup\{\|T(x)\| : x \in X \ \textit{satisfies} \ \|x\| = 1\}$.

If T is contraction then by definition there exist θ, $0 \leq \theta < 1$ such that $\|T(x) - T(y)\| \leq \theta \|x - y\|$ for every $x, y \in X$. If we take y = 0 then for every $x \in X$ with $\|x\| = 1$ we have $\|T(x)\| \leq \theta$, thus $\|T\| \leq \theta < 1$.

Conversely: if $\|T\| < 1$ then take $\theta = \|T\|$ and hence $d(T(x), T(y)) = \|T(x) - T(y)\| = \|T(x - y)\|$

$\leq \|T\|\|x-y\| = \theta d(x,y) \quad \forall x,y \in X$. Thus T is contraction.

Corollary (2.18):

If $T : X \to X$ is contraction operator then $L(x) = J^{mix}(x) = J(x) = \{0\}$ for every $x \in X$ and hence T is not J – class operator.

Proof:

Since T is contraction operator then $\|T\| < 1$ (theorem (2.17)) that is $\|T^n\| \leq \|T\|^n < 1$ for every n so that T is power bounded. If $y \in J(x)$ then there exists a strictly increasing sequence of positive integers $\{k_n\}$ and a sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to x$ and $\{T^{k_n}(x_n)\} \to y$. On the other hand $\|T^{k_n}(x_n)\| \leq \|T^{k_n}\|\|x_n\| \leq \|T\|^{k_n}\|x_n\|$. Since $\|T\| < 1$ then $\|T\|^{k_n} \to 0$ and hence $\|T^{k_n}x_n\| \to 0$, this implies that $\{T^{k_n}(x_n)\} \to 0$, so that y = 0. Thus $J(x) = \{0\}$ for every $x \in X$. Since $L(x), J^{mix}(x) \subseteq J(x)$ for every $x \in X$ (remark (2.5)) and by definitions $0 \in L(x), J^{mix}(x) \ \forall x \in X$ then $L(x) = J^{mix}(x) = J(x) = \{0\}$ for every $x \in X$

Definition (2.19) [3]:

Let φ be a holomorphic self map of U (U is the unit ball of the complex numbers) then φ is said to be automorphism if it is one – to – one and onto.

Recall that if p is fixed point for φ (i.e. $\varphi(p) = p$) then p is called interior fixed point if $p \in U$ and boundary fixed point if $p \in \partial U = \{z \in \mathbb{C} : |z| = 1\}$.

Remark (2.20):

Let φ be a linear fractional self map of U then

1- φ is called elliptic if it is automorphisim with interior fixed point in U.
2- φ is called parabolic if it has only one fixed point see [3] , [7] for more details.

Lemma (2.21) [7]:

Let φ be a holomorphic self map of U, then φ is automorphism with interior fixed point (i.e. φ is elliptic) if and only if there exist $\alpha \in \partial U$ such that $\varphi(z) = \alpha z$ for every $z \in U$.

Remark (2.22):

We can prove easily that $C_\varphi^n = C_{\varphi_n}$ for every positive integer n, where $C_\varphi^n = \underbrace{C_\varphi o C_\varphi o \cdots o C_\varphi}_{n-th\ time}$.

Theorem (Dejoy – Wolff) (2.23) [3]:

Suppose that φ is holomorphic self map of U that is not an elliptic automorphism.

(a) If φ has a fixed point $p \in U$ then $\varphi_n \xrightarrow{k} p$ and $|\varphi'(p)| < 1$.

(b) If φ has no fixed point in U, then there is a point $p \in \partial U$ such that $\varphi_n \xrightarrow{k} p$. Furthermore :
- p is a boundary fixed point of φ .
- $0 < \varphi'(p) \leq 1$.

(c) Conversely, if φ has a boundary fixed point p at which $\varphi'(p) \leq 1$ then φ has no fixed points in U, and $\varphi_n \xrightarrow{k} p$.

The fixed point p to which the iterates of φ converge is called the Denjoy – Wolff point of φ .

Remark (2.24):

Let φ be a holomorphic self map of U and $C_\varphi : H^2 \to H^2$ is the composition operator then every constant function is fixed point for C_φ .

The following proposition consequence from the contraction mapping theorem and the above remark.

Proposition (2.25):

The composition operator on H^2 is not contraction mapping.

Theorem (2.26) [3]:

Let φ be a holomorphic self mapping of U then C_φ is bounded operator on H^2 and $\|C_\varphi\| \leq \sqrt{\dfrac{1+\varphi(0)}{1-\varphi(0)}}$.

From the above theorem and theorem (2.17), proposition (2.25) we have the following proposition.

Proposition (2.27):

Let φ be a holomorphic self mapping on U then $1 \leq \|C_\varphi\| \leq \sqrt{\dfrac{1+\varphi(0)}{1-\varphi(0)}}$.

The following theorem appeared in [2].

Theorem (2.28):

Let $T : X \to X$ be an operator acting on a separable Banach space X . The following are equivalent.

(i) T is hypercyclic .
(ii) For every $x \in X$ it holds that $J(x) = X$
(iii) The set $A = \{x \in X : J(x) = X\}$ is dence in X .
(iv) The set $A = \{x \in X : J(x) = X\}$ has non – empty interior.

3. The limit and extended limit set for the composition operators.

In this section we study the limit set and the extended limit set for the composition operators C_φ where φ is holomorphic self map on U.

Theorem (3.1) [7]:
If φ is automorphism with no interior fixed point in U then C_φ is hypercyclic operator.

The following proposition consequence from theorem (3.1) and theorem (2.28)

Proposition (3.2):
Let φ be a linear fractional self map of U. If φ is automorphism with no interior fixed point then $J_{C_\varphi}(f) = H^2$ for every $f \in H^2$, i.e. C_φ is J – class operator.

We prove the following theorem.

Theorem (3.3):
Suppose that α is complex number such that $|\alpha| \le 1$ then the sequence $\{\alpha^n\}$ has a converge subsequence.

Proof:
We know from Hiene-Borel theorem that the unit disk $D_1(0) = \{x \in ₵ : |x| \le 1\}$ is compact set. Since $\{\alpha^n\}$ is sequence in $D_1(0)$ then $\{\alpha^n\}$ has a converge subsequence (every infinite subset of compact set has limit point).

The following corollary consequence from theorem (3.3).

Corollary (3.4):
For every complex number α such that $|\alpha| \le 1$ the set $M_\alpha = \{r \in ₵$: there exists a subsequence of $\{\alpha^n\}$ that converge to $r\}$ is non – empty.

Lemma (3.5):
Let φ be a holomorphic self map of U. If φ has interior fixed point $p \in U$ then $L_{C_\varphi}(f) = J_{C_\varphi}(f) \quad \forall f \in H^2$, where $L_{C_\varphi}(f)$, $J_{C_\varphi}(f)$ are the limit set and the extended limit set of f under C_φ respectively.

Proof:
We know from Theorem (2.26) that $\|C_\varphi^n\| = \|C_{\varphi_n}\| \le \sqrt{\dfrac{1+\varphi_n(0)}{1-\varphi_n(0)}}$.

If φ is automorphism then 0 is fixed point for φ lemma (2.21), hence $\|C_\varphi^n\| \le 1$ for all n . If φ is not automorphism then $\{\varphi_n(0)\} \to p$ (theorem(2.23)) and hence $\left\{\sqrt{\dfrac{1+\varphi_n(0)}{1-\varphi_n(0)}}\right\} \to \sqrt{\dfrac{1+p}{1-p}}$ so that the sequence $\left\{\sqrt{\dfrac{1+\varphi_n(0)}{1-\varphi_n(0)}}\right\}$ is bounded , this implies that there exist $M \ge 0$ such that $\|C_\varphi^n\| \le M \ \forall n$,so that C_φ is power bounded , therefore proposition(2.8) shows that $L_{C_\varphi}(f) = J_{C_\varphi}(f) \quad \forall f \in H^2$.

We prove the following theorem.

Theorem (3.6):
If $\varphi(z) = \alpha z \ \forall z \in U$ where $|\alpha| \le 1$ then $L_{C_\varphi}(f) = J_{C_\varphi}(f) = \{f(rz) : r \in M_\alpha\} \forall f \in H^2$

Proof:
Since $\varphi(z) = \alpha z \ \forall z \in U$ then φ has interior fixed point and hence $L_{C_\varphi}(f) = J_{C_\varphi}(f) \quad \forall f \in H^2$ (lemma (3.5)).

Suppose that $g \in L_{C_\varphi}(f)$ where $f, g \in H^2$

i.e. $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and

$g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ then there exist a strictly increasing sequence of positive integers $\{k_n\}$ such that $\|C_\varphi^{k_n} f - g\| \to 0$ that is

$$\|C_\varphi^{k_n} f - g\|^2 = |a_0 - b_0|^2 + |a_1\alpha^{k_n} - b_1|^2 + \cdots \to 0$$
.

$\because |a_i\alpha^{ik_n} - b_i|^2 \le \|C_\varphi^{k_n} f - g\|^2 \to 0 \quad i = 1,2,\cdots$

then $a_i\alpha^{ik_n} \to b_i \quad i = 1,2,\cdots$ ....(1)

We know from theorem (3.3) that $\{\alpha^{k_n}\}$ has a converge subsequence ,say $\{\alpha^{k'_n}\}$ which converge to r , hence $r \in M_\alpha$, i.e. $\{a_i\alpha^{ik'_n}\} \to a_i r^i$ as $n \to \infty$, i=1,2,… , but from (1) $\{a_i\alpha^{ik'_n}\} \to b_i$ i=1,2,… , therefore $b_i = a_i r^i \quad \forall i$

$\therefore g(z) = b_0 + b_1 z + \cdots = a_0 + a_1 rz + \cdots = f(rz)$

Thus $L_{C_\varphi}(f) \subseteq \{f(rz) : r \in M_\alpha\} \forall f \in H^2$.

Conversely: suppose that $r \in M_\alpha$ we must prove that the function g define by $g(z) = f(rz) \ \forall z$ is belong to $L_{C_\varphi}(f)$

Case (1): f is non – constant function.

i.e. $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ so that $g(z) = a_0 + a_1 rz + a_2 r^2 z^2 + \cdots$, suppose that $f_m(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$ and

$g_m(z)=a_0+a_1rz+a_2r^2z^2+\cdots+a_mr^mz^m$, m=1,2,… .We claim that $g_m\in L_{C_\varphi}(f_m)\ \forall m$

$\because r\in M_\alpha$ then there exist a strictly increasing sequence of positive integers $\{k_n\}$ such that $\{\alpha^{k_n}\}\to r$, therefore

$\left\|C_\varphi^{k_n}f_m-g_m\right\|^2=\left\|f_m(\varphi_{k_n})-g_m\right\|^2=|a_1|^2\left|\alpha^{k_n}-r\right|^2+\cdots+|a_m|^2\left|\alpha^{mk_n}-r^m\right|^2\to 0$ as $n\to\infty$, so that $g_m\in L_{C_\varphi}(f_m)\quad m=0,1,2,...$.

Since C_φ is power bounded (see the prove of lemma (3.5)) and $\{g_m\}\to g$, $\{f_m\}\to f$ then lemma (2.9) shows that $g\in L_{C_\varphi}(f)$.

Case (2): f is constant function i.e. $f(z)=a_0\quad\forall z$ then it is clear that $(C_\varphi^n f)(z)=f(\varphi_n(z))=a_0=f(z)\ \forall z$, that is $\{C_\varphi^n f\}\to f$. hence $f\in L_{C_\varphi}(f)$. Thus $\{f(rz):r\in M_\alpha\}\subseteq L_{C_\varphi}(f)\quad\forall f\in H^2$

Corollary (3.7):

If $\varphi(z)=\alpha z\ \forall z\in U$ where $|\alpha|<1$ then $L_{C_\varphi}(f)=J_{C_\varphi}(f)=\{f(0)\}\quad\forall f\in H^2$

Proof:

Since $|\alpha|<1$ then $M_\alpha=\{0\}$ and hence from theorem (3.6) that

$L_{C_\varphi}(f)=J_{C_\varphi}(f)=\{f(0)\}\quad\forall f\in H^2$.

The following corollary consequence from lemma (2.21) and theorem (3.6)

Corollary (3.8):

If φ is elliptic (i.e. φ is automorphism with interior fixed point)

then

$L_{C_\varphi}(f)=J_{C_\varphi}(f)=\{f(rz):r\in M_\alpha\}\quad\forall f\in H^2$

Lemma (3.9) [3]:

For each $f\in H^2$,

$$|f(z)|\le\frac{\|f\|}{\sqrt{1-|z|^2}}\text{ for each } z\in U.$$

We prove the following theorem.

Theorem (3.10):

Suppose that φ is non – elliptic. If φ has interior fixed point p then

$L_{C_\varphi}(f)=J_{C_\varphi}(f)\subseteq\{f(p)\}\quad\forall f\in H^2$

Proof:

Since φ has interior fixed point p then theorem (2.23) shows that for every $z\in U$, $\varphi_n(z)\to p$ as $n\to\infty$ also $L_{C_\varphi}(f)=J_{C_\varphi}(f)\quad\forall f\in H^2$ (lemma (3.5)).

Let $g\in L_{C_\varphi}(f)$ then there exist a strictly increasing sequence of positive integers $\{k_n\}$ such that $C_\varphi^{k_n}f\to g$, so that $\|f(\varphi_{k_n})-g\|\to 0$ as $n\to\infty$.

For each $z\in U$

$$\left|f(\varphi_{k_n}(z))-g(z)\right|\le\frac{\|f(\varphi_{k_n})-g\|}{\sqrt{1-|z|^2}}\to 0$$

therefore $f(\varphi_{k_n}(z))\to g(z)$ as $n\to\infty$ for all $z\in U$, but $f(\varphi_{k_n}(z))\to f(p)$ (since $\varphi_n(z)\to p$). Thus $g(z)=f(p)$ for each $z\in U$.

Example (3.11):

We can prove easily that $\varphi(z)=\dfrac{z}{2-z}$ is linear fractional self map of U and 0 is the interior fixed point for φ therefore

$L_{C_\varphi}(f)=J_{C_\varphi}(f)\subseteq\{f(0)\}\quad\forall f\in H^2$.

Theorem (3.12) [7]:

If φ has no fixed point in U then C_φ is hypercyclic unless φ is a parabolic non – automorphism . In this latter case C_φ is strongly non hypercyclic in the sense that the only possible limit points of C_φ – orbits are constant functions.

In the following theorem we summarized the previous results.

Theorem (3.13):

Let φ be holomorphic self map of U.

1- If φ has interior fixed point then C_φ is not J – class operator
2- If φ has no interior fixed point then
   (i) If φ is not parabolic then C_φ is J – class operator.
   (ii) If φ is parabolic non – automorphism then $L_{C_\varphi}(f)\subseteq K\quad\forall f\in H^2$, where K is the set of constant functions.

(iii) If φ is parabolic automorphism then C_φ is J – class operator.

Proof:
For proof (1) see Corollary (3.8) and theorem (3.10).
For proof (2), (i) and (iii) see theorem (2.28) and theorem (3.12).
Proof (2), (ii):
Suppose that $f \in H^2$ and $g \in L_{C_\varphi}(f)$ then there exist a strictly increasing sequence of positive integers $\{k_n\}$ such that $C_\varphi^{k_n} f \to g$. Thus g is constant function (theorem (3.12)), so that $g \in K$.
We end this paper by the following problem.

Problem:
Is C_φ J – class operator when φ is parabolic non – automorphism?

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المؤثرات التركيبية من الصنف J

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الملخص :

ليكن X فضاء باناخ قابل للفصل ، لكل $x \in X$ ، $L(x)$ ، $J(x)$ ، $J^{mix}(x)$ تشير الى المجموعة المحددة ، المجموعة المحددة الموسعة ، مجموعة الخلط المحددة الموسعة للعنصر x بالنسبة للمؤثر T على الترتيب .في هذا البحث أعطينا عدة خواص لهذه المجموعات. كذلك درسنا هذه المجموعات بالنسبة للمؤثر التركيبي C_φ وأعطينا شروط تجعل المؤثر التركيبي C_φ من الصنف J.