Pseudo-prime radical submodule

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Abstract:

Let R be acommutative ring with identity ,and M be a unitary R-module .Aproper submodule N of M is called pseudo-prime submodule if whenever abm∈N for a,b∈R and m∈M ,then either $a^n m \in N$ or $b^k \in N$ for $n,k \in Z^n$. In this paper we Introduce pseudo-prime radical of asubmodule N as ageneralization of aprime –radical of asubmoudule N where a pseudo-prime radical of asubmodule N denoted by PP-rad_M (N) is define to the as the intersection of all pseudo prime submodule of M that contain N. also ,we Introduce the concept pseudo –prime radical submodule where aproper submodule N of M which satisfies PP-rad_M (N) = N is called pseudo –prime radical submodule ofM.We give some basic properties of these concepts.

Introduction:

Let R be a commutative ring with unity, and M be a unitary R-module. A Proper sub module N of an R-module ,M is called a prime if rm ∈ N for r∈R and m∈M implies that either m∈N or r∈[N:M].the prime radical of an R-sub module N of M denoted By rad_m(N) is defined as the intersection of all prime submodule of M containing N,if them is no prime sub module of M containing N, then rad_m (N) = M [1]. pseudo –prime sub module are generalization of a prime submodule are introduce in [4], where a proper sub module N of an R-module M is called pseudo-prime if xym∈N,where x,y∈R ,m∈M then $x^n m$∈N or $y^x m$∈N n,k∈Z^+ , in this paper we introduce the concept of pseudo-prime radical of a sub module as a generalization of prime radical of sub module in section one ,and in section two we introduce the concept of pseudo radical sub module as a generalization of a prime radical sub module and give some basic proportion of concept.

Section 1:pseudo-prime radical of a submodule

In this section, we introduce the concept of pseudo-prime radical of asubmodule as a generalization of prime radical of asubmodule and gives some of its basic properties:

Definition 1-1

Let M be an R-module ,and let N be asubmodule of M ,the pseudo –prime radical of N denoted by pp-rad_M(N) is the intersection of all pseudo-prime submodule of M containing N .If there is no pseudo-prime submodule of M containing N ,we write pp-rad_M(N)=M.

We start this section by the following proposition:

Proposition 1.2

Let M and M' be two R-modules with $f: M \to M'$ is an epi-morphism N is asubmodule of M with kerf⊆ N , then the following are satisfiy

1-f(pp-rad_M(N\)=PP-$rad_{M'}$(f(N))

2-f'(pp-rad_M(N\)=PP-rad_M($f^{-1}(N')$,whene N' is asubmodule of M'.

Proof(1)

Since pp-rad(n)=∩L ,whene the intersection is over all pseudo-prime submodule of M with N⊆L ,so that f(pp-rad_M(N))=f($\cap L$).Since kerf⊆N⊆L then by[1] we have f(pp-rad_M(N))=∩F(L) where the intersection is over all pseudo-prime submodules f(L) of M' With f(n)⊆f(L) .This f(pp-rad_M(N))=PP-rad_M(N)) .

(2):let N'be a submodule of M', then pp-rad_M (N') =∩L', Whene the intersection is over all pseudo-prime submodule L' of M' with N'⊆L' .Then by [3] ,f^{-1}(pp-$rad_{M'}$ (N'))=f^{-1}(∩L'), where the intersection is over all pseudo-prime submodule f^{-1}(L') of M with f^{-1}(N')⊆ f^{-1}(L') .

Thus $f^{-1}(PP-rad_{M'}(N')) = PP-rad_M(f^{-1}(N'))$.

The following proposition gives some basic properties of pseudo-prime radical ofasubmodule.

Proposition 1.3

Let N and L be asubmodules of anR-module M Then the following are satisfy

1-N⊆PP-rad_M(N)

2-PP-rad_M(PP-rad_M(N))=PP-rad_M(N)

3-PP-rad_M(N∩ L)⊆PP-rad_M(N)∩PP-rad_M(L)

Proof(1)

Since pp-rad_M(N)=∩L,Where the intersection is over all psudo-prime submodules L of M with N⊆L ,then N⊆PP-rad_M(N).

(2):From part (1)we have pp-rad_M(N)⊆PP-rad_M(PP-rad_M(N)).From definition of pseudo-prime radical of asubmodule we have pp-rad_M (PP-rad_M (N))= ∩L, where the intersection is over all pseudo-prime submodule L withpp-rad_M(N)⊆L. Then by part(1) we have N⊆PP-rad_M(N)and hence pp-rad_M(N))⊆PP-rad_M(N).Hence pp-rad_M(N).

(3)let P be apseudo-prime submodule of M containing L.But N∩L⊆L⊆P, Then pp-rad_M (N∩L)⊆P. Hence pp-rad_M (N∩L) ⊆ PP-rad_M (L). Also,we have pp-rad_M (N∩L)⊆ PP-rad_M (N). Hence pp-rad_M (N∩L)⊆ PP-rad_M (N)∩ PP-rad_M (L).

The following proposition gives condition where which the equality of prop.1.3.(3) hold.

Proposition1.4

Let N and L are asubmodules of an R-module M such that every pseudo –prime submodule ofM with contains is completely irreducible submodule Then pp-rad$_M$(N∩L)=PP-rad$_M$(N)∩ PP-rad$_M$(L).

Proof

From prop-1.3 we have pp-rad$_M$ (N∩ L)⊆ PP-rad$_M$(N)∩PP-rad$_M$ (L).Now if pp-rad$_M$ (N∩L)=M, then pp-rad$_M$ (N)=PP-rad$_M$ (L)=M, then pp-rad$_M$(N)=PP-rad$_M$ (L)=M. But ,if pp-rad$_M$(N)≠M,then there exist apsudo-prime submodule k of M such that N∩L⊆K .Hence pp-rad$_M$(N)⊆K or pp-rad$_M$(L)⊆K ,and then pp-rad$_M$(N)⊆ ∩K or pp-rad$_M$(L)⊆∩K ,where N∩L⊆K. It follows that pp-rad$_M$(N)⊆PP-rad$_M$(N∩L) or pp-rad$_M$(L)⊆PP-rad$_M$(N∩K)and then pp-rad$_M$(N)∩PP-rad$_M$(L)⊆PP-rad$_M$(N∩L) Thus pp-rad$_M$(N∩L)=PP-rad$_M$(N∩L)=PP-rad$_M$(N)∩PP-rad$_M$(L).

We need to introduce the following lemma.

Lemma 1.5

Let P be a pseudo-prime submodule of M, and L,N be submodule of M such that N∩L⊆P such that [N:M]=R. Then L ⊆P.

Proof

Since[N:M]+[P:M]=R, then there exist a∈ [N;M] and b∈[P;M] such that a+b=1 Now,let x∈L,then ax+bx=x,but ax∈N and ax∈L.Hence ax∈N∩L and bx∈P,then x=ax+bx∈P. Thenfor L⊆K.

proposition 1.6

let N and L be any two submodule of an R-module M,such that [N:M]+[K:M]=R for each pseudo-prime submodule k containing N∩L .Thenpp-rad$_M$(N∩L) =PP-rad$_M$(N)∩PP-rad$_M$(L).

Proof

Let N∩L⊆K ,hence by Lemma 1.5 ,we have L⊆K .Thus k is completely irreducible and hence by prop.1.4 we have pp-rad$_M$(N∩L)=PP-rad$_M$(N)∩PP-rad$_M$(L).

Section 2: pseudo-prime radical submodules

In this section we introduce the definition of pseudo-prime radical submodule and gives some of it's basic properties .

Defenetion 2.1

A proper submodule k of an R-module M is called pseudo-prime radical submodule if pp-rad$_M$(N)=N

Recall that an R-module M satisfies the ascending chain condition if everyAscending chain of submodule of M is finite [2]

Proposition 2.2

Let M be an R-module such that M satisfies the ascending chain condition for pseudo-prime radical submodule, then every pseudo-prime radical submodule of M is the intersection of finite number of pseudo-prime submodules.

Proof

Let N be a pseudo-prime radical submodule of M ,and let N=$\cap_{i\in\lambda}$N$_i$, where N$_i$ is pseudo-prime submodule of M ,for each i∈ λ ,and the expression isReduced .

Assume that λ is infinite induce set .without loss of generality we may assume that λ is countable then N=$\cap_{i=1}^{\infty}$N$_i$⊆$\cap_{i=1}^{\infty} N_i$ ⊆$\cap_{i=1}^{\infty} N_i$ ⊆… is an ascending chain of pseudo-prime radical submodules Then by prop-1.3 we have $\cap_{i\in\lambda} N_i \subseteq PP-rad_M(\cap_{i\in\lambda} N_i) \subseteq PP-rad_M(N_i) = \cap_{i\in\lambda} N_i$.
But this ascending chain must terminate ,so there exist j∈ λ such that$\cap_{i=j}^{\infty}$ N$_i$ =$\cap_{i=j+1}^{\infty}$ N$_i$ .Therefore $\cap_{i=j+1}$N$_i$⊆ N$_j$ which contradicts that the expression N=$\cap_{i=1}^{\infty}$N$_i$ is reduced ,therefore λ must be finite and hence N=$\cap_{i=1}^{n} N_i$

Proposition 2.3

Let M be an R-module such that M satisfies the ascending chain condition for pseudo-prime radical submodule.then every proper submodule of M is a pseudo-prime radical submodule of finitely generated submodule .

Proof

Assume that there exist aproper submodule N OF M which is not pseudo-prime radical of finitely generated submodule of it .let m_1 ∈N and N_1=pp-rad$_M$(Rm$_1$) ,then N$_1$⊂N ,Thus there exist m$_2$∈N-N$_1$,let N$_2$=PP-rad$_M$(Rm$_1$+Rm$_2$),then N$_1$⊂ N$_2$⊂N ,hence there exist m$_3$∈N-N$_3$.This implies that an ascending chain of pseudo-prime radical submodule N$_1$⊂N$_2$⊂N$_3$ … Which does not terminate and this is contradiction .

We end this section by the following proposition .

Proposition 2.4

Let M be afinitely generated R-module .if every pseudo-prime submodule of M is a pseudo-prime submodule of M is apseudo-prime radical of a finitely generated submodule of it. Then M satisfies the ascending chainCondition for pseudo-prime submodules .

Proof

Let N$_1$⊆ N$_2$⊆ N$_3$⊆ … be ascending chain of pseudo-prime submodule of M.Since M.since M is finitely generated ,then N=∪ N$_j$ is a pseudo-prime submodule of M.Thus by hypothesis ,N is pseudo-prime radical submodule of M for some finitely generated submodule L=Rm$_1$+Rm$_2$+…+Rm$_n$,where m$_i$∈M for all i=1,2,…,n Hence L⊆ PP-rad$_M$(L)=N=∪ N_i,then there exist jϵI Such that L⊆ N_j.there for ∪ $N_{i=} N_i$

hence the chain of pseudo-prime submoudule N_i terminates

Before we introduce the next resut we introduce the following definition

Definition 2.5

An R-module M is called pseudo-compactly packed if every proper submodule of M is pseudo-compactly packed submodule where a proper submodule N of M is called pseudo-compactly packed if for each family $\{N_\alpha\}_{\alpha\in\lambda}$ of pseudo-prime-submodule of M with N$\subseteq\cup_{\alpha\in\lambda} N_\alpha$,there exists $\alpha_1,\alpha_2,\dots\alpha_n\epsilon\lambda$ such that N$\subseteq\cup_{i=1}^{n} N_\alpha$.

Proposition 2.6

If M is pseudo-compactly packed R-module with J(M)$\neq M$ then M satisfies the ascending chain condition for pseudo-prime submodules

Proof

Let $N_1 \subseteq N_2 \subseteq \cdots$ be ascending chain of pseudo-prime submodule of M ,and N=$\cup_{i=1}^{\infty} N_i$,we prove that N=M and H is a maximal submodule of M,then H$\subseteq\cup_{i=1}^{\infty k} N$.But M is pseudo-compactly packed H$\subseteq N_{n_m}$ But H is maximal submodule of M ,then H=N_{n_m} and hence M=$\cap_{i=1}^{k} N_{n_i} = N_{n_m}$ where is a contradiction Thus N is a proper submodule of M.Thus ,there exist $N_1 \subseteq N_2 \subseteq \cdots$ is a scending chain ,so there exist m$\in\{1,2,3,\dots,k\}$ such that $\cup_{i=1}^{k} N_{n_i} = N_{n_m}$, that is $\cup_{i=1}^{k} N_{n_m}$,so $N_1 \subseteq N_2 \subseteq \dots N_n$. therefor M satisfies the ascending chain condition For pseudo-prime submodule.

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المقاسات الجزئية الجذرية الكاذبة اولياً

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الملخص

لتكن R حلقة إبداليه بمحايد و M مقاساً احادياً. المقاس الجزئي الفعلي N من M يدعى مقاس اولي كاذب اذا كان N $\in$abm حيث $a,b,m\in R$ و $m\in M$ فانه اما a^nm $\in$ N او $a^k m\in N$ و Z^+ $\in$n,k. في هذا البحث قدمنا الجزء الاولي الكاذب للمقاس الجزء N كأعمام للجذر الاولي حيث يرمز له ب((PP-rad$_M$(N)) ويعرف بانه تقاطع كل القاسات الجزئية الاولية الكاذبة والتي تحتوي N كذلك قدمنا مفهوم المقاس الجزئي للجذر الاولي الكاذب .حيث ان يقال لمقاس جزئي N من M الذي يحقق الخاصية PP-rad$_M$(N)=N بالمقاس الجزئي للجذر الاولى الكاذبة .اعطينا بعض الخواص الاساسية لهذه المفاهيم.