The χ-subgroup of the Alternating groups An

Afrah M. Ibraheem , Mahdi S. Nayef
Dep. of Mathematics, College of Education, Al-Mustansiriyah University, Baghdad, Iraq

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Abstract

In this paper we study the structure of irreducible characters of Symmetric groups S_n and the Alternating groups A_n to find χ-subgroup for the Alternating groups when χ is a non-linear character of degree less than 32.

1. Introduction

Let G be a finite group,if χ is an irreducible character of G then a subgroup H of G is called a χ-subgroup if there exist a linear character θ of H such that $<\chi_H, \theta>=1$ where <,> is the inner product of restriction of χ to H and θ ,this is define by DIXEN in [2],[3] and [4],when he using the character restriction method of χ-subgroup to construct a representation of G affording χ .

In this paper we using the Dixon's definition to find χ-subgroups of A_n when χ have degree less than 32 .In the following sections we use the classification of S_n and the atlas of finite groups in[1] to show that for all characters χ of degree less than 32 there always exist χ-subgroup ,and in most cases these can be chosen as p-subgroups .

This paper has two sections .The **section 2** discuss the irreducible characters of Symmetric groups S_n ,and in **section 3** we find χ-subgroups of alternating groups A_n .Finally we summarize our results with in the table -1-.

2.The irreducible characters of S_n

In this section we can label the irreducible characters of S_n by partitions of n ,since the number of irreducible characters of a group is equal to the number of conjugate classes .We denote the irreducible character labeled by the partition λ by [λ] ,and the set of all irreducible characters of S_n by $Irr(S_n)$,so $Irr(S_n)=\{ [\lambda] : \lambda \vdash n \}$, where ⊢ is a partition of n by λ .The notations used in this section can be found in [5] and [6] .

Definition (2.1) :

A partition $\lambda=(\lambda_1,\ldots,\lambda_I)$ of n ,n∈N is a decreasing sequence $\lambda_1 \geq \ldots \geq \lambda_I > 0$ of integers with $|\lambda| = \sum_{i=1}^{I} \lambda_i = n$,for short we write $\lambda \vdash n$ .The integer $I=I(\lambda)$ is the length of λ ,the number λ_i are the parts of λ .We also write the partition exponentially as $\lambda = (I_1^{a_1},\ldots,I_m^{a_m})$, $I_1 > \ldots > I_m > 0$,when we have a_i parts of size I_i .

Definition (2.2) :

Let $\lambda=(\lambda_1,\ldots,\lambda_I)$ be a partition of n .The ***Ferrers diagram*** of λ is an array of n dots having I left–justified rows with row i containing λ_i dots for $1 \leq i \leq I$ .The dot in row i and column j has coordinates (i,j) ,as in a matrix .

Example (2.2.1)

The partition $\lambda=(4,4,2,1)=(4^2,2,1)$ has Ferrers diagram

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Given a partition λ ,one obtains another partition λ\`, the conjugate partition of λ ,by transposing the Ferrers diagram about its main diagonal .So for $\lambda=(4^2,2,1)$ the Ferrers diagram for λ\` is

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and $\lambda`=(4,3,2^2)$.

Definition (2.3) :

If λ is a partition of n ,then each inner corner dot of the Ferrers diagram of λ is ***a node*** whose removal leaves a diagram ,the Ferrers diagram of a partition of n-1. Note that the inner corner of λ are exactly those nodes at the end of a row and column of the diagram of λ .Thus the nodes at the end of rows of $\lambda=(4^2,2,1)$ have coordinates (1,4),(2,4),(3,2) and(4,1) .

Definition (2.4) :

If $\lambda=(\lambda_1,\ldots,\lambda_I)$ is a partition of n .We say the partition has ***level*** k if $k = \lambda_2+\ldots+\lambda_I$ $(= n-\lambda_1)$ .Similarly we say that the corresponding irreducible character [λ] of S_n has level k .

Example (2.4.1)

The principal Character **1** = [n] is a character of level 0 .

By the definition above the number of irreducible characters of level k of S_n is equal to the number of partitions of k so when n=9 ,the characters [6,2,1] ,[6,3] and $[6,1^3]$ have level 3 .

Now the following theorem shows the behavior of irreducible characters of S_n when we restrict them to the subgroup S_{n-1} , and we can see it in [6] .

Theorem (2.5) (Branching theorem) :

If $\lambda=(\lambda_1, \lambda_2 ,\ldots,\lambda_I) \vdash n$,then

$$[\lambda]_{S_{n-1}} = \sum_{i=1}^{I} \epsilon_i [\lambda_1,\ldots,\lambda_i^{\backslash},\ldots,\lambda_I] \quad where \quad \lambda_i^{\backslash} = \begin{cases} \lambda_{i-1} & if \quad \lambda_i \rangle \lambda_{i+1} \quad or \quad i=1 \\ \lambda_i & if \quad \lambda_i = \lambda_{i+1} \end{cases}$$

And $\epsilon_i=0$ if $\lambda_i^{\backslash} = \lambda_i$ other wise $\epsilon_i=1$.

Lemma (2.6) :

If $r \geq 1$ and $[\lambda]$ is an irreducible character of S_n of level $k \geq r$ then the level of each constituent of $[\lambda]_{S_{n-r}}$ is at most k .Furthermore $[\lambda]_{S_{n-r}}$ has at least one constituent of level k-r .
In particular for r=k the principal character is a constituent of $[\lambda]_{S_{n-r}}$.

Proof :
By the definition (2,4) and the Branching theorem the constituents of $[\lambda]_{S_{n-r}}$ have level *k*, in case $\lambda_1^{\backslash} = \lambda_1 - 1$, and level *k* -1 in the other cases.
On the other hand, if $k > 0$ then $[\lambda_1, \ldots, \lambda_{I-1}, \lambda_I - 1]$ is a constituent of level *k*-1 of $[\lambda]_{S_{n-r}}$.
Now by induction on *r* the constituents of $[\lambda]_{S_{n-r}}$ have level at most *k* and $[\lambda]_{S_{n-r}}$ has a constituent of level $k-r$. If $r = k$ then $[\lambda]_{S_{n-r}}$ has a constituent of level 0 which is the principal character.

Definition (2.7) :
If $v = (i, j)$ is a node in the diagram of λ ,then its ***hook*** is
$H_v = H_{i,j} := \{(i, j`) : j` \geq j\} \cup \{(i`, j) : i` \geq i\}$
with corresponding ***hooklength*** $h_v = h_{i,j} := |H_{i,j}|$.

Example (2.7.1)
If $\lambda = (4^2, 2, 1)$, then the circle dots in

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are the hook $H_{2,2}$ with hooklength $h_{2,2} = 4$.
Now the following theorem gives the degrees of irreducible characters of S_n, which we can see it in [6].

Theorem (2.8) (Hook Formula):
If λ is a partition of n , then

$$[\lambda](1) = \frac{n\cdot}{\prod_{(i,j)\in\lambda} h_{i,j}} \quad .$$

Theorem (2.9) :
Let $k \geq 0$ be fixed, and suppose $(\lambda_2, \ldots, \lambda_I) \vdash k$.
Consider the irreducible character $[\lambda] = [n-k, \lambda_2, \ldots, \lambda_I]$ of S_n of level k. Then $[\lambda](1)$ is a polynomial in n of degree k.

Proof:
Let H_{ij} be the hooks of the diagram of $[\lambda]$ corresponding to the nodes (i, j). By definition (2.7) we have $|H_{ij}| = h_{ij} \leq k$ for $i \geq 2$. Also there exist n-k hooks, H_{1j} , such that $|H_{1j}| = h_{1j}$ has a value of the form $(n - m_j)$ with $m_1 < m_2 < \ldots < m_{n-k}$ for $1 \leq j \leq n-k$.
Simplifying the Hook Formula,

$$[\lambda](1) = \frac{n\cdot}{\prod_{(i,j)\in\lambda} h_{i,j}} = \frac{n\cdot}{(\prod_{(1,j)\in\lambda} h_{1,j})(\prod_{(i\geq 2,j)\in\lambda} h_{i,j})}$$

only k factors remain in the numerator. This means $[\lambda](1)$ is a polynomial in n of degree k .
Now we can see theorem(2.10) and theorem(2.11) in [6] .

Theorem (2.10):
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_I)$ be a partition of n of level k . If $0 \leq r \leq \lambda_1 - \lambda_2$, then $[\lambda_1 - r, \lambda_2, \ldots, \lambda_I]$ is a constituent with multiplicity one in $[\lambda]_{S_{n-r}}$ and all other constituents have level < k .

Proof:
If , $\lambda_1 - \lambda_2 = 0$ then r = 0 .
Now suppose $\lambda_1 - \lambda_2 > 0$. By induction on r.
(i) If r =1 the Branching Theorem shows

$$[\lambda]_{S_{n-1}} = [\lambda_1 - 1, \lambda_2, \ldots, \lambda_I] + \sum_{i=2}^{I} \epsilon_i [\lambda_1, \lambda_2, \ldots, \lambda_i^{\backslash}, \ldots, \lambda_I]$$

where $\lambda_i^{\backslash} = \begin{cases} \lambda_{i-1} & if \quad \lambda_i \rangle \lambda_{i+1} \quad or \quad i = 1 \\ \lambda_i & if \quad \lambda_i = \lambda_{i+1} \end{cases}$,

and $\epsilon_i = 1$ if $\lambda_i^{\backslash} = \lambda_i - 1$ other wise $\epsilon_i = 0$.
The constituent $[\lambda_1 - 1, \lambda_2, \ldots, \lambda_I]$ has level k and has multiplicity one in $[\lambda]_{S_{n-r}}$ and the other constituents have level k-1 .
(ii) Suppose the theorem is true for $r \geq 1$ and we have

$$[\lambda]_{S_{n-r}} = [\lambda_1 - r, \lambda_2, \ldots, \lambda_I] + \sum_j e_j \psi_j \quad \text{,where}$$

$[\lambda_1 - r, \lambda_2, \ldots, \lambda_I]$ is the constituent of level k ,and $\psi_j \in Irr(S_{n-r})$ have level< k .
(iii)We prove the theorem for r + 1.
If we restrict $[\lambda]$ to $S_{n-(r+1)}$ then we get

$$[\lambda]_{S_{n-(r+1)}} = [\lambda_1 - r, \lambda_2, \ldots, \lambda_I]_{S_{n-(r+1)}} + \sum_j e_j (\psi_j)_{S_{n-(r+1)}}$$

.Using the Branching theorem we have

$$[\lambda_1 - r, \lambda_2, \ldots, \lambda_I]_{S_{n-(r+1)}} = [\lambda_1 - (r+1), \lambda_2, \ldots, \lambda_I]_{S_{n-(r+1)}} + \sum_{i=2}^{I} \epsilon_i [\lambda_1 - r, \lambda_2, \ldots, \lambda_i', \ldots, \lambda_I]$$

where $[\lambda_1 - (r+1), \lambda_2, \ldots, \lambda_I]$ has level k and $[\lambda_1 - r, \lambda_2, \ldots, \lambda_i', \ldots, \lambda_I]$ have level k-1.
On the other hand since the characters ψ_j have level < k so by Lemma (2.6) the constituents of $(\psi_j)_{S_{n-(r+1)}}$ have level < k . This means none of the constituents of $(\psi_j)_{S_{n-(r+1)}}$ are equal $[\lambda_1 - (r+1), \lambda_2, \ldots, \lambda_I]$.
Therefore $[\lambda_1 - (r+1), \lambda_2, \ldots, \lambda_I]$ is the only constituent of $[\lambda]_{S_{n-(r+1)}}$ of level k and has multiplicity one.

Theorem (2.11) :
Let $\lambda = (\lambda_1, \ldots, \lambda_I)$ be a partition of n have level k .Suppose $\lambda_1 = n-k \geq k + \lambda_2$,then for all $m \geq n$ the characters$[m-k, \lambda_2, \ldots, \lambda_I]_{Sn-k}$ and $[n-k, \lambda_2, \ldots, \lambda_I]_{S_{n-k}}$ have the same constituents .Furthermore $[n-k, \lambda_2, \ldots, \lambda_I]$ is a constituent with multiplicity one for both of these characters.

Proof:
We use induction on *m:* If *m* = *n* then there is nothing to prove .

Suppose $m > n$ and $[m\text{-}k, \lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ and $[n-k, \lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ have the same constituents and $[n\text{-}2k, \lambda_2,\ldots,\lambda_I]$ is the only constituent of level k with multiplicity one for both of these characters. We prove the Theorem for m+1.

By using the Branching Theorem we have

$$[m+1-k,\lambda_2,\ldots,\lambda_I]_{S_m} = [m-k,\lambda_2,\ldots,\lambda_I]_{S_{n-(r+1)}} + \sum_{i=2}^{I}\in_i[m+1-k,\lambda_2,\ldots,\lambda_i{}',\ldots,\lambda_I]$$ where

$$\lambda_i{}' = \begin{cases} \lambda_{i-1} & \text{if } \lambda_i \rangle \lambda_{i+1} \text{ or } i=1 \\ \lambda_i & \text{if } \lambda_i = \lambda_{i+1} \end{cases}$$

Since $[m\text{-}k, \lambda_2,\ldots,\lambda_I]$ is a constituent of $[m+1-k,\lambda_2,\ldots,\lambda_I]_{S_m}$ and by assumption $[m-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ and $[n-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ have the same constituents so each constituent of $[n-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ is a constituent of $[m+1-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$.

Now we show each constituent of $[m+1-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ is a constituent of $[n-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$.

Let $1 \le r \le k$. We claim each constituent of $[m+1-k,\lambda_2,\ldots,\lambda_I]_{S_{(m+1)-r}}$ is either a constituent of $[m-k,\lambda_2,\ldots,\lambda_I]_{S_{m-(r-1)}}$ or a character of level k -r (note that $(m + 1) - r = m - (r - 1)$).

Using the Branching Theorem, if

(i) $r = 1$ then $[m+1-k,\lambda_2,\ldots,\lambda_I]_{S_{(m+1)-1}}$ has two different constituents,

$[m-k,\lambda_2,\ldots,\lambda_I] = [m-k,\lambda_2,\ldots,\lambda_I]_{S_{m-(1-1)}}$ and $[m+1-k,\mu_2,\ldots,\mu_I]$ where $\mu_2+\ldots+\mu_I = k\text{-}1$.

(ii) $r = 2$ then the constituents of $[m+1-k,\mu_2,\ldots,\mu_I]_{S_{(m+1)-2}}$ are either $[m-k,\mu_2,\ldots,\mu_I]$ which is a constituent of $[m-k,\lambda_2,\ldots,\lambda_I]_{S_{m-(2-1)}}$ or $[m+1-k,\eta_2,\ldots,\eta_I]$ where $\eta_2+\ldots+\eta_I = k\text{-}2$.

This means each constituent of $[m+1-k,\lambda_2,\ldots,\lambda_I]_{S_{(m+1)-2}}$ is either a constituent of $[m-k,\lambda_2,\ldots,\lambda_I]_{S_{m-(2-1)}}$ or a character of level k-2.

Therefore for $1 \le r \le k$ if $[m+1-k,\tau_2,\ldots,\tau_I]$ is a constituent of $[m+1-k,\lambda_2,\ldots,\lambda_I]_{S_{(m+1)-r}}$ of level r-1 then each constituent of $[m+1-k,\tau_2,\ldots,\tau_I]_{S_{(m+1)-r}}$ is either $[m-k,\tau_2,\ldots,\tau_I]$ which is a constituent of $[m-k,\lambda_2,\ldots,\lambda_I]_{S_{m-(r-1)}}$ or $[m+1-k,\delta_2,\ldots,\delta_I]$ where $\delta_2+\ldots+\delta_I = k\text{-}r$.

(iii) if $r = k$ then each constituent of $[m+1-k,\lambda_2,\ldots,\lambda_I]_{S_{(m+1)-k}}$ is either a constituent of $[m-k,\lambda_2,\ldots,\lambda_I]_{S_{m-(k-1)}}$ or a character of level 0 which is the trivial character **1**. Therefore by restricting the character $[m+1-k,\lambda_2,\ldots,\lambda_I]$ on S_{n-k} we have each constituent of $[m+1-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ is either a constituent of $[m-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ or the trivial character **1**. Now by assumption $[m-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ and $[n-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ have the same constituents and on the other hand by lemma (2.6) the principal character is a constituent of $[n-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ .Thus each constituent of $[m+1-k,\lambda_2,\ldots,\lambda_I]$ is a constituent of $[n-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ .By using theorem (2.10) the character $[n-2k,\lambda_2,\ldots,\lambda_I]$ is a constituent of $[m+1-k,\lambda_2,\ldots,\lambda_I]_{S_{n-k}}$ with S_{n-k} multiplicity one. This completes the proof .

Suppose $n - k \ge k + \lambda_2$ in $[\lambda] = [n\text{-}k,\lambda_2 ,\ldots,\lambda_I]$ for $k = \lambda_1,\ldots,\lambda_I$ then using the Branching Theorem we have

$$[\lambda]_{S_{n-r}} = [n-k-1,\lambda_2,\ldots,\lambda_I]_{S_{n-r}} + \sum_{i=2}^{I}\in_i[n-k,\lambda_2,\ldots,\lambda_i{}',\ldots,\lambda_I]_{S_{n-r}}$$ for $r \ge 1$.

Now if $n - r \ge k + \lambda_2$ then the constituent of level k has multiplicity one and if we want to know the structure of $[\lambda]_{S_{n-r}}$ we need to know the structure of $[n-k,\lambda_2,\ldots,\lambda_i{}',\ldots,\lambda_I]_{S_{n-r}}$ .It means that if we want to know the structure of characters of level k, we should know the structure of characters of level k-1. Then by this recursive method we can get the structure of all irreducible characters of S_n when we restrict them on S_{n-r}.

Now in the following theorem we describe the restriction of the irreducible characters $[\lambda]$ of S_n of level k = 1, 2, 3.

Theorem (2.12): Let $r \ge 0$ then ,

1. If $[\lambda] = [n-1, 1]$ then for $n - r \ge 2$

$[\lambda]_{S_{n-r}} = [n - (r + 1), 1] + r.1$

2. If $[\lambda] = [n-2, 2]$ then for $n - r \ge 4$

$[\lambda]_{S_{n-r}} = [n - (r + 2), 2] + r\,[n-(r+1),1] + \frac{r(r-1)}{2}.1$

3. If $[\lambda] = [n-2, 1^2]$ then for $n - r \ge 3$

$[\lambda]_{S_{n-r}} = [n - (r + 2), 1^2] + r\,[n-(r+1),1] + \frac{r(r-1)}{2}.1$

4. If $[\lambda] = [n-3, 3]$ then for $n - r \ge 6$

$[\lambda]_{S_{n-r}} = [n - (r + 3), 3] + r\,[n-(r+2),2] + \frac{r(r-1)}{2}[n-(r+1),1] + \frac{r(r-1)(r-2)}{6}.1$

5. If $[\lambda] = [n-3, 1^3]$ then for $n - r \ge 4$

$[\lambda]_{S_{n-r}}$ = [n-(r + 3),1^3] + r[n-(r+2),1^2]+ $\frac{r(r-1)}{2}$ [n-(r+1),1]+ $\frac{r(r-1)(r-2)}{6}$.1

6. If [λ] = [n -3, 2,1] then for n - r ≥ 5

$[\lambda]_{S_{n-r}}$ = [n - (r + 3),2,1] + r([n-(r+2),2]+[n-(r+2),1^2]) + r(r-1) [n-(r+1),1]+ $\frac{r(r-1)}{2}$ [n-(r+1),1]+ $\frac{r(r-1)(r-2)}{3}$.1

Proof:

It is clear by induction on r .

Example(2.12.1)

Let we consider the irreducible character [n- 4,3,1] of level 4 of S_n then by induction on r ≥ 1 we can show

$$[n-4,3,1]_{S_{n-r}} = [n-(r+4),3,1] + \sum_{i=1}^{r}([n-(r+4-i),2,1]_{S_{n-r}} + [n-(r+4-i),3]_{S_{n-r}})$$

which by using parts (4) and (6) of the theorem above gives us all constituents and their multiplicities.

Similarly for the other characters of level 4 of S_n we have

$$[n-4,4]_{S_{n-r}} = [n-(r+4),4] + \sum_{i=1}^{r}([n-(r+4-i),3]_{S_{n-r}}$$

$$[n-4,2,1^2]_{S_{n-r}} = [n-(r+4),2,1^2] + \sum_{i=1}^{r}([n-(r+4-i),1^3]_{S_{n-r}} + [n-(r+4-i),2,1]_{S_{n-r}})$$

$$[n-4,2^2]_{S_{n-r}} = [n-(r+4),2^2] + \sum_{i=1}^{r}[n-(r+4-i),2,1]_{S_{n-r}}$$,and

$$[n-4,1^4]_{S_{n-r}} = [n-(r+4),1^4] + \sum_{i=1}^{r}[n-(r+4-i),1^3]_{S_{n-r}}.$$

3.The χ–Subgroups of A_n

In this section for all irreducible characters χ of degree less than 32 of alternating group A_n there exists a χ –subgroup ,and then in most cases this can be chosen as a p-subgroup .We list our results in the table -1- .

Definition (3.1) :

Let G be a finite group and χ be an irreducible character of G .The subgroup H of G is called a χ-subgroup if there exists a linear character θ of H such that $\langle \chi_H, \theta \rangle = 1$.

Since the alternating group A_n is a normal subgroup of index 2 in S_n, so by Clifford's Theorem the restriction of each irreducible character of S_n to A_n is either irreducible or splits into two conjugate irreducible characters of A_n.

Let λ⊢n and λ\` be the conjugate partition of $\lambda$ .If λ=($\lambda_1,\ldots,\lambda_I$) ⊢n with $\lambda_1 \neq I$ , then clearly λ ≠ λ\` .Under the condition λ ≠ λ\`, $[\lambda]_{A_n}$ is irreducible. Therefore the following characters are irreducible $[n-1,1]_{A_n}$ for $n \geq 4$, $[n-2,2]_{A_n}$ for $n \geq 5$, $[n-2,1^2]_{A_n}$, $[n-3,3]_{A_n}$ for $n \geq 6$, $[n-3,1^3]_{A_n}$ for $n \geq 8$ and $[n-3,2,1]_{A_n}$ for $n \geq 7$.

This proves the following theorem .

Theorem (3.2) :

Let r ≥ 0. Then the restrictions of irreducible characters of A_n decompose as follows:

1. For n - r ≥ 4

$$[n-1,1]_{A_{n-r}} = [n-(r+1),1]_{A_{n-r}} + r.1 .$$

2. For n - r ≥ 4

$$[n-2,2]_{A_{n-r}} = [n-(r+2),2]_{A_{n-r}} + r[n-(r+1),1]_{A_{n-r}} + \frac{r(r-1)}{2}.1$$

3. For n - r ≥ 6

$$[n-2,1^2]_{A_{n-r}} = [n-(r+2),1^2]_{A_{n-r}} + r[n-(r+1),1]_{A_{n-r}} + \frac{r(r-1)}{2}.1 .$$

4. For n - r ≥ 6

$$[n-3,3]_{A_{n-r}} = [n-(r+3),3]_{A_{n-r}} + r[n-(r+2),2]_{A_{n-r}} + \frac{r(r-1)}{2}[n-(r+1),1] + \frac{r(r-1)(r-2)}{6}.1$$

5. For n - r ≥ 8

$$[n-3,1^3]_{A_{n-r}} = [n-(r+3),1^3]_{A_{n-r}} + r[n-(r+2),1^2]_{A_{n-r}} + \frac{r(r-1)}{2}[n-(r+1),1] + \frac{r(r-1)(r-2)}{6}.1$$

6. For n - r ≥ 7

$$[n-3,2,1]_{A_{n-r}} = [n-(r+3),2,1]_{A_{n-r}} + r([n-(r+2),2]_{A_{n-r}} + [n-(r+2),1^2]_{A_{n-r}}) + r(r-1)[n-(r+1),1]_{A_{n-r}} + \frac{r(r-1)}{2}[n-(r+1),1] + \frac{r(r-1)(r-2)}{3}.1$$

Now consider the characters χ := [n-1, 1] , [n-2,2] and [n-2,1^2] of degrees n-1, (n^2-3n)/2 and (n^2-3n+2)/2, respectively.

Theorem (3.3) :

1. If n ≥ 4 and χ = $[n-1,1]_{A_n}$, then A_3 is a χ-subgroup.
2. If n ≥ 6 and χ = $[n-2,2]_{A_n}$ or $[n-2,1^2]_{A_n}$,then $Syl_{A6}(3)$, a Sylow 3-subgroup of A_6 ,is a χ-subgroup.

Proof :

Suppose χ = $[n-1,1]_{A_n}$. By theorem (3.2) with *n - r* =4 $\chi_{A_n} = [3,1]_{A_n} + (n-r).1$ .On the other hand a simple calculation shows that $[3,1]_{A_n} = 1 + \varphi_1 + \varphi_2$ where φ_1 and φ_2 are distinct non-trivial linear characters of A_3. Therefore $\chi_{A_3} = (n-3).1 + \varphi_1 + \varphi_2$ .This means $\langle \chi_{A_3}, \varphi_1 \rangle = \langle \chi_{A_3}, \varphi_2 \rangle = 1$ and A_3 is a χ–subgroup by definition (3.1) .

Let H := $Syl_{A6}(3)$ be a Sylow 3-subgroup of A_6. Then H is an abelian subgroup of order 9 and Irr(H) = { φ_1 ,..., φ_2 } where φ_1 = 1 and φ_2 ,..., φ_9 are non-trivial linear characters of *H*. If χ = $[n-2,2]_{A_n}$, then by

theorem (3.2) with $n - r = 6$ shows $\chi_{A_6} = [4,2]_{A_6} + (n-6)[5,1]_{A_6} + \frac{(n-6)(n-7)}{2}.1$.
On the other hand a simple calculation shows that $[4,2]_H = \sum_{i=1}^{9}\varphi_i$ and $[5,1]_H = \sum_{i=1}^{5}\varphi_i$ for a suitable ordering of the characters φ_i . Therefore

$$\chi_H = \sum_{i=1}^{9}\varphi_i + (n-6)\sum_{i=1}^{5}\varphi_i + \frac{(n-6)(n-7)}{2}.1$$
$$= \sum_{i=1}^{9}\varphi_i + (n-5)\sum_{i=2}^{5}\varphi_i + \frac{n^2-11n+32}{2}.1$$

and this implies $\langle \chi_H, \varphi_i \rangle = 1$ for $6 \le i \le 9$ as required.

Now consider χ = $[n-2,1^2]_{A_n}$. Then by theorem (3.2) with $n - r = 6$,

$$\chi_{A_6} = [4,1^2]_{A_6} + (n-6)[5,1]_{A_6} + \frac{(n-6)(n-7)}{2}.1$$ where $[4,1^2]_H = 2\varphi_1 + \sum_{i=1}^{9}\varphi_i$ and $[5,1]_H = \sum_{i=1}^{5}\varphi_i$ .Therefore

$$\chi_H = 2\varphi_1 + \sum_{i=1}^{9}\varphi_i + (n-6)\sum_{i=1}^{5}\varphi_i + \frac{(n-6)(n-7)}{2}.1$$
$$= \sum_{i=6}^{9}\varphi_i + (n-5)\sum_{i=2}^{5}\varphi_i + \frac{n^2-11n+34}{2}.1$$

This implies $\langle \chi_H, \varphi_i \rangle = 1$ for $6 \le i \le 9$, and by definition (3.1) the proof of (2) is complete .

(3.4)Summary

Table -1- χ-Subgroups of A_n which are p-subgroup

G	|G|	1< χ (1) < 32	χ- subgroup
A_5	$2^2.3.5$	3 ,4 ,5	Syl(3)
A_6	$2^3.3^2.5$	5 ,8 ,9 ,10	Syl(2)
A_7	$2^3.3^2.5.7$	6 ,10 ,14 ,15 ,21	Syl(3)
A_8	$2^6.3^2.5.7$	7 ,14 ,20 ,21 ,28	Syl(3) for χ(1) <28 Syl(2) for χ(1) =28
A_9	$2^6.3^4.5.7$	8 ,21 ,27 ,28	A_3 for χ(1) =28 Syl(2) for χ(1) ≠28
A_{10}	$2^7.3^4.5^2.7$	9	A_3
A_{11}	$2^7.3^4.5^2.7.11$	10	A_3
A_{12}	$2^9.3^5.5^2.7.11$	11	A_3
A_{13}	$2^9.3^5.5^2.7.11.13$	12	A_3

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شخوص الزمرة الجزئية للزمرة المتناوبة An

افراح محمد ابراهيم ، مهدي صالح نايف

قسم الرياضيات ، كلية التربية ، الجامعة المستنصرية ، بغداد ، العراق

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الملخص :

في هذا البحث قمنا بدراسة بنية الشخوص غير القابلة للاختزال للزمر التناظرية S_n والمتناوبة A_n لإيجاد χ -للزمر الجزئية من الزمر المتناوبة عندما تكون χ غير خطية ذات درجة اقل من 32 .