

Construction Character Table of the Symmetric Group S_4 by Using Permutation Module

Assi-lecture Bassim Kareem Mihsin, Assi-Lecture Haider Baker Ameen,
department of mathematics college of mathematics and computer science . kufa university.

Abstract

In this paper, we study calculation the irreducible character table for symmetric group S_4 , by using the permutation module and the Hook-length for semi-standard young tableaux to obtain on reducible character from permutation module .

الخلاصة

في هذا البحث قدما طريقة جديدة لحساب جدول شواخص السمات لزمرة التباديل S_4 معتمد بذلك على مفهوم speath modeal الشواخص المركبة و كيفية تحليلها الى شواخص غير قابلة للتحليل.

1. Introduction

We give the background on representations of finite groups without proofs. It is perfectly possible to use these results

And this procedure to explicate on how to construct character tables of symmetric groups. The focus isn't going to be on why it works, but rather how. Recall that the conjugacy classes of the symmetric group S_n were in correspondence with partitions.

In 1903 Frobenius used young tableau for the first time when he investigated representations of the symmetric group [see [1]]

2. Young diagrams

First we need to settle some definitions and notations regarding partitions and young diagrams. [2][3][4][9][10][11][12][13]

Definition-1 [2]

A partition of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ satisfying $\lambda_1 \geq \lambda_2 \geq \dots \lambda_k > 0$ and $n = \lambda_1 + \lambda_2 + \dots + \lambda_k$ we write $\lambda \vdash n$ to denoted that λ is a partition of n .

Definition -2 [2]

A Young diagram is a finite collection of boxes arranged in left-justified rows with the row size weakly decreasing , the Young diagram associated to the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is the one that has k rows and λ_i boxes on the i th row.

Definition -3 [3]

suppose $\lambda \vdash n$,A Young tableau t of shape λ , is obtained by filling in the boxes of a Young diagram of λ with $1, 2, \dots, n$ with each number occurring exactly once . in this case, we say that t is a λ - tableau.

Example -4

let $\lambda = (2,1)$, then the Young tableau corresponding to the partition $(2,1)$ are :

1	2	2	1	1	3	2	3
3		3		2		1	1

3. Tabloids & Permutation Module M^λ [4]

We would like to consider certain permutation representations of S_n on the elements $\{1,2,\dots,n\}$, which extends to the defining representation. In this merits, we construct other representation of S_n using equivalence classes of tableaux, known as tabloids.

And we introduce tabloids and use them to construct a representation of S_n known as the permutation module M^λ , however, permutation modules are generally reducible.

Definition -5 [5]

Two λ -tableaux t_1 and t_2 are row – equivalent, denoted $t_1 \sim t_2$, if the corresponding rows of the two tableaux contain the same elements, a tabloid of a shape λ , or λ – tabloid is such an equivalence class, denoted by $\{t\} = \{t_1 | t_1 \sim t\}$ where t is a λ – tabloid, the tabloid $\{t\}$ is drawn as the tableau t without vertical bars separating the entries within each row.

Example -6

if $t =$

1	2
3	

 Then $\{t\}$ is the tabloid drawn as $\{t\} =$

1	2
3	

which represents the equivalence class containing the following two tableaux:

1	2	2	1
3		3	

Definition -7 [8]

Let $\lambda \vdash n$. Then $M^\lambda := C\{\{t_1\}, \dots, \{t_k\}\}$.

where $\{t_1\}, \dots, \{t_k\}$ is the complete list of λ -tabloids, is called the permutation module corresponding to λ .

Definition -8 [7]

For a tableau t of size n , the row group of t , denoted R_t , is the sub group of S_n consisting of permutations which only permutes the elements within each row of t . Similarly, the column group C_t is the subgroup of S_n consisting of permutations which only permutes the elements within each column of t .

Example -9

let $t =$

1	2	4
3	5	

Then the row group is:

$$R_t = S_{\{1,2,4\}} \times S_{\{3,5\}} \\ \cong S_3 \times S_2,$$

And column group is:

$$C_t = S_{\{1,3\}} \times S_{\{2,5\}} \times S_{\{4\}} \\ \cong S_2 \times S_2 \times S_1.$$

Definition -10 [7]

The associated polytabloid to a tableau t is

$$e_t = k_t \{t\} := \sum_{\pi \in C_t} \text{sg}(\pi) \pi \{t\}.$$

Example -11

if
 $t = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}$

Then
 $e_t = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline 4 & 2 & 3 \\ \hline 1 & & \\ \hline \end{array}$

Definition -12 [7]

Suppose $\lambda \vdash n$, Let M^λ denoted the vector space whose basis is the set of λ – tabloids, then M^λ is a representation of S_n known as the permutation module corresponding to λ .

Remark -13

The M^λ corresponding to the young diagram are in fact familiar representations.

Proposition -14 [7]

if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, then

$$\dim M^\lambda = \frac{n!}{\lambda_1! \lambda_2! \dots \lambda_r!}.$$

Proposition -15 [7]

Suppose $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ & $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ are partitions of n , the character of M^λ evaluated at an element of S_n with cycle type μ is equal to the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_r^{\lambda_r}$ in $\prod_{i=1}^k (x_1^{\mu_i} + x_2^{\mu_i} + \dots + x_r^{\mu_i})$.

Example -16

let use compute the full list of the characters of the permutation modules for S_4 , the character at the identity element is equal to the dimension, and it can found through [**proposition-14**], For instance, the character of $M^{(2,1,1)} = \frac{4!}{2!1!1!} = 12$.

Say we want to compute the character of $M^{\lambda=(2,2)}$ at the permutation $\mu_{(2,1^2)}$ which has cycle type $(2,1,1)$, by using [**proposition-15**], we see that the character is equal to the coefficient of $x_1^2 x_2^2$ in:

$$(x_1^2 + x_2^2)(x_1^1 + x_2^1)(x_1^1 + x_2^1) = x_1^4 + x_1^3 x_2^1 + x_1^3 x_2^2 + x_1^2 x_2^2 + x_1^2 x_2^3 + x_1^1 x_2^3 + x_1^1 x_2^4 + x_2^4$$

other characters can be similarly computed, and the result is shown in the following table:

$\lambda \backslash \mu$	$\mu_{(1^4)}$	$\mu_{(2,1^2)}$	$\mu_{(2^2)}$	$\mu_{(3,1)}$	$\mu_{(4)}$
$M^{\lambda=(4)}$	1	1	1	1	1
$M^{\lambda=(3,1)}$	4	2	0	1	0
$M^{\lambda=(2,2)}$	6	2	2	0	0
$M^{\lambda=(2,1,1)}$	12	2	0	0	0
$M^{\lambda=(1,1,1,1)}$	24	0	0	0	0

Note that in the above example, we did not construct the character table S_4 , as all the M^λ are in fact reducible with the exception of $M^{(4)}$. in the next facts, we take a step further and construct the irreducible representation of S_n .

The table, which was constructed in the above was done depending on the [**proposition-15**], and a compound character biodegradable to irreducible characters, by using the method of subtraction, and as we will show that later. The establishment of such a table to the higher degree of group of the clique is very complex and not easy to get him so the item will show another way to, calculation the irreducible characters.

4. Main procedure

this method depended on the inner product formula [see [5][6][7][8]] ,to inference irreducible character indicator in the example below. The trivial representation [8], is already irreducible, so the top row is an irreducible character; let's call it $\chi_4 = M^{(4)}$ We can figure out how many copies of χ_4 each of the lower characters contains by taking inner products.

$$\begin{aligned}\langle \chi_4, M^{(3,1)} \rangle &= 1 \\ \langle \chi_4, M^{(2,2)} \rangle &= 1 \\ \langle \chi_4, M^{(2,1,1)} \rangle &= 1 \\ \langle \chi_4, M^{(1,1,1,1)} \rangle &= 1\end{aligned}$$

Then, since we know how many copies of χ_4 occur in the lower representations, we can subtract them off and get a new table:

Cycle type	(1,1,1,1)	(2,1,1)	(2,2)	(3,1)	(4)
χ_4	1	1	1	1	1
$\chi_{(3,1)}$	3	1	-1	0	-1
$M^{(2,2)'}$	5	1	1	-1	-1
$M^{(2,1,1)'}$	11	1	-1	-1	-1
$M^{(1,1,1,1)'}$	23	-1	-1	-1	-1

Now row 2 is an irreducible character $\chi_{(3,1)}$; you can see this by taking its inner product with itself. We can now repeat by taking the inner product of $\chi_{(3,1)}$ with the M characters and subtracting them off.

$$\begin{aligned}\langle \chi_{(3,1)}, M^{(2,2)} \rangle &= 1 \\ \langle \chi_{(3,1)}, M^{(2,1,1)} \rangle &= 1 \\ \langle \chi_{(3,1)}, M^{(1,1,1,1)} \rangle &= 1\end{aligned}$$

Cycle type	(1,1,1,1)	(2,1,1)	(2,2)	(3,1)	(4)
χ_4	1	1	1	1	1
$\chi_{(3,1)}$	3	1	-1	0	-1
$\chi_{(2,2)}$	2	0	2	-1	0
$M^{(2,1,1)''}$	5	-1	1	-1	1
$M^{(1,1,1,1)''}$	14	-4	2	-1	2

Once again, something my stereos has happened, and row 3 is irreducible.

Let's call it $\chi_{(2,2)}$, and subtract it off from the lower rows

$$\begin{aligned}\langle \chi_{(2,2)}, M^{(2,1,1)} \rangle &= 1 \\ \langle \chi_{(2,2)}, M^{(1,1,1,1)} \rangle &= 2\end{aligned}$$

Cycle type	(1,1,1,1)	(2,1,1)	(2,2)	(3,1)	(4)
χ_4	1	1	1	1	1
$\chi_{(3,1)}$	3	1	-1	0	-1
$\chi_{(2,2)}$	2	0	2	-1	0
$\chi_{(2,1,1)}$	3	-1	-1	0	1
$M^{(1,1,1,1)'''}$	10	-4	-2	1	2

As you might guess by now, the new row 4 is irreducible, so we can call it $\chi_{(2,1,1)}$ and subtract it off from the last row.

$$\langle \chi_{(2,1,1)}, M^{(1,1,1,1)} \rangle = 3$$

Cycle type	(1,1,1,1)	(2,1,1)	(2,2)	(3,1)	(4)
χ_4	1	1	1	1	1
$\chi_{(3,1)}$	3	1	-1	0	-1
$\chi_{(2,2)}$	2	0	2	-1	0
$\chi_{(2,1,1)}$	3	-1	-1	0	1
$\chi_{(1,1,1,1)}$	1	-1	1	1	-1

we've ended with the character table of S_4 .

5. References

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