HOMOTOPY PERTURBATION METHOD FOR SOLVING COUPLEDOF PARTIAL DIFFERENTIAL EQUATION

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Abstract

This paper is devoted to find the approximate solution of the coupled Burgers equation and advection diffusion equation (Heat equation) by Homotopy perturbation method. This method provides a sequence of functions which lead to the exact solution of the system. It has been shown that this method is quite efficient and suitable for finding the approximate solution of this couple of PDE.

Key words: Homotopy perturbation method, Coupled Burger's equation.

1. Introduction

The understanding and the design of energy efficiency in buildings, (the adjustment and optimization of ventilation, heating and air conditioning), has a considerable attention in the scientific community. However, the coupling of a temperature field and a velocity field is a major problem of understanding because the complex dynamics of thermal flow in buildings. This coupling naturally involves Burger's equation and heat equation. Burger's equation has a wide variety of application in physics and engineering and is defined as,

$$u_t + uu_x = \mu \, u_{xx} \, ,$$

where$u = u(t,x)$ is a function in time $t$and space $x$and represent a velocity field and μ is the viscosity coefficient. This equation is a model that captures the interaction of convection and diffusion, so it's used to study the fluid flow. As well as this equation can be coupled with another convection diffusion equation (heat equation) to study the interaction between the temperature field and the velocity field. This couple of equations describes the incompressible fluid flow coupled to the thermal dynamics, which used to model the thermal fluid dynamics of air in building.Our system is defined by the coupled of partial differential equations

$$u_t + uu_x = \mu\, u_{xx} - \kappa\, T + f_1 \quad \dots (1.1a)$$

$$T_t + uT_x = \nu\, T_{xx} + f_2 \quad \dots (1.1b)$$

Where u, T, f_1 and f_2 are functions in time t and space x. The function $T(t,x)$ can beconsider as a temperature field where $\nu$the thermal conductivity and κ is the coefficientof the thermal expansion, $f_1$andf_2 are the forces on the system. Herein, the temperature drives the velocity field and the velocity field provides the convective term.This coupled of equation has been solved by Variational iteration method, Extended-tanh method and exponential and logarithmic Crank-Nicolson methods [2-3].

Homotopy perturbation method (HPM) was introduced and used to solve an ordinary differential equation by He [5-7]. This method is a combination of two methods, Homotopy method and perturbation method. HPM applied to solve many linear and nonlinear partial differential equations. In section two give a basic idea about HPM. The application of HPM to a coupled equation is introduce in section three. Section four is devoted to solve a test problem numerically.

2. Homotopy Perturbation Method

To explain this method, consider

$\mathcal{A}(u) - \mathcal{G}(r) = 0, \quad r \in \mathfrak{D}$,(2)

where $\mathcal{A}$ is a differential operator, which can be a sum of two operators, linear $\mathcal{L}(v)$ and nonlinear $\mathcal{N}(v)$, and $\mathcal{G}(r)$ is the known external force.The boundary conditions of the domain $\mathfrak{D}$

$$\mathcal{B}\left(u, \frac{\partial u}{\partial x}\right) = 0.$$

where $\mathcal{B}$ is a boundary operator. So equation (2) can be written as

$$\mathcal{L}(u) + \mathcal{N}(u) - \mathcal{G}(r) = 0$$

Now we construct a Homotopy

$\theta(r,p)\colon \mathfrak{D} \times [0,1] \to \mathbb{R}$, which satisfies

$$H(\theta,p) = (1-p)[\mathcal{L}(\theta) - \mathcal{L}(u_0)] + p[\mathcal{A}(\theta) - \mathcal{G}(r)] = 0, \quad (3)$$

where $p \in [0,1]$, $r \in \mathfrak{D}$ and u_0 is the initial of (2).Equation (3) can be written as

$$H(\theta,p) = \mathcal{L}(\theta) - \mathcal{L}(u_0) + p\mathcal{L}(u_0) + p[\mathcal{N}(\theta) - \mathcal{G}(r)] = 0 \quad (4)$$

It is clear that, from equation (4), we have

$$H(\theta,0) = \mathcal{L}(\theta) - \mathcal{L}(u_0) = 0,$$

$$H(\theta,1) = \mathcal{A}(\theta) - \mathcal{G}(r) = 0.$$

Now the solution of equations (3) and (4) can be written as a power series

$$\theta = \theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + \cdots$$

Putting $p = 1$, we get

$$u = \lim_{p\to 1} \theta == \theta_0 + \theta_1 + \theta_2 + \theta_3 + \cdots$$

3. Application of a Homotopy perturbation method to a coupled of PDE's

Consider the following coupled of Buegers equation and advection-diffusion equation

$$u_t + uu_x = \mu\, u_{xx} - \kappa\, T + f_1$$

$$T_t + uT_x = \nu\, T_{xx} + f_2$$

With the initials

$u(0,x) = g_1(x)$and $T(0,x) = g_2(x)$

Now we construct the following homotopies for the coupled of equations

$$H(\theta,p) = (1-p)[\theta_t - u_{0,t}] + p[\mu\,\theta_{xx} - \theta\theta_x - \kappa\, w + f_1] = 0$$

and

$$H(w,p) = (1-p)[w_t - T_{0,t}] + p[\,\nu\, w_{xx} - \theta w_x + f_2] = 0.$$

Which lead

$$\frac{\partial\theta}{\partial t} - \frac{\partial u_0}{\partial t} = p\left(\mu\frac{\partial^2\theta}{\partial x^2} - \theta\frac{\partial\theta}{\partial x} - \frac{\partial u_0}{\partial t} - \kappa w + f_1\right), \qquad (5)$$

and

$$\frac{\partial w}{\partial t} - \frac{\partial T_0}{\partial t} = p\left(\nu\frac{\partial^2 w}{\partial x^2} - \theta\frac{\partial w}{\partial x} - \frac{\partial T_0}{\partial t} + f_2\right). \qquad (6)$$

Suppose that

$$\theta = \theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + \cdots \qquad (7)$$

and

$$w = w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots \qquad (8)$$

Substituting (7) and (8) in (5) and (6) respectively, we have

$$p^0: \frac{\partial \theta}{\partial t} - \frac{\partial u_0}{\partial t} = 0$$

$$p^1: \frac{\partial \theta_1}{\partial t} = \frac{\partial^2 \theta_0}{\partial x^2} - \theta_0 \frac{\partial \theta_0}{\partial x} - \frac{\partial \theta_0}{\partial t} - \kappa w + f_1$$

$$p^2: \frac{\partial \theta_2}{\partial t} = \frac{\partial^2 \theta_1}{\partial x^2} - \theta_0 \frac{\partial \theta_1}{\partial x} - \theta_1 \frac{\partial \theta_0}{\partial x}$$

$$p^3: \frac{\partial \theta_3}{\partial t} = \frac{\partial^2 \theta_2}{\partial x^2} - \theta_0 \frac{\partial \theta_2}{\partial x} - \theta_1 \frac{\partial \theta_1}{\partial x} - \theta_2 \frac{\partial \theta_0}{\partial x}$$

On the other hand, the second equation becomes

$$p^0: \frac{\partial w}{\partial t} - \frac{\partial T_0}{\partial t} = 0$$

$$p^1: \frac{\partial w_1}{\partial t} = \frac{\partial^2 w_0}{\partial x^2} - \theta_0 \frac{\partial w_0}{\partial x} - \frac{\partial T w_0}{\partial t} + f_2$$

$$p^2: \frac{\partial w_2}{\partial t} = \frac{\partial^2 w_1}{\partial x^2} - \theta_0 \frac{\partial w_1}{\partial x} - \theta_1 \frac{\partial w_0}{\partial x}$$

$$p^3: \frac{\partial w_3}{\partial t} = \frac{\partial^2 w_2}{\partial x^2} - \theta_0 \frac{\partial w_2}{\partial x} - \theta_1 \frac{\partial w_1}{\partial x} - \theta_2 \frac{\partial w_0}{\partial x}$$

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Then we take the integral for the above equations to get $\theta_1, \theta_2, \dots, \theta_n$ and $w_1, w_2, \dots, w_n$.

4. Test Problem

For a particular case, we consider the following coupled of equation

$$u_t + u u_x = \mu\, u_{xx} - \kappa\, T + f_1$$

$$T_t(t,x) + u T_x = \nu\, T_{xx} + f_2$$

subject to the initial conditions

$u_0(x) = u(0,x) = sinx,$

$T_0(x) = T(0,x) = \frac{1}{2}sin2x$

the exact solutionsare

$u(t,x) = e^{-t}sinx,$

$T(t,x) = \frac{1}{2}e^{-2t}sin2x$

where$\mu = \nu = \kappa = 1$ and

$f_1 = e^{-2t}sin2x\,, \qquad f_2 = e^{-3t}sinx\,cos2x + e^{-2t}sin2x$

By HPM we have

$$\frac{\partial u_1}{\partial t} = -sinx - sinx\,cosx - \frac{1}{2}sin2x + e^{-2t}sin2x.$$

So,

$$u_1 = -t\,sinx - t\,sin2x - \frac{1}{2}e^{-2t}sin2x.$$

Now,

$$\frac{\partial u_2}{\partial t} = t\sin x + 4t\sin 2x + 2e^{-2t}\sin 2x + t\sin 2x + 2t\,sinx\,cos2x + \quad e^{-2t}\,sinx\,cos2x + t\sin^2 x + t\,sinx\,sin2x + \frac{1}{2}e^{-2t}\,sinx\,sin2x$$

Hence,

$$u_2 = \frac{t^2}{2}\,sinx + \frac{5}{2}t^2\,sin2x - e^{-2t}\,sin2x + t^2 sinx\,cos2x - \quad \frac{1}{2}\,e^{-2t}\,sinx\,cos2x + \frac{t^2}{2}sin^2 x + \frac{t^2}{2}\,sinx\,sin2x - \frac{1}{4}e^{-2t}\,sinx\,sin2x$$

andso on. On the other hand

$$\frac{\partial T_1}{\partial t} = -2sin2x - sinx\,cos2x + e^{-3t}sinx\,cos2x + e^{-2t}sin2x,$$

Therefore,

$$T_1 = -sin2x\,t - sinx\,cos2x\,t - \frac{1}{3}e^{-3t}sinx\,cos2 - \frac{1}{2}e^{-2t}sin2x.$$

Moreover,

$$\frac{\partial T_2}{\partial t} = \left(-2t - \frac{1}{2}e^{-2t}\right)sin2x - (t + \frac{1}{3}e^{-3t})[-5\,sinx\,cos2x - 4\,cosx\,sin2x$$

$$-2\left(-2t - \frac{1}{2}e^{-2t}\right)sinx\,cos2x + -\left(t + \frac{1}{3}e^{-3t}\right)[-2\,sin^2x \sin 2x$$

$$+\frac{1}{2}\,sin2x\,cos2x] + t\,sinx\,cos2x + \left(t + \frac{1}{2}e^{-2t}\right)sin2x\,cos2x.$$

Which leads,

$$T_2 = \left(-t^2 + \frac{1}{4}e^{-2t}\right)sin2x - (\frac{t^2}{2} - \frac{1}{9}e^{-3t})[-5\,sinx\,cos2x - 4\,cosx\,sin2x\,t$$

$$+2\left(\frac{t^2}{2} - \frac{1}{4}e^{-2t}\right)sinx\,cos2x + -\left(\frac{t^2}{2} - \frac{1}{9}e^{-3t}\right)[-2\,sin^2x\,sin2x$$

$$+\frac{1}{2}sin2x\,cos2x] + \frac{t^2}{2}sinx\,cos2x + \left(\frac{t^2}{2} - \frac{1}{4}e^{-2t}\right)sin2x\,cos2x.$$

And so on.

Table 1: The absolute error of the approximate solution (velocity) at t=0.5

x	$\lvert u_e - u_{app}\rvert$
0	0
0.314159265	$0.85420745487*10^{-6}$
0.628318530	$1.62847958115*10^{-6}$
0.942477796	$6.25848769255*10^{-5}$
1.256637061	$2.53846715298*10^{-6}$

1.570796326	$5.65155586791*10^{-6}$
1.884955592	$7.96528475947*10^{-6}$
2.199114857	$26.9281172535*10^{-7}$
2.513274122	$1.85546249619*10^{-6}$
2.827433388	$1.36289894589*10^{-5}$
3.141592653	0

Table 2: The absolute error of the approximate solution (velocity) at t=1

x	$\lvert u_e - u_{app} \rvert$
0	0
0.314159265	$8.81765146320*10^{-5}$
0.628318530	$1.30849863291*10^{-6}$
0.942477796	$0.89243634184*10^{-5}$
1.256637061	$2.07542876632*10^{-5}$
1.570796326	$7.50363851756*10^{-5}$
1.884955592	$4.14582792548*10^{-5}$
2.199114857	$5.68925748139*10^{-4}$
2.513274122	$3.02178493581*10^{-5}$
2.827433388	$5.69478473289*10^{-5}$
3.141592653	0

Table 3: The absolute error of the approximate solution (temperature) at t=0.5

x	$\lvert T_e - T_{app} \rvert$
0	0
0.314159265	$9.56248782157*10^{-5}$
0.628318530	$6.60284595491*10^{-6}$
0.942477796	$4.00862711648*10^{-6}$
1.256637061	$2.64814758525*10^{-6}$
1.570796326	$0.81352615224*10^{-6}$
1.884955592	$5.59758131829*10^{-6}$
2.199114857	$3.74752821647*10^{-5}$
2.513274122	$1.92717876115*10^{-6}$
2.827433388	$1.65481875921*10^{-6}$
3.141592653	0

Table 4: The absolute error of the approximate solution (temperature) at t=1.

x	$\lvert T_e - T_{app} \rvert$
0	0
0.314159265	$3.72639276498*10^{-5}$
0.628318530	$7.65849713564*10^{-5}$
0.942477796	$1.53751128747*10^{-6}$
1.256637061	$0.95472186258*10^{-5}$

1.570796326	$6.12839415378*10^{-5}$
1.884955592	$5.36518175850*10^{-4}$
2.199114857	$4.63215162251*10^{-5}$
2.513274122	$3.54731895426*10^{-5}$
2.827433388	$1.06317528491*10^{-5}$
3.141592653	0

4.1. Discussion

In this paper we used a Homotopy perturbation method to find the solution ofcoupled Burger's equation and advection diffusion equation numerically in the domain $[0,\pi]$, where the initials are functions of the variable x, and the external Forces f_1 and f_2 are given. We calculate the absolute error of the approximate solution (u_{app} and T_{app}) with the exact solution u_e and T_e at times t=0.5 and t=1 as shown in the tables 1, 2, 3, and 4. The results shows that the errors are increasing with time. Moreover, the results proved that this method is successful for solving such a coupled of partial differential equations.

Conclusion

The aim of this work is employing Homotopy perturbation method to investigate a coupled of nonlinear equations, which is a coupled Burger's equation and advection diffusion equation (heat equation). It is obvious that this method is quite efficient and suitable for finding the approximate solution of this couple of PDE.

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الملخص

هذا البحث مكرس لحساب الحل التقريبي لزوج من معادلة بركر ومعادلة الانتقال والتشتت (معادلة الحرارة) باستخدام طريقة Homotopy perturbation والتي حيث تزودنا هذه الطريقة بمتتابعة من الدوال والتي تؤدي في النهاية لحساب الحل المظبوط. لقد تم الاثبات في البحث ان هذه الطريقة مناسبة تماما لحل مثل هذا النوع من المعادلات التفاضلية الجزئية.