Topological Properties of Some Types of Spaces

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Abstract

The aim of this paper is to introduce the relation with some type of spaces ($T_{1/2}$, $T^*_{1/2}$, T_c , ${}_\alpha T_b$, ${}_\alpha T_c$, ${}_g\hat{T}_{\delta g}$, complemented , $T_{\alpha gs}$) spaces . At last we study the topological properties on them .

1- Introduction

Levine [14,15] introduced semi open sets and g-closed sets. Njastad [16] introduced α - open sets. Abd El-Monsef et.al [1] invented β - open .Maki et.al [7,8] introduced αg - closed sets and $g\alpha$ - closed sets . Bhattacharya and Lahiri [3] , Arya and Nour [2] , Dontchev [12] , Gnanambal [13] and Chandrasekhara Rao and Joseph [4] investigated sg - closed sets , gs - closed sets , gsp – closed sets gpr – closed sets and $s*g$ – closed sets respectively .Dunham [11], Bhattacharya and Lahiri [3], Duntchev [12], Gnanambal[13] were familiar with $T_{1/2}$, semi – $T_{1/2}$. Devi et.al [9,10] introduced T_b , T_d and ${}_\alpha T_b$, ${}_\alpha T_d$ - spaces respectively . Veera Kumar [17-19] introduced $T_{1/2}^*$, ${}^*T_{1/2}$, T_p^* , ${}^*T_p$ - spaces.Chandrasekhara Rao and Narasimhan [5] introduced T_S - spaces .

In this paper we introduce the relation among some types of these spaces by diagram and proof them moreover we give counter example . And we study the topological properties on them .

2- Preliminaries and basic definitions

Let (X,τ) be a topological space .For any subset $A \subseteq X$, the closure [resp. δ -closure , α - closure] of a subset A of a space (X,τ) is the intersection of all closed sets that contain A and is denoted by *cl(A)* [resp. cl δ (A) , α cl(A)].

Let (X,τ) and $(Y,\Im)$ be topological spaces . A function $h: X \to Y$ is said to be a homeomorphism if and only if the following conditions are satisfied :

- The function $h: X \to Y$ is both injective and surjective (so that the function $h: X \to Y$ has a well-defined inverse $h^{-1}: X \to Y$
- The function $h: X \to Y$ and its inverse $h^{-1}: X \to Y$ are both continuous [6] .

Definition 2.1: A set A of a topological space (X,τ) is called :

(1) semi open if there exists an open set U such that $U \subseteq A \subseteq cl(U)$.

(2) semi closed if $X - A$ is semi open. Equivalently , a set A of a topological space (X,τ) is called semi closed if there exists a closed set F such that $\text{int}(F) \subseteq A \subseteq F$.

(3) generalized closed (g - closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

(4) generalized open (g - open) if $X - A$ is *g-closed.* Equivalently , a set *A* of a topological space (X,τ) is called (g - open) If $F \subseteq \text{int}(A)$, *whenever* $F \subseteq A$ *and* F is closed in X .

(4) generalized semi open (gs - open) if $F \subseteq \text{int}(A)$ whenever $F \subseteq A$ and F is closed in X .

(6) generalized semi closed (gs - closed) if $X - A$ is gs - open. Equivalently , a set A of a topological space (X,τ) is called (gs - closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

(7) semi star generalized closed (*s*g* - closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X .

(8) semi star generalized open (*s*g* - open) if $X - A$ is *s*g* - closed in *X*. Equivalently , a set *A* of a topological space (X,τ) is called (*s*g* - open) if $F \subseteq \text{int}(A)$, whenever $F \subseteq A$ and *F* is semi closed in *X.*

(9) α - open if $A \subseteq \text{int}\{cl[\text{int}(A)]\}$.

(10) α - closed if $cl\{\text{int}[cl(A)]\} \subseteq A$.

(11) αg - closed if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

(12) αgs - closed if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X .

(13) $g*$ - closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g - open in X .

(14) δ - g closed if $cl_{\delta}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

(15) $\delta - \hat{g}$ closed if $cl_{\delta}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X.

Definition 2.2: A topological space (X, τ) is called :

(a) $T_{1/2}$ - space if every g - closed set is closed .

(b) $T^*_{1/2}$ - space if every g^* - closed set is closed .

(c) T_c - space if every gs - closed set is g^*-closed.

(d) ${}_{\alpha}T_b$ - space if every αg - closed set is closed .

(e) ${}_{g}\hat{T}_{\delta g}$ - spaces if every g - closed is $\delta - \hat{g}$ - closed .

(f) ${}_{\alpha}T_c$ - *space* if every αg - closed set is g^*-closed .

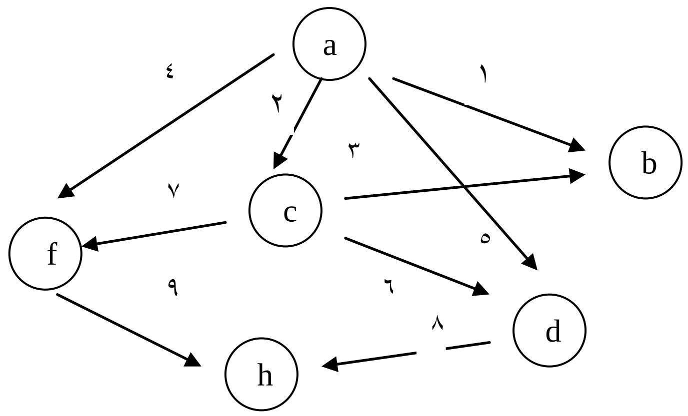
(g) complemented space if every open set is closed .

(h) $T_{\alpha gs}$ - *space* if every αgs - closed set is closed .

3- Topological Properties and Relations among Some Types of Spaces

In this section we introduce the relations between all above definition (2.2) by the following diagram:

Theorem 3.1:



Proof:

1- Every $T_{1/2}$ space is a $T^*_{1/2}$ space.

Let X be a $T_{1/2}$ space , let $A \subseteq X$ and g - closed set in X .It is very sufficient to prove that A is g^* - closed .

Now we have $cl(A) \subseteq U$, $A \subseteq U$, U is open .

Since X is $T_{1/2}$ space , we have A is closed set in X .

Suppose that $B = A$ then we have $cl(B) \subseteq U$, $B \subseteq U$, B is closed set in X .

$\therefore$ U is g - open in X.

$\therefore$ B is g^* - closed.

i.e. A *is* g^* *- closed.*

$\therefore X$ is $T^*_{1/2}$ space.

2- Every $T_{1/2}$ space is a T_c space.

Let X be a $T_{1/2}$ space , let $A \subseteq X$ and g - closed set in X . It is very sufficient to prove that A is g^* - closed .

Now we have $cl(A) \subseteq U$, $A \subseteq U$, U is open .

Since X is $T_{1/2}$ *space* , we have A is closed set in X .

$\therefore$ A is gs - closed .

Since X is $T_{1/2}$ space , we have A is closed set in X.

Suppose that $B = A$ then we have $cl(B) \subseteq U$, $B \subseteq U$, B is closed set in X .

$\therefore$ U is g - open in X.

$\therefore$ B is g^* - closed.

i.e. A is g^* - closed.

$\therefore$ X is T_c space .

3-Every T_c space is a $T^*_{1/2}$ space.

Let X be a T_c space , let $A \subseteq X$ and gs - closed set in X . It is very sufficient to prove that A is closed .

Now we have $cl(A) \subseteq U$, $A \subseteq U$, U is open .

Since X is T_c space , we have A is g^* - closed .

We have $cl(A) \subseteq U$, $A \subseteq U$, U is g - open set in X .

Suppose that $B = A$, then we have $cl(B) \subseteq U$, $B \subseteq U$, B is closed set in X .

$\therefore$ A is closed in X.

$\therefore$ X is $T^*_{1/2}$ space.

4- - Every $T_{1/2}$ space is ${}_{\alpha}T_c$ space. □

Let X be a $T_{1/2}$ *space* , let $A \subseteq X$ and *g-closed* set in X .It is very sufficient to prove that A is *g*-closed* .

Now we have $cl(A) \subseteq U$, $A \subseteq U$, U is open .

Since $\alpha cl(A) \subseteq cl(A)$, we hav, $\alpha cl(A) \subseteq U$, $A \subseteq U$, U is open.

$\therefore$ A is αg - closed .

Since X is $T_{1/2}$ space , we have A is closed set in X .

we have $cl(A) \subseteq U$, $A \subseteq U$, A is closed set in X .

$\therefore$ U is g - open in X.

$\therefore$ A is g^* - closed.

$\therefore X$ is ${}_{\alpha}T_{c}$ space

5- - Every $T_{\frac{1}{2}}$ space is ${}_{\alpha}T_{b}$ space.

Let X be a $T_{\frac{1}{2}}$ space , let $A \subseteq X$ and g - closed set in X .It is very sufficient to prove that A is closed .
Now we have $cl(A) \subseteq U$, $A \subseteq U$, U is open .
Since $\alpha cl(A) \subseteq cl(A)$, we have $\alpha cl(A) \subseteq U$, $A \subseteq U$, U is open.
$\therefore$ A is αg - closed .

Since X is $T_{\frac{1}{2}}$ space , we have A is closed set in X .

$\therefore X$ is ${}_{\alpha}T_{b}$ space.

6- Every T_{c} space is ${}_{\alpha}T_{b}$ space.

Let X be a T_{c} space , let $A \subseteq X$ and gs - closed set in X . It is very sufficient to prove that A is closed .
Now we have $cl(A) \subseteq U$, $A \subseteq U$, U is open .
Since $\alpha cl(A) \subseteq cl(A)$, we have $\alpha cl(A) \subseteq U$, $A \subseteq U$, U is open.
$\therefore$ A is αg - closed .

Since X is T_{c} space , we have A is g^* - closed .

We have $cl(A) \subseteq U$, $A \subseteq U$, U is g - open set in X .
Suppose that $B = A$, then we have $cl(B) \subseteq U$, $B \subseteq U$, B is closed set in X
$\therefore$ A is closed in X.

$\therefore$ X is ${}_{\alpha}T_{b}$ space.

7- Every T_{c} space is ${}_{\alpha}T_{c}$ space.

Let X be a T_{c} space , let $A \subseteq X$ and gs - closed set in X . It is very sufficient to prove that A is g^*- closed .
Now we have $cl(A) \subseteq U$, $A \subseteq U$, U is open .
Since $\alpha cl(A) \subseteq cl(A)$, we have $\alpha cl(A) \subseteq U$, $A \subseteq U$, U is open.
$\therefore$ A is αg - closed .

Since X is T_{c} space , we have A is g^* - closed .

$\therefore$ X is ${}_{\alpha}T_{c}$ space.

8- Every ${}_{\alpha}T_{b}$ space is $T_{\alpha gs}$ space.

Let X be a ${}_{\alpha}T_{b}$ space , let $A \subseteq X$ and αg - closed set in X . It is very sufficient to prove that A is closed .
We have $\alpha cl(A) \subseteq U$, $A \subseteq U$, U is open.
Since every open set is semi open , then U is semi open.
So that $\alpha cl(A) \subseteq U$, $A \subseteq U$, U is semi open.
$\therefore A$ is αgs - closed.

Since X is ${}_{\alpha}T_{b}$ space , we have A is closed .

$\therefore X$ is $T_{\alpha gs}$ space .

9- Every ${}_{\alpha}T_{c}$ space is $T_{\alpha gs}$ space.

Let X be a ${}_{\alpha}T_{c}$ space , let $A \subseteq X$ and αg - closed set in X . It is very sufficient to prove that A is closed .
We have $\alpha cl(A) \subseteq U$, $A \subseteq U$, U is open.
Since every open set is semi open , then U is semi open.
So that $\alpha cl(A) \subseteq U$, $A \subseteq U$, U is semi open.
$\therefore A$ is αgs - closed..

Since X is ${}_{\alpha}T_{c}$ space , we have A *is* g^* - closed.
We have $cl(A) \subseteq U$, $A \subseteq U$, U is g - open set in X .
Suppose that $B = A$, then we have $cl(B) \subseteq U$, $B \subseteq U$, B is closed set in X .
$\therefore$ A is closed in X.

$\therefore X$ is $T_{\alpha gs}$ space.

Now we want to present the topological property of some types of spaces.
Theorem 3.2 : $T_{\frac{1}{2}}$ space property is a topological property .
Proof :
Let h be a homeomorphism from a topological space (X,τ) in to a topological space $(Y,\Im)$, and let (X,τ) be a $T_{\frac{1}{2}}$ space.
To prove that $(Y,\Im)$ is $T_{\frac{1}{2}}$ space.
Suppose $B \subseteq Y$ be any g - closed set , since h is onto then there exist A g - closed in X such □at $B = h(A)$.
Since (X,τ) is $T_{\frac{1}{2}}$ space
$\therefore$ A is closed set in X .
Since h is open function.
$\therefore h(X - A)$ is open in Y.
Now B is closed in Y.
$\therefore$ $(Y,\Im)$ is $T_{\frac{1}{2}}$ space .

Theorem 3.3 : $T^*_{\frac{1}{2}}$ - space property is a topological property .
Proof :
Let h be a homeomorphism from a topological space (X,τ) in to a topological space $(Y,\Im)$, and let (X,τ) be a $T^*_{\frac{1}{2}}$ - space.
To prove that $(Y,\Im)$ is $T^*_{\frac{1}{2}}$ - space.
Suppose $B \subseteq Y$ be any g^* - closed set , since h is onto then there exist A g^* - closed in X such that $B = h(A)$.
Since (X,τ) is $T^*_{\frac{1}{2}}$ - space.
$\therefore$ A is closed set in X.
Since h is open function.
$\therefore h(X - A)$ is open in Y.
Now B is closed in Y.
$\therefore$ $(Y,\Im)$ is $T^*_{\frac{1}{2}}$ - space.

Theorem 3.4 : T_{c} - space property is a topological property .
Proof :

Let h be a homeomorphism from a topological space (X,τ) in to a topological space $(Y,\mathfrak{I})$, and let (X,τ) be a T_c - space.

To prove that $(Y,\mathfrak{I})$ is T_c - space.

Suppose $B \subseteq Y$ be any gs - closed set , since h is onto then there exist A gs - closed in X such that $B = h(A)$.

Since (X,τ) is T_c - space.

$\therefore$ A is g^* - closed set in X.
Since h is open function.
$\therefore h(X-A)$ is g^* - open in Y.
Now B is g^* - closed in Y.

$\therefore$ $(Y,\mathfrak{I})$ is T_c - space.

Theorem 3.5 : $_\alpha T_b$ - space property is a topological property .
Proof :
Let h be a homeomorphism from a topological space (X,τ) in to a topological space $(Y,\mathfrak{I})$, and let (X,τ) be a $_\alpha T_b$ - space.

To prove that $(Y,\mathfrak{I})$ is $_\alpha T_b$ - space .

Suppose $B \subseteq Y$ be any αg - closed set , since h is onto then there exist A αg -closed in X such that $B = h(A)$.

Since (X,τ) is $_\alpha T_b$ - space.

$\therefore$ A is closed set in X.
Since h is open function.
$\therefore h(X-A)$ is open in Y.
Now B is closed in Y.

$\therefore$ $(Y,\mathfrak{I})$ is $_\alpha T_b$ - space.

Theorem 3.6 : $_g\hat{T}_{\delta g}$ - space property is a topological property.
Proof :
Let h be a homeomorphism from a topological space (X,τ) in to a topological space $(Y,\mathfrak{I})$, and let (X,τ) be $_g\hat{T}_{\delta g}$ - space .

To prove that $(Y,\mathfrak{I})$ is $_g\hat{T}_{\delta g}$ - space .

Suppose $B \subseteq Y$ be any g - closed set , since h is onto then there exist A g - closed in X such that $B = h(A)$.

Since (X,τ) is $_g\hat{T}_{\delta g}$ - space .

$\therefore$ A is $\delta - \hat{g}$ - closed set in X.
Since h is open function.

$\therefore h(X-A)$ is $\delta - \hat{g}$ - open in Y.

Now B is $\delta - \hat{g}$ closed in Y.

$\therefore$ $(Y,\mathfrak{I})$ is $_g\hat{T}_{\delta g}$ - space.

Theorem 3.7 : $_\alpha T_c$ space property is a topological property .
Proof :
Let h be a homeomorphism from a topological space (X,τ) in to a topological space $(Y,\mathfrak{I})$, and let (X,τ) be a $_\alpha T_b$ space.

To prove that $(Y,\mathfrak{I})$ is $_\alpha T_c$ space.

Suppose $B \subseteq Y$ be any αg - closed set , since h is onto then there exist A αg - closed in X such that $B = h(A)$.

Since (X,τ) is $_\alpha T_c$ space. □

$\therefore$ A is g^* - closed *set* in X.
Since h is open function.
$\therefore h(X-A)$ is g^* - open in Y.
Now B is g^* - closed in Y.

$\therefore$ $(Y,\mathfrak{I})$ is $_\alpha T_c$ space.

Theorem 3.8 : complemented space property is a topological property .
Proof :
Let h be a homeomorphism from a topological space (X,τ) in to a topological space $(Y,\mathfrak{I})$, and let (X,τ) be a complemented space . To prove that $(Y,\mathfrak{I})$ is be a complemented space .

Suppose $B \subseteq Y$ be any open set , since h is onto then there exist A open in X such that $B = h(A)$.
Since (X,τ) is complemented space .

$\therefore$ A *is* closed *set in* X .
$\therefore h(X-A)$ *is* open *in* Y . □
Now B is closed in Y .
$\therefore$ $(Y,\mathfrak{I})$ is complemented space.

Theorem 3.9 : $T_{\alpha gs}$ space property is a topological property .
Proof :
Let h be a homeomorphism from a topological space (X,τ) in to a topological space $(Y,\mathfrak{I})$, and let (X,τ) be a $T_{\alpha gs}$ space.

To prove that $(Y,\mathfrak{I})$ *is* $T_{\alpha gs}$ space.

Suppose $B \subseteq Y$ be any αgs - closed set , since h is onto then there exist A αgs - closed in X such that $B = h(A)$.

Since (X,τ) is $T_{\alpha gs}$ space.

$\therefore$ A is closed set in X.
Since h is open function.
$\therefore h(X-A)$ is open in Y.
Now B is closed in Y.

$\therefore$ $(Y,\mathfrak{I})$ is $T_{\alpha gs}$ space.

Now it is very suitable to give an example which is prove that T_c - space in not $T\frac{1}{2}$ space .

Example : Let $X = \{1,2,3,4\}$

$\tau_x = \{\phi, X, \{1\}, \{3\}, \{1,3\}\}$

let $A = \{2\}$, $U = X$ is open

$cl(A) = \{2,4\} \subseteq U$, $A \subseteq U$

$\therefore A$ is gs - closed

since X is T_c - space then A is g^*- closed

since $cl(A) \subseteq U$, $A \subseteq U$, U is open , then A is g - closed .

but A is not closed set in X .

$\therefore$ X is not $T\frac{1}{2}$ space .

Remark: It is very easy to show another counter examples .

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الخواص التبولوجية لبعض انواع الفضاءات

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الملخص

الهدف من هذا البحث هو تقديم العلاقات بين بعض أنواع الفضاءات من النوع ($T\frac{1}{2}$, $T^*\frac{1}{2}$, T_c , $_\alpha T_b$, $_\alpha T_c$, $_g\hat{T}_{\alpha g}$, $T_{\alpha gs}$) complemented وكذلك دراسة الصفات التبولوجية على تلك الفضاءات .