Solution of Fractional Integral Equation with Weaker Boundary Conditions

Abdul – Kareem I. Sheet, Ikram H. Ammory
Dep. of Mathematics, College of Basic Education, Univ. of Mosul, Mosul, Iraq
(**Received:** 7 / 6 / 2009 ---- **Accepted:** 27 / 4 / 2010)

Abstract:
The present work investigate the solution of certain fractional integral equation with boundary conditions by using the condition of Caratheodory theorem. In addition, to extended some results of Dennis (2001), in order to prove the existence theorem of boundary condition by using weaker conditions .

1- Introduction:
In recent years, there has been a growing interest in the formulation of many physical problems in terms of integral equations, and this has fastered a parallel rapid growth of the literature on their numerical solution. An integral equation is an equation in which the unknown function appears within an integral, while a fractional integral equation is a fractional order integral equation.
The solution of a fractional integral equation is founded in many references for example [1], [4], [3]. The main result of this paper is to extend some result of Dennis (2001) [5], in order to prove the existence theorem by using the conditions of Caratheodory theorem for the following fractional integral equation:

$$x(t) = A + \frac{1}{\Gamma(\alpha)}\int_a^t (b-s)^{\propto-1} f(s,x(s))ds \quad \ldots\ldots\ldots (1)$$

$a \le t \le b, 0 \le \propto \le 1$

with boundary conditions

$x(a) = A$;

$$x(b) = A + \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\propto-1} f(s,x(s))ds$$

2. Preliminary Concept:
1. Gamma Function:
If $\propto > 0$ then the gamma function $\Gamma(\alpha)$ is defined by [5]:

$$\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\propto-1}dx$$

if $\propto \in R$ then the $\int_0^\infty e^{-x}x^{\propto-1}dx$ is defined for each $\propto$.

2. LT space:
Let $0 < \propto \le 1$ and $I_h = [a, a+h]$, $h > 0$, define the space $LT_\propto(I_h)$ as the set of all Lebague integrable function m in I_h such that [3]:

i) $\left\{x \to \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\propto-1}|m(y)|dy, x \in (a, a+h)\right\} \in c(a, a+h)$,

ii) $\lim_{x\to a} \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\propto-1}|m(y)|dy = 0$

Lemma 1: Let f be a continuous function on a closed bounded region G in R^2 and $K \in LT_\propto[a,b]$, then $Kf \in LT_\propto[a,b]$ [3].

Proof :

Since f is continuous function on closed bounded region G then f is bounded on G , i.e. there exist apossitive number M such that $|f(z,y)| \le M$, for all (z,y) in G .

Now $|K(y)f(z,y)| = |k(y)||f(z,y)| \le M|k(y)|$ then by the definition 2 we get that

$Mk(y) \in LT_\propto[a,b] \Rightarrow Kf \in LT_\propto[a,b]$.

Lemma 2:
Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions defined on the set $E \subset R$ such that $|f_n| \le M_n$, M_n is any positive number, then $\sum_{n=1}^\infty f_n$ is uniformly convergent [6].

3- Main Result:
The main result of this paper is formulated in the following theorem

Theorem:
Suppose equation(1) has the assumption

$f(t, x(t)) \in LT_\propto[a,b] \quad \forall t \in [a,b]$, ……..(2)

then $x(t)$ is a solution and

$\|x(t)\| \le a(t)\ \forall t \in [a,b]$, $a(t)$ is a continuous function on *R*.

Proof:
In a Banach space E we shall prove that $S = \{x(t) : \|x(t)\| \le a(t), \forall t \in [a,b]\}$ is a closed convex set and to do this let $x(t), y(t) \in S, 0 < q < 1$, then

$$\begin{aligned}\|qx(t) + (1-q)y(t)\| &\le |q|\|x(t)\| + |1-q|\|y(t)\| \\ &\le qa(t) + (1-q)a(t) \\ &\le qa(t) + a(t) - qa(t) \\ &\le a(t)\end{aligned}$$

Hence S is a closed convex set (see appendix 1).
For any x(t) $\in$ S we define the operator T on E by:

$$(Tx)(t) = A + \frac{1}{\Gamma(\alpha)}\int_a^t (b-s)^{\propto-1} f(s,x(s))ds \,; S \in [a,b] \ldots (٣)$$

provided that operator T is continuous from S to E. A is continuous (every constant function is continuous), from lemma 1 and assumption 2 we get $f(t,x(t))$ is continuous on a closed bounded interval therefor by definition 2 of LT space we have

$\frac{1}{\Gamma(\alpha)}\int_a^t (b-s)^{\propto-1} f(s,x(s))ds$ is continuous on $[a,b]$.

thus T is continuous operator from S and E, also we find that:

$$\|(Tx)(t)\| = \left\| A + \frac{1}{\Gamma(\alpha)}\int_a^t (b-s)^{\propto-1} f(s,x(s))ds \right\|$$

$$\leq \|A\| + \frac{1}{\Gamma(\alpha)}\int_a^t (b-s)^{\propto-1}\|f(s,x(s))\|ds$$

since A is a constant $\Rightarrow \|A\| = |A|$ *and* $\in S$

$$\|(Tx)(t)\| \leq A + \frac{1}{\Gamma(\alpha)}\int_a^t (b-s)^{\propto-1}\|f(s,x(s))ds\|$$

$$\leq a(t)\ \forall a \leq t \leq b \Rightarrow TS \subseteq S$$

Next to prove t TS is relatively compact set in E, we Define

$\{(Tx_n)(t)\}_1^\infty$ as follows:

$$(Tx_n)(t) = A + \frac{1}{\Gamma(\alpha)}\int_a^t (b-s)^{\propto-1} f(s,x_n(s))ds$$

$; S \in [a,b], n = 1,2,......$ (٤)

Now in order to prove that $\{(Tx_n)(t)\}_1^\infty$ is uniformly bounded on [a,b], we substitute:

$$\|(Tx_n)(t)\| = \left\| A + \frac{1}{\Gamma(\alpha)}\int_a^t (b-s)^{\propto-1} f(s,x_n)(s)ds \right\|$$

$$\leq |A| + \frac{1}{\Gamma(\alpha)}\int_a^t (b-s)^{\propto-1}\|f(s,x_n)(s)\|ds$$

$$\leq |A| + \frac{1}{\Gamma(\alpha)}\int_a^t (b-s)^{\propto-1}\|f(s,x_n)(s)\|ds \leq a(t)$$

since a(t) is a bounded function on [a,b], there exist a positive number k

such that:

$$\|a(t)\| \leq k \quad \forall t \in [a,b]$$

So $\|(Tx_n)(t)\| \leq \|a(t)\| \leq k \quad ; k > 0, \forall n$

$\Rightarrow \{(Tx_n)(t)\}_1^\infty$ is uniformly bounded on [a,b].

Finally to prove that $\{(Tx_n)(t)\}_1^\infty$ is equi-continuous on [a,b], let $t_1, t_2 \in [a,b]$ (choose $t_2 > t_1$, the prove is similar when $t_2 \leq t_1$).

$$\|(Tx_n)(t_2) - (Tx_n)(t_1)\| = \left\| A + \frac{1}{\Gamma(\alpha)}\int_a^{t2} (b-s)^{\propto-1} f(s,x_n(s))ds - A - \frac{1}{\Gamma(\alpha)}\int_a^{t1} (b-s)^{\propto-1} f(s,x_n(s))ds \right\|$$

$$= \left\| \frac{1}{\Gamma(\alpha)}\int_a^{t2} (b-s)^{\propto-1} f(s,x_n(s))ds - \frac{1}{\Gamma(\alpha)}\int_a^{t1} (b-s)^{\propto-1} f(s,x_n(s))ds \right\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\left\| \int_a^{t2} (b-s)^{\propto-1} f(s,x_n)(s)ds - \int_a^{t1} (b-s)^{\propto-1} f(s,x_n(s)ds) \right\|$$

Since $f \in LT_\alpha$ and $x_n(t) \in S$ implies that $x_n \in$ C[a,b] also from Lemma 1,

we obtain $f x_n \in LT_\alpha$ on [a,b], by definition 2 we get: $\frac{1}{\Gamma(\alpha)}\int_a^t (b-s)^{\propto-1} f(s,x_n(t))ds \in C[a,b], \forall n$

so that

$$\frac{1}{\Gamma(\alpha)}\left\| \int_a^{t2} (b-s)^{\propto-1} f(s,x_n(s))ds - \int_a^{t1} (b-s)^{\propto-1} f(s,x_n(s))ds \right\| \leq \in$$

when $|t_2 - t_1| < \delta_\in$, therefore

$$\|(Tx_n)(t_2) - (Tx_n)(t_1)\| < \in \quad \text{when } t_2 - t_1 < \delta_\in$$

thus $\forall \in > 0$ there exist $\delta_\in > 0$ such that:

$\|(Tx_n)(t_2) - (Tx_n)(t_1)\| < \in$

when $t_2 - t_1 < \delta_\in$ therefore $\{(Tx_n)(t)\}_1^\infty$ is equi-continuous on [a,b] hence Ts is relatively compact set from S to E and from schauder fixed point theorem [6] we have:

$(T_x)(t)$ has a fixed point

$$\Rightarrow (T_x)(t) = x(t)$$

and so $x(t)$ is a solution of (1), which completes the prove.

Appendix 1:

Def:

A subset A of a vector space X is said to be convex if [6]

$x, y \in A \Rightarrow M = \{z \in X : z =$

$\alpha x + (1-\alpha)y\} \subseteq A, \quad 0 \leq \alpha \leq 1$

Def:

A subset M of a metric space X is open if it contains a ball about each of its points and it is said to be closed if its complement in X is open [6].

References:

1- Al abedeen A. Z. and Arora H., A global existence and uniqueness theorem for ordinary differential equation at generalized order,Canada , Math., Bull, 21, 1978.

2- Apostol Tem M., Mathematical Analysis 2nd edition, Addison-Wesley Publishing company Inc, London, 1974.

3- Barrett J.H, Differential equation of non-integer order,Canada, J. math., vol.6, 1954.

4- Bassam M. A., Some existence theorem on differential equation of generalized order, J. Fur die reine und angewandte mathematic, Band 218, 1965.

5- Dennis G. Zill, Differential Equations with boundary-value problems,5th edition, printed in U.S.A, 2001.

6- Royden H. L, Real analysis, Macmillan Publishing Co., Inc., New york, 1968.

حل معادلة تكاملية كسرية بشروط حدية أضعف

عبد الكريم إبراهيم شيت ، اكرام حازم عموري

قسم الرياضيات ، كلية التربية الأساسية ، جامعة الموصل ، الموصل ، العراق

(تاريخ الاستلام: ٧ / ٦ / ٢٠٠٩ ---- **تاريخ القبول:** ٢٧ / ٤ / ٢٠١٠)

الملخص:

يتضمن البحث دراسة الحل لمعادلة تكاملية كسرية ذات شروط حدية وذلك باستخدام شروط نظرية كاراثيودري، بالإضافة إلى توسيع بعض نتائج Dennis (2001) من أجل برهان نظرية الوجود للشروط الحدية باستخدام شروط أضعف.