Finitely Pseudo-N-Injective Modules

Ali S. Mijbass[1], Hibat K. Mohammadali[2], Nagham A.Hussein[2]
[1]*Department of Mathematics ,College of Computers Sciences and Mathematics , University of Tikrit,Tikrit,Iraq.*
[2] *Department of Mathematics, Education College, University of Tikrit,Tikrit, Iraq.*

(**Received:** 13 / 7 / 2009 ---- **Accepted:** 13 / 12 / 2010)

Abstract

In this work, we introduce the concept of finitely pseudo-N-injective modules as a generalization for the concepts of pseudo-N-injective modules and finitely N-injective modules. Many characterizations and properties of finitely pseudo-N-injective modules are obtained. Relationships between finitely pseudo-injective modules and other classes of modules are given. New characterizations of semi-simple artinain rings and strongly regular rings are given by finitely pseudo-injectivity property. Furthermore, Endomorphisms rings of finitely pseudo-injective modules are studied.

Introduction

Throughout this paper, R will denote an associative commutative ring with identity ,and all R-modules are unitary(left) R-modules. Given two R-modules M and N, M is called pseudo-N-injective if for any R-submodule A of N and every R-monomorphism from A into M can be extended to an R-homomorphism from N into M[13]. An R-module M is called pseudo-injective if M is pseudo-M-injective[13]. An R-module M is called finitely N-injective if for any finitely generated R-submodule B of N and every R-homomorphism from B into M can be extended to an R-homomorphism from N into M , and M is called finitely quasi-injective if M is finitely M-injective [14]. For an R-module M , E(M) stand for the injective envelope of M and Hom(N,M) is the set of all R-homomorphism from an R-module N into an R-module M. This paper is based on M .Sc. thesis written by the third author under supervision of the first and second authors and submitted to the college of Education ,university of Tikrit in September 2007.

§1:Basic properties of finitely pseudo-N-injective modules. Definition 1.1

Let M and N be two R-modules. M is said to be finitely pseudo-N-injective if for any finitely generated R-submodule H of N, and any R-monomorphism f: H $\rightarrow$ M can be extended to an R-homomorphism from N into M. An R-module M is called finitely pseudo-injective if M is finitely pseudo-M-injective.

A ring R is called finitely pseudo-injective if R is finitely pseudo-R-injective R-module.

Examples and Remarks 1.2

1-Every pseudo-N-injective module is finitely-pseudo-N-injective for any R-module N. We do not have a finitely pseudo-N-injective module which is not pseudo-N-injective.

2-Every finitely N-injective module is finitely pseudo-N-injective module. The converse need not be true .The Z-module Z_2 is a finitely pseudo-$Q\oplus Z_2$-injective, but Z_2 is not finitely $Q\oplus Z_2$-injective, where Q is the set of all rational numbers.

3-A finitely-pseudo-N-injective R-module is not closed under direct sum. For example: Z_2 and Z_4 are finitely-pseudo-injective Z-modules, but $Z_2\oplus Z_4$ is not finitely- pseudo-injective Z-module.

In the following theorem, we give many characterizations of finitely-pseudo-N-injective R-modules.

Theorem 1.3 Let M and N be two R-modules and S=End(M). Then the following statements are equivalent.

1- M is finitely-pseudo-N-injective.

2-For each finitely generated R-submodule L=$\sum_{\lambda=1}^{s} Rm_\lambda$ $\subseteq$M and for each finitely generated R-submodule K=$\sum_{\lambda=1}^{s} Rn_\lambda$ $\subseteq$N , where $s\in Z$,and for each finite subset $\{r_1, r_2, \ldots, r_s\}$ of R, $\sum_{\lambda=1}^{s} r_\lambda n_\lambda = 0$ if and only if $\sum_{\lambda=1}^{s} r_\lambda m_\lambda = 0$ there exists an R-homomorphism g: N$\rightarrow$M such that $g(n_\lambda)=m_\lambda$,for all λ =1,2,…,s.

3-For each finitely generated R-submodule L=$\sum_{\lambda=1}^{s} Rm_\lambda$ $\subseteq$M and for each finitely generated R-submodule K=$\sum_{\lambda=1}^{s} Rn_\lambda$ $\subseteq$N where $s\in Z$ and for each finite subset $\{r_1, r_2, \ldots, r_s\}$ of R, $\sum_{\lambda=1}^{s} rn_\lambda = 0$ if and only if $\sum_{\lambda=1}^{s} rm_\lambda = 0$, then for each $f\in S$, there exists R-homomorphism $h\in$Hom(N,M) such that $f(m_\lambda)=h(n_\lambda)$, for all λ =1,2,…,s.

4-For each R-monomorphism f: A$\rightarrow$M , where A is any R-submodule of N, and for each finite set $\{a_1, a_2, \ldots, a_s\}$, there exists an R-homomorphism g:N$\rightarrow$M such that $g(a_\lambda)=f(a_\lambda)$, for all λ =1,2,…,s.

Proof: (1)$\Rightarrow$(2) Let f: K$\rightarrow$M defined by f $(\sum_{\lambda=1}^{s} r_\lambda n_\lambda) = \sum_{\lambda=1}^{s} r_\lambda m_\lambda$.

It is easily proved that f is an R-monomorphism. Since M is a finitely-pseudo-N-injective R-module, so there exists an R-homomorphism g:N$\rightarrow$M such that g(k)=f(k), for each k $\in$K. In particular,

$g(n_\lambda)=f(n_\lambda)=m_\lambda$ for all $n_\lambda\in K$

(2) $\Rightarrow$(3) Let $\alpha\in$ S. By hypothesis, there exists an R-homomorphism g:N$\rightarrow$ M such that $g(n_\lambda)=m_\lambda$,where $n_\lambda\in K$ and $m_\lambda\in L$ .Thus $\alpha(m_\lambda)=\alpha(g(n_\lambda))=(\alpha \circ g)(n_\lambda)$, for all

$\lambda=1,2,\dots,s$. Hence $\alpha(L)\subseteq(\alpha \circ g)(K)$. Since $\alpha \circ g\in Hom(N,M)$, so $\alpha(L) \subseteq Hom(N,M)(K)$, for all $\alpha\in S$. Therefore $S(L) \subseteq Hom(N,M)(K)$.(3)$\Rightarrow$(4) Let f: A$\rightarrow$M be any R-monomorphism. Put K= $\sum_{\lambda=1}^{s} Ra_\lambda$ and f(a_λ)=m_λ, where $m_\lambda \in M$. Thus L= $\sum_{\lambda=1}^{s} Rm_\lambda \subseteq M$, and for each finite set {r_1 ,r_2 ,…,r_s } of R, $\sum_{\lambda=1}^{s} ra_\lambda = 0$ if and only if $\sum_{\lambda=1}^{s} rm_\lambda = 0$ .Let I:M$\rightarrow$M be the identity R-homomorphism. But I$\in$S, so there exists an R-homomorphism g$\in$Hom(N,M) such that g(a_λ)=m_λ =f(a_λ), for all $\lambda=1,2,\dots,s$.

(4) $\Rightarrow$(1) Let A= $\sum_{\lambda=1}^{s} Ra_\lambda$ be any finitely generated R-submodule of N and f: A$\rightarrow$M ,be any R-monomorphism. By hypothesis there exists an R-homomorphism g: N$\rightarrow$M ,such that g(a_λ)=f(a_λ), for all $\lambda=1,2,\dots,s$. For each x $\in$A , x= $\sum_{\lambda=1}^{s} r_\lambda a_\lambda$ where $r_\lambda \in R$, for all $\lambda=1,2,\dots,s$. Thus g(x)= $\sum_{\lambda=1}^{s} r_\lambda g(a_\lambda)$ $\sum_{\lambda=1}^{s} r_\lambda f(a_\lambda)$ =f(x) , proving that g is an extension of f. Therefore M is finitely-pseudo-N-injective .

As an immediate consequence of Th. 1.3, we have the following corollary in which we get many characterizations of finitely-pseudo-injective modules.

Corollary 1.4

The following statements are equivalent for an R-module M.

1-M is finitely-pseudo-injective.

2-For each finitely generated R-submodule L= $\sum_{\lambda=1}^{s} Rm_\lambda$ and K= $\sum_{\lambda=1}^{s} Rk_\lambda$ of M, where m_λ , $k_\lambda \in M$ and $s\in Z$, and for each finite subset {r_1 ,r_2 ,…,r_s } of R, $\sum_{\lambda=1}^{s} r_\lambda n_\lambda$ =0 if and only if $\sum_{\lambda=1}^{s} rm_\lambda = 0$ there exists an R-homomorphism g:M$\rightarrow$M such that g(k_λ)=m_λ ,for all $k_\lambda \in K$, $m_\lambda \in L$ and $\lambda=1,2,\dots,s$.

3-For each finitely generated R-submodule L= $\sum_{\lambda=1}^{s} Rm_\lambda$ and K= $\sum_{\lambda=1}^{s} Rk_\lambda$ of M where m_λ , $k_\lambda \in M$ and s $\in$Z,and for each finite subset {r_1 ,r_2 ,…,r_s } of R , $\sum_{\lambda=1}^{s} r_\lambda k_\lambda$ =0 if and only if $\sum_{\lambda=1}^{s} r_\lambda m_\lambda$ =0, then for each f $\in$S, there exists h $\in$S such that f(m_λ)=h(k_λ), for all $\lambda=1,2,\dots,s$,and hence S(L) $\subseteq$ S(K).

4-For each R-monomorphism f: A $\rightarrow$M , where A be any R-submodule of M, and for each finite subset {a_1 ,a_2 ,…,a_s } of A ,where s $\in$Z, there exists an R-homomorphism g $\in$S such that g(a_λ)=f(a_λ) ,for all $\lambda=1,2,\dots,s$.

Recall that a function f: N $\rightarrow$M is split, if there exists a function g: M $\rightarrow$N such that g $\circ$f=I_N [10].

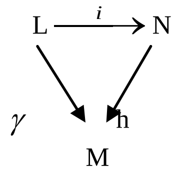
Before we give the following proposition, we define the concept of finitely-split.

Definition1.5

An R-monomorphism f : N $\rightarrow$M, where N and M are R-modules, is called finitely-split if for each finite subset B={b_1 ,b_2 ,…,b_s }of N, where s $\in$Z, there exists an R-homomorphism g_B :M $\rightarrow$N (g_B may depend on B) such that ($g_B \circ$f)(a_λ)=a_λ ,for all $\lambda=1,2,\dots,s$.

Proposition 1.6:Let M and N be two R-modules. If M is finitely-pseudo-N-injective, then every R-monomorphism α : M$\rightarrow$N is finitely -split.

Proof: Let α :M$\rightarrow$N be any R-monomorphism, and a_1, a_2,…,$a_s \in$M. Define $\beta:\alpha(M) \rightarrow M$ by $\beta(\alpha(m))=m$, for all m$\in$M. It is easily proved that β is an R-monomorphism. Let L be the R-submodule of N generated by $\alpha(a_1)$, $\alpha(a_2)$,…, $\alpha(a_s)$ and let $\gamma=\beta|_L$:L$\rightarrow$M. Consider the following diagram



Where i: L$\rightarrow$N is the inclusion map. Since M is finitely -pseudo-N-injective, and $\alpha(a_\lambda) \in \alpha(M)$, for all $\lambda=1,2,\dots,s$, hence by Th.1.3 there exists an R-homomorphism h: N$\rightarrow$M such that (h$\circ$I) ($\alpha(a_\lambda)$) = $\gamma(\alpha(a_\lambda))$, for all $\lambda=1,2,\dots,s$ .Whence h ($\alpha(a_\lambda)$)= $\gamma(\alpha(a_\lambda))$, for all $\lambda=1,2,\dots,s$. But γ ($\alpha(a_\lambda)$)=a_λ , for all $\lambda=1,2,\dots,s$, so (h $\circ$ α) (a_λ) =a_λ , $\lambda=1,2,\dots,s$. Therefore α is finitely -split .

The following corollaries are immediate consequences of Prop.1.6.

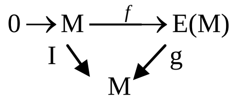
Corollary 1.7 If M is a finitely -pseudo-injective R-module, then every R-monomorphism α :M$\rightarrow$M is finitely -split.

Corollary 1.8 If M is a finitely -pseudo –E(M)-injective R-module, then every R-monomorphism α :M$\rightarrow$E(M) is finitely -split.

In the following result, we characterize finitely generated injective modules by finitely -pseudo-injectivity

Proposition 1.9 Let M be a finitely generated R-module. Then M is injective if and only if M is finitely -pseudo-E(M)-injective.

Proof: The only if part is clear. Let M be finitely -pseudo-E(M)-injective and let f: M$\rightarrow$E(M) be a monomorphism. Consider the following diagram:



where I:M$\rightarrow$M is the identity R-homomorphism. Since M is finitely -pseudo-E(M)-injective, thus there exists an R-homomorphism g: E(M) $\rightarrow$M such that g$\circ$f=I which implies that f is split. Hence E(M)=f(M)$\oplus$A´ where A´ is a R-submodule of E(M). Since E(M) is injective, then f(M) is injective [13]. But f(M)$\simeq$ M , so M is injective.

As a particular case of Prop.1.9, we have the following corollary

Corollary 1.10 A ring R is self injective if and only if R is a finitely-pseudo-E(R)-injective R-Module.

The proof of the following proposition is left as an easy exercise to the reader.

Proposition1.11 Let M, N be any two R-modules. If M is finitely-pseudo-N-injective, then M is finitely -pseudo-A-injective for each submodule A of N.

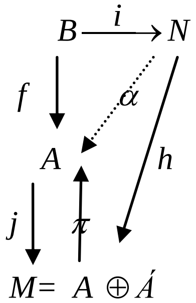
As an immediate consequence of proposition 1.11 we have the following corollary.

Corollary1.12 Let N be any submodule of an R-module M. If N is finitely -pseudo-M-injective, then N is finitely -pseudo-injective.

The next proposition shows that the finitely -pseudo-N-injectivity is inherited by direct summands.

Proposition 1.13 Any direct summand of finitely -pseudo-N-injective R-module is finitely -pseudo-N-injective.

Proof: Let M be any finitely -pseudo-N-injective R-module, and A be any direct summand R-submodule of M. Thus there exists an R-submodule A of M such that M=A$\oplus$ Á. Let B be any finitely generated R-submodule of N, and let f: B$\rightarrow$A be an R-monomorphism. Let g=j$\circ$f ,where j:A$\rightarrow$M is the injection mapping. It is clear that g is an R-monomorphism. Consider the following diagram



where i: B $\rightarrow$N is the inclusion map. Since M is finitely -pseudo-N-injective R-module, then there exists an R-homomorphism h: N $\rightarrow$M such that h$\circ$i=g. Let π: M$\rightarrow$A be the natural projective R-homomorphism .Put $\alpha=\pi\circ h$. For each b$\in$B, α (b)=

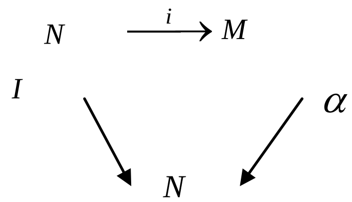
(π $\circ$h)(b)= π (h(b))= π (g(b))= π (f(b),0)=f(b).

Therefore A is finitely-pseudo-N-injective .By virtue of Prop.1.13 and Cor.1.12, we have the following result.

Corollary 1.14 Any direct summand of a finitely-pseudo-injective R-module is also finitely-pseudo-injective.

Proposition 1.15 Let N be a finitely generated R-submodule of an R-module M. If N is finitely-pseudo-M-injective, then N is a direct summand of M.

Proof: Let {a_1 ,a_2 ,…,a_s} be a set of generator of N and let I: N$\rightarrow$N be the identity R-homomorphism. Since N is finitely-pseudo-N-injective , thus there exists an R-homomorphism α:M$\rightarrow$N such that α (a_λ)=I(a_λ), for all λ =1,2,…,s (Th.1.3(4)). Consider the following diagram



where i: N$\rightarrow$M is the inclusion map. α $\circ$i=I , thus i split .Hence N is a direct summand of M [10].

Proposition 1.16

1-Isomorphic R-module to finitely-pseudo-N-injective is finitely-pseudo-N-injective for any R-module N.

2-Let N_1 and N_2 be two R-modules such that $N_1 \simeq N_2$. If M is finitely-pseudo-N_1-injective, then M is finitely-pseudo-N_2-injective

Recall that an R-module M satisfies(FC_2) if each finitely generated R-submodule of M which is isomorphic to a direct summand of M is a direct summand of M.

Proposition1.17 Any finitely-pseudo-injective R-module satisfies(FC_2).

Proof: Let M be an finitely-pseudo-injective R-module, and A be any finitely generated R-submodule of M which is isomorphic to a direct summand B of M. Since M is finitely-peudo-injective, thus by Prop.1.13 , B is finitely-pseudo-injective . And by Prop.1.16 A is finitely-pseudo-M -injective. Also since A is finitely generated thus A is a direct summand of M by Prop.1.15. Therefore, M is satisfies(FC_2).

§2:Endomorphisms rings of finitely-pseudo-injective modules

In this section, we study some properties of Endomorphisms rings of finitely-pseudo-injective modules.

A ring R is regular (in the sense of Von-Neumann) if for each element x in R , there exists an element y in R such that x=xyx [8],and a non-zero R-submodule K of M is called essential in M If $K \cap L \neq 0$ for each non-zero R-submodule L of M [8].

We preface the section by the following lemma which appears in [8].

Lemma 2.1 Let M be an R-module, S=End(M) and W(S)={ $\alpha \in$ S: ker(α) is essential in M}, thus W(S) is a two sided ideal of S.The Jacobson radical J(R) of a ring R is the intersection of all maximal ideals of R[6]. A ring R is called quasi-regular if for each $a \in R$,1-a has an inverse in R [4]. Let N be a submodule of an R-module M. A relative complement of N in M is a submodule H of M which is maximal with respect to the property $H \cap N=0$ [10].An R-module M is called Noetherian if every R-submodule of M is finitely generated[9].

Theorem2.2 Let M be a finitely -pseudo- injective Noetherian R-module, S=End(M) and let W(S)={ $\alpha \in$ S: ker(α) is essential in M}. Then J(S) =W(S) and S/J(S) is a regular ring.

Proof:

Let f+W(S) $\in$ S/W(S),where f $\in$ S. Put K=ker(f) and let L be the relative complement of K in M. Let{$x_1,x_2,x_3,\dots,x_n$} be a set of generators of L. Define θ:f(L) $\rightarrow$ M by θ(f(x))=x, for all $x \in L$. We prove that θ is well defined. For that, let f(a)=f(b), where a ,b $\in$ L. Thus f(a –b)=0 and hence a –b $\in$ ker(f)=K which means that a =b . It follows that θ (f(a))= θ (f(b)).Therefore θ is well defined. It is easily seen that θ is an R-monomorphism. Since M is finitely -pseudo-injective , there exists an R-homomorphism α :M $\rightarrow$ M such that α (f(x_i))= θ (f(x_i)),where i=1,2,3,…,n. If $u=x+y \in L \oplus K$ where $x \in L$, and $y \in K$, then ($f-f\circ\alpha\circ f$)(u)=f(x+y)-($f\circ\alpha\circ f$)(x+y)=f(x)+f(y)-($f\circ\alpha\circ f$)(x)-($f\circ\alpha\circ f$)(y) . But $y \in K$, so f(y)=0.Let x= $\sum_{i=1}^{x} r_i x_i$,where $r_i \in R$. Thereby (f-f α f)(u)=f(x)-(f α f)(x)=f(x)-($f\circ\alpha$)(f(x))=f(x)-f(α (f($\sum_{i=1}^{n} r_i x_i$)))=f(x)-f($\sum_{i=1}^{n} r_i \alpha(f(x_i))$)=f(x)-f($\sum_{i=1}^{n} r_i \theta(f(x_i))$)=f(x)-f(x)=0 which implies that $u \in$ ker($f-f\circ\alpha\circ f$). Therefore $L \oplus K \subseteq$ ker($f-f\circ\alpha\circ f$). Since $L \oplus K$ is an essential R-submodule of M, thus ker($f-f\circ\alpha\circ f$) is an essential submodule of M[9]. This prove that $f-f\circ\alpha\circ f$ $\in$ W(S) and hence f+W(S)=(f+W(S))(α +W(S))(f+W(S)). Therefore S/W(S) is a regular ring.

Let α $\in$ J(S). By (1), S/W(S) is a regular ring, thus there exists f $\in$ S such that $\alpha - \alpha\circ f\circ\alpha$ $\in$ W(S).Put $\beta = \alpha - \alpha\circ f\circ\alpha$. Since J(S) is a two sided ideal of S, thus $\alpha\circ f \in$ J(S) .Also since J(S) is quasi-regular, then $(I-\alpha\circ f)^{-1}$ exists where I is the identity R-homomorphism from M into M. Hence $(I-\alpha\circ f)^{-1}(\alpha - \alpha\circ f\circ\alpha)= (I-\alpha\circ f)^{-1}(I-\alpha\circ f)\alpha=\alpha$, thus $(I-\alpha\circ f)^{-1}\circ\beta=\alpha$. Since $\beta \in$ W(S) , $(I-\alpha\circ f)^{-1} \in$ S and W(S) is two sided ideal, then by Lemma 2.1, $\alpha \in$ W(S). Therefore J(S) $\subseteq$ W(S).Given any $f \in$ W(S),we have ker f is essential in M and ker(I-f) $\cap$ kerf=0 ;hence ker(I-f)=0. Then I-f provides an isomorphism of M onto (I-f)M, and the inverse isomorphism (I-f)M→M extends to a map $g \in S$ such that g(I-f)=I. Thus f is a left quasi-regular element of S. Now W(S) is a left quasi-regular ideal of S, and so W(S) $\subseteq$ J(S).Thereby J(S)=W(S).

Corollary 2.3 Let M be a finitely -pseudo-injective Noetherian R-module. Then $H \cap K$=HK+W(S) $\cap$ ($H \cap K$) for each two sided ideals H and K of S.

Proof: By Th.2.2, S/W(S) is a regular ring. Let $f \in H \cap K$. Then there exist α +W(S) $\in$ S/W(S) such that f+W(S)=$f\circ\alpha\circ f$+W(S), and hence ($f-f\circ\alpha\circ f$) $\in$ W(S). From that, we have ($f-f\circ\alpha\circ f$) $\in$ W(S) $\cap$ ($H \cap K$). Put β =$f-f\circ\alpha\circ f$, then f=$f\circ\alpha\circ f$+ $\beta \in$ HK+W(S) $\cap$ ($H \cap K$). It follows that $H \cap K \subseteq$ HK+W(S) $\cap$ ($H \cap K$). Since HK $\subseteq H \cap K$ and W(S) $\cap$ ($H \cap K$) $\subseteq H \cap K$,so HK+W(S) $\cap$ ($H \cap K$) $\subseteq H \cap K$. From previous argument, we have $H \cap K$=HK+W(S) $\cap$ ($H \cap K$).

The following corollary is direct from corollary 2.3 .

Corollary 2.4 If M is a finitely -pseudo-injective Noetherian R-module, then K=K^2+W(S) $\cap$ K for each two sided ideal K of S.

Proposition 2.5 If M is a finitely-pseudo-injective R-module and S=End(M), then SA=SB for each isomorphic R-submodules A,B of M.

Proof: There exists an R- isomorphism α :A $\rightarrow$ B. Let $b \in B$. Thus there exists an element $a \in A$ such that α (a)=b. It is clear that for each $r \in R$, ra=0 if and only if rb=0. Since M is finitely -pseudo-injective, then by Cor.1.4(3), Sb $\subseteq$ Sa and hence Sb $\subseteq$ SA for each b in B. Thus SB $\subseteq$ SA. Similarly, we can prove that SA $\subseteq$ SB . Therefore SA=SB.

As an immediate consequence of proposition 2.5, we have the following result.

Corollary 2.6 If R is a finitely -pseudo-injective ring and A,B are two isomorphic ideals of R, then A=B.

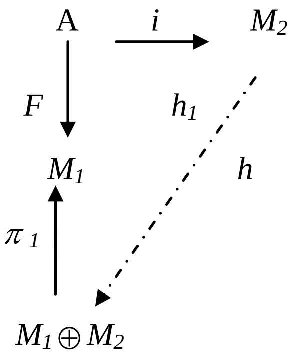
§3:Relationships between finitely -pseudo-injective R-modules and other classes of Modules

This section is devoted to study finitely-pseudo-injectivity property in some classes of modules such as fully invariant submodules, multiplication modules and uniform modules among others.

We preface our section by the following theorem which gives the relationship between two direct summands of finitely -pseudo-injective R-modules.

Theorem 3.1 If $M_1 \oplus M_2$ is a finitely-pseudo-injective R-module, then M_α is finitely- M_β-injective for all α, β=1,2 and $\alpha \neq \beta$.

Proof: We show that M_1 is finitely -M_2-injective . Let A be any finitely generated R-submodule of M_2, and let f:A$\rightarrow$ M_1 be any R-monomorphism. Define g:A$\rightarrow M_1 \oplus M_2$ by g(a)=(f(a),a) ,for all a$\in$A. It is easily proved that g is monomorphism. Since $M_1 \oplus M_2$ is finitely-pseudo- $M_1 \oplus M_2$-injective R-module, thus $M_1 \oplus M_2$ is finitely-pseudo-M_2-injective R-module(Prop.1.10).Then there exists an R-homomorphism h: $M_2 \rightarrow M_1 \oplus M_2$ such that h(a)=g(a), for all a$\in$A. Consider the following diagram



where i: A$\rightarrow M_2$ be the inclusion map. Let π_1: $M_1 \oplus M_2 \rightarrow M_1$ be the canonical projection. put $h_1 = \pi_1 \circ h$: $M_2 \rightarrow M_1$ .Thus for all a $\in$A, we have that h_1 (a)= π_1(h(a))= π_1(g(a))=π_1(f(a),a)=f(a). Therefore M_1 finitely-M_2-injective R-module.

The converse of Prop.3.1 is not true in general as the following example declare that.

Example3.2 Let Z_2 and Z_6 be Z-modules. It is easy to prove that Z_2 is finitely-Z_6-injective, and Z_6 is finitely-Z_2-injective. But $Z_2 \oplus Z_6$ is not finitely-pseudo-$Z_2 \oplus Z_6$ injective

The following result is concluded from Th.3.1.

Corollary 3.3 If $\oplus \sum_{\lambda \in \Lambda} M_\lambda$ is a finitely -pseudo-injective R-module, then M_α is finitely-M_β -injective for all distinct α ,β $\in \Lambda$.

The following proposition gives a condition under which finitely-pseudo-injective module is finitely quasi-injective.

Proposition 3.4 Any uniform finitely-pseudo-injective R-module is finitely quasi-injective.

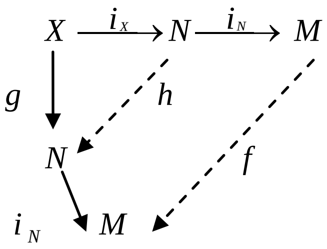
Proof: Let f:N$\rightarrow$M be any R-homomorphism, where N be a finitely generated R-submodule of M. If ker(f)=(0), then f is R-monomorphism. Since M is finitely-pseudo-injective R-module, so there exists an R-homomorphism f_1:M$\rightarrow$M such that f_1(n)=f(n), for all n$\in$N.If ker(f)$\neq$(0), let i:N$\rightarrow$M be the inclusion R-homomorphism, and let α=i+f:N$\rightarrow$M. It is clear that α is an R-homomorphism and ker(f) I ker(α)=(0). But ker(f) is an essential R-submodule of M, so ker(α)=(0). Therefore α is R-monomorphism. Since M is finitely-pseudo-injective R-module, thus there exists an R-homomorphism h: M$\rightarrow$M such that h(n)= α (n), for all n$\in$N. Put g=h-I:M$\rightarrow$M, where I:M$\rightarrow$M is the identity homomorphism, g is an R-homomorphism. Now, let n$\in$N, then g(n)=(h-I)(n)= α (n)-n=(i+f)(n)-n=f(n).Hence g is an extension of f. Therefore, M is a finitely quasi-injective R-module.

The class of finitely-pseudo-injective R-modules is not closed under submodule in general, as we mentioned in section one (examples and remarks 1.2).In the next proposition, we give a condition under which the class of finitely-pseudo-injective modules becomes closed under submodule.

Recall that a submodule N of R-module M is fully invariant submodule of M if f(N) $\subseteq$N, for all f$\in$End(M)[10].

Proposition 3.5 Every fully invariant submodule of finitely-pseudo-injective module is finitely-pseudo-injective.

Proof: Let M be a finitely-pseudo-injective module, and let N be a fully invariant submodule of M. To prove that N is finitely-pseudo-injective module, let X be any finitely generated submodule of N, and let g: X$\rightarrow$N be an R-monomorphism. Consider the following diagram



where i_X: X$\rightarrow$N ,i_N: N$\rightarrow$M are the inclusion mappings. Since M is finitely-pseudo-injective, then there exists a homomorphism f: M$\rightarrow$M such that $f \circ i_N \circ i_X = i_N \circ g$.

Since N is fully invariant in M, then f(N) $\subseteq$N. Let $f|_N$=h, then for all x in X $(h \circ i_X)(x)=f(x)=(i_N \circ g)(x)=g(x)$.Thus $h \circ i_X = g$. Therefore N is finitely-pseudo-injective.

Recall that an R-module M is called duo if every R-submodule of M is fully invariant[10] .

Corollary 3.6 If M is a finitely -pseudo-injective duo R-module, then every R-submodule of M is finitely -pseudo-injective.

An R-module M is called multiplication module if every R-submodule of M is of the form AM for some ideal A of R[5],and every R-submodule of multiplication R-module is fully invariant[3] .

Corollary 3.7 If M is a finitely -pseudo-injective multiplication R-module, then every R- submodule of M is finitely -pseudo-injective module.

Recall that an R-submodule N of an R-module M satisfies Baer criterion , if for each R-homomorphism $f:N\rightarrow M$ there exists $r\in R$ such that f(n)=rn, for all $n\in N$[1]. And R-module M is said to be satisfied Baer criterion if each R-submodule of M satisfies Baer criterion [1].

Corollary 3.8 If M is a finitely-pseudo-injective R-module which satisfies Baer criterion, then every R-submodule of M is a finitely -pseudo-injective module.

Proof: It follows from R-submodule which satisfies Baer criterion is fully invariant and Prop.3.6.

An R-submodule N of an R-module M is annihilator, if $N=ann_M(A)$ for some ideal A of R [1].And every annihilator R-submodule N is fully invariant

Proposition3.9 If M is a finitely-pseudo-injective R-module in which every R-submodule is annihilator, then every submodule of M is finitely -pseudo-injective module.

§4: finitely-pseudo-N-injective and finitely-setwise-injective modules

In this section, we introduce finitely-setwise-injective and finitely-setwise-ker-injective concepts. We study the relations between those concepts and finitely-pseudo-injective concept. Also we study the relations among them.

Definition 4.1 An R-module M is called finitely setwise-injective, if for each R-monomorphism $f: A\rightarrow B$ where A and B are two R-modules, and for each R-homomorphism $g: A\rightarrow M$ and for each finite set $D=\{a_1, a_2, \ldots, a_s\}\subseteq A$, there exists an R-homomorphism $h_D:B\rightarrow M$ (h_D may depend on D) such that $(h_D \circ f)(a_\lambda)=g(a_\lambda)$ for each λ=1,2,3,…,s.

The following proposition shows that the class of finitely-pseudo-N-injective modules contains the class of finitely setwise-injective modules.

Proposition4.2 Every finitely setwise-injective R-module is finitely-pseudo-N-injective for all R-module N.

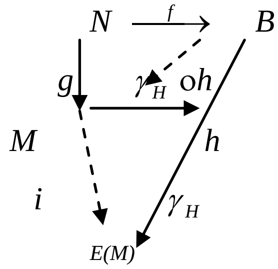
As an immediate consequence of proposition 4.2, we get the following corollary.

Corollary 4.3 Every finitely setwise-injective R-module is finitely-pseudo-E(M)-injective.The next proposition gives a characterization of finitely setwise injective module by means of finitely-split.

Proposition 4.4 An R-module M is finitely setwise-injective if and only if every R-monomorphism $\alpha:M\rightarrow E(M)$ is finitely-split.

Proof: By Cor. 4.3, M is finitely-pseudo-E(M)-injective, and hence by Prop.1.8 every R-monomorphism $\alpha:M\rightarrow E(M)$ is finitely-split.

Conversely ; assume that every R-monomorphism $\alpha:M\rightarrow E(M)$ is finitely-split. Let N be an R-module and let $D=\{a_1, a_2, \ldots, a_s\}\subseteq N$, where $s\in Z$. Assume that $f:N\rightarrow B$ is an R-monomorphism, where B is an R-module, and $g: N\rightarrow M$ is R-homomorphism. Consider the following diagram



where $i:M\rightarrow E(M)$ is the inclusion map .Since E(M) is injective, there exists an R-homomorphism $h:B\rightarrow E(M)$ such that $h\circ f=i\circ g$ By hypothesis, $i:M\rightarrow E(M)$ is finitely-split ,so for the set $H=\{g(a_1), g(a_2), \ldots, g(a_s)\}\subseteq M$ there exists an R-homomorphism $\gamma_H:E(M)\rightarrow M$ such that $(\gamma_H \circ i)(g(a_\lambda))=g(a_\lambda)$, for each λ=1,2,3,…,s.. Thus $\gamma_H \circ h:B\rightarrow M$ and $(\gamma_H \circ h \circ f)(a_\lambda)=\gamma_H(h \circ f)(a_\lambda)=(\gamma_H \circ i)(g(a_\lambda))=g(a_\lambda)$, for each λ=1,2,3,…,s. . Therefore M is finitely setwise-injective.

In the following proposition, we give a characterization of finitely setwise-injective modules by means of finitely-pseudo-injectivity.

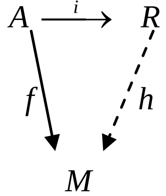
Proposition 4.5

An R-module M is finitely setwise-injective if and only if M is finitely-pseudo-E(M)-injective.

Proof: The only if part follows from Cor.4.3. To prove if part, Let M be a finitely-pseudo-E(M)-injective . By Cor.1.8, every R-monomorphism $\alpha:M\rightarrow E(M)$ is finitely-split, and hence by Prop. 4.4, M is finitely setwise-injective. A ring R is Noetherian if every ideal of R is finitely generated [2].

Corollary 4.6 Let R be a Noetherian ring. Then any R-module M is injective if and only if M is finitely-pseudo-E(M)-injective.

Proof: The only if part is clear. To prove the if part, suppose that M is finitely-pseudo-E(M)-injective. Then by Prop.4.5 M is finitely setwise-injective R-module. Let A be an ideal of R. Since R is Noetherian ring, so there exists $a_1, a_2, \ldots, a_s \in A$ such that $A=\sum_{\lambda=1}^{s} Ra_\lambda$. Let $f: A\rightarrow M$ be an R-homomorphism. Consider the following diagram

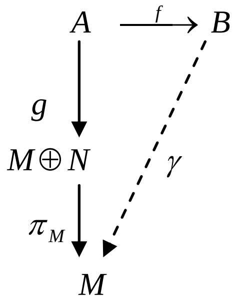


Since M is finitely setwise-injective , so there exists an R-homomorphism $h:R\rightarrow M$ such that $(h\circ i)(a_\lambda)=f(a_\lambda)$ for each λ=1,2,3,…,s. . We claim that $(h \circ i)(x)=f(x)$ for each $x \in A$. Let $x \in A$, then $x=\sum_{\lambda=1}^{s} r_\lambda a_\lambda$, where $r_\lambda \in R$,for each λ=1,2,3,…,s. Thus $(h \circ i)(x)=(h \circ i)(\sum_{\lambda=1}^{s} r_\lambda a_\lambda)=\sum_{\lambda=1}^{s} r_\lambda f(a_\lambda)=f(x)$, and hence by Bear's criterion theorem M is injective.

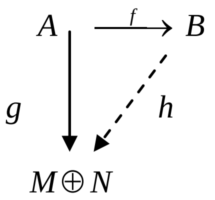
Before we give another characterization of finitely setwise-injective R-module, we present the following lemma.

Lemma 4.7 If M and N are finitely setwise-injective R-modules, then $M \oplus N$ is a finitely setwise-injective R-module.

Proof: Let f: $A \to B$ be an R-monomorphism, where A and B are R-modules, and let g: $A \to M \oplus N$ be an R-homomorphism. Assume that $D=\{a_1, a_2, \ldots, a_s\} \subseteq A$,where $s \in Z$. Consider the diagram below



where $\pi_M : M \oplus N \to M$ is the canonical projection. Since M is finitely setwise-injective R-module, thus there exists an R-homomorphism $\gamma : B \to M$ such that$(\gamma \circ f)(a_\lambda)=(\pi_M \circ g)(a_\lambda)$, λ=1,2,3,…,s. Similarly, there exists an R-homomorphism β: B $\to$N such that $(\beta \circ f)(a_\lambda)=(\pi_N \circ g)(a_\lambda)$, for each λ=1,2,3,…,s, where $\pi_N : M \oplus N \to N$ is the canonical projection .Now consider the following diagram



Define $h: B \to M \oplus N$ by $h(b)=(\gamma(b), \beta(b))$, for all $b \in B$.Then for each $a_\lambda \in D$, we have that $(h \circ f)(a_\lambda)=((\gamma \circ f)(a_\lambda),(\beta \circ f)(a_\lambda))=((\pi_M \circ g)(a_\lambda), (\pi_N \circ g)(a_\lambda))=(\pi_M(g(a_\lambda)), \pi_N(g(a_\lambda)))=g(a_\lambda)$, for each λ=1,2,3,…,s. Therefore $M \oplus N$ is a finitely setwise-injective R-module. The next lemma shows that the class of finitely quasi-injective modules contains the class of finitely setwise-injective modules.

Lemma 4.8 Every finitely setwise-injective R-module M is finitely quasi-injective.

Theorem 4.9 The following statements are equivalent for an R-module M.

1-M is finitely setwise-injective.

2- $M \oplus E(M)$ is finitely quasi-injective.

3-$M \oplus E(M)$ is finitely-pseudo-injective.

Proof: (1) $\Rightarrow$(2) By Lemma 4.7, $M \oplus E(M)$ is finitely setwise-injective , and by Lemma 4.8, $M \oplus E(M)$ is finitely quasi-injective .

(2) $\Rightarrow$(3) trivial.

(3)$\Rightarrow$(1) By Th.3.1, M is finitely –E(M)-injective, and hence M is a finitely -pseudo-E(M)-injective R-module. Therefore by Prop.4.5, M is finitely setwise-injective . From Th.4.9 we conclude the following corollaries.

Corollary 4.10 Let M be a finitely generated R-module. Then, M is injective if and only if $M \oplus E(M)$ is a finitely-pseudo-injective R-module.

Proof: The only if part is direct from Th.4.9. To prove if part, suppose that $M \oplus E(M)$ is finitely -pseudo-injective R-module, then by Th.4.9, M is finitely setwise-injective, and hence by Prop.4.5 M is finitely -pseudo-E(M)-injective R-module. Therefore M is injective (Prop.1.9)

Corollary 4.11 Let R be a Noetherian ring. Then an R-module M is injective if and only if $M \oplus E(M)$ is a finitely -pseudo-injective R-module.

Proof: The only if part is trivial . To prove the if part, suppose that $M \oplus E(M)$ is finitely -pseudo-injective R-module. By Th.4.9, M is a finitely setwise-injective R-module, and by Prop.4.5 M is a finitely -pseudo-E(M)-injective R-module. Therefore M is injective (Cor.4.6). It is known that every finitely generated Z-module is not injective [13].Thus by Cor.4.11, we have the following result.

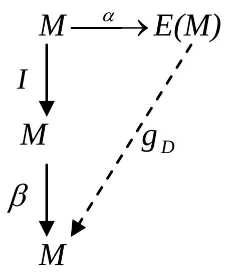
Corollary 4.12 If M is a finitely generated Z-module then $M \oplus E(M)$ is not finitely-pseudo-injective Z-module. Before we give other new characterizations of finitely setwise-injectivity, we introduce the following definitions.

Definition 4.13 An R-module M is called finitely setwise ker-injective if for each R-monomorphism $f: A \to B$,where A and B are R-modules, and for each R-homomorphism $g: A \to M$, and for each finite set $D=\{a_1, a_2, \ldots, a_s\} \subseteq A$ there exists an R-monomorphism $\alpha : M \to M$, and R-homomorphism $\beta_D : B \to M$ (β_D may depend on D) such that $(\beta_D \circ f)(a_\lambda)=(\alpha \circ g)(a_\lambda)$, for each λ=1,2,3,…,s.

Definition 4.14 An R-monomorphism $f: N \to M$ is called finitely setwise ker-split if fore each finite set $B=\{b_1, b_2, \ldots, b_s\}$ there exists an R-monomorphism $\alpha : N \to N$, and an R-homomorphism $g_B : M \to N$ (g_B may depend on B) such that $(g_B \circ f)(a_\lambda)=\alpha(a_\lambda)$, for each λ=1,2,3,…,s.

Proposition 4.15 If M is a finitely setwise ker-injective R-module, then every R-monomorphism α:M$\rightarrow$E(M) is finitely setwise ker-split.

Proof: Assume that α:M$\rightarrow$E(M) be an R-monomorphism, and D={m_1 ,m_2 ,…,m_s}$\subseteq$M. Let I: M$\rightarrow$M be the identity R-homomorphism. Consider the following diagram



Since M is finitely setwise ker-injective, then there exists an R-monomorphism β:M$\rightarrow$M, and an R-homomorphism g_D:E(M)$\rightarrow$M, such that (g_D o α)(m_λ)=(β oI)(m_λ)= β(m_λ), for each λ=1,2,3,…,s. Therefore α finitely setwise ker-split.

Now we give characterizations of finitely setwise-injective and finitely setwise ker-injective.

Theorem 4.16 The following statements are equivalent for an R-module M.

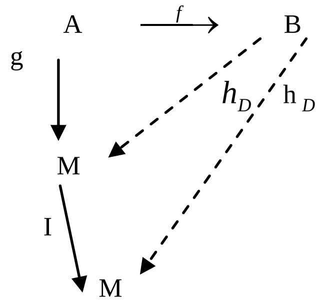
1-M is finitely setwise-injective.

2-M is finitely- quasi-injective and finitely setwise ker-injective.

3-M is finitely-pseudo-injective and finitely setwise ker-injective.

Proof: (1)$\Rightarrow$(2) Since M is finitely setwise-injective R-module, so by Lemma 4.8, M is finitely quasi-injective

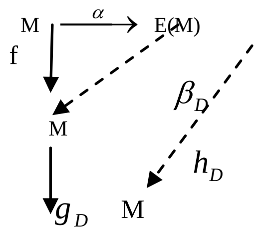
Now, let f: A$\rightarrow$B be an R-monomorphism, where A and B are R-modules, and let g: A$\rightarrow$M be an R-homomorphism. Assume D={a_1 ,a_2 ,…,a_s }$\subseteq$A. Consider the following diagram



Since M is finitely setwise-injective R-module, thus there exists an R-homomorphism h_D: B$\rightarrow$M such that (h_D of)(a_λ)=g(a_λ).Suppose I:M$\rightarrow$M is the identity R-homomorphism, and β_D=h_D:B$\rightarrow$M. For each $a_\lambda \in$D ,(β_D of)(a_λ)= (h_D of)(a_λ) =g(a_λ).Thus M is finitely setwise ker-injective R-module which complete the proof.

(2) $\Rightarrow$(3) trivial.

(3) $\Rightarrow$(1) Let α:M$\rightarrow$E(M) be an R-monomorphism. By assumption, M is finitely setwise ker-injective,thus by Prop.4.15,α is finitely setwise-ker-split. Then for each finite set D={a_1 ,a_2 ,…,a_s }$\subseteq$M, there exists an R-monomorphism f:M$\rightarrow$M and an R-homomorphism β_D: E(M) $\rightarrow$M such that (β_D oα)(a_λ)=f(a_λ), for each λ=1,2,3,…,s. Whence we have the following diagram



Since M is a finitely- pseudo-injective R-module and f: M$\rightarrow$M be an R-monomorphism, thus by Cor.1.7 f is finitely-split. Thus there exists an R-homomorphism g_D: M$\rightarrow$M such that (g_D of)(a_λ)=a_λ, for each λ=1,2,3,…,s. Put h_D=g_D oβ_D :E(M)→M, hence (h_D oα)(a_λ)=(g_D o β_D o α)(a_λ)=g_D ((β_D oα)((a_λ))=(g_D o f)(a_λ)= a_λ , for all λ=1,2,3,…,s. Therefore each R-monomorphism α:M$\rightarrow$E(M) is finitely-split and hence by Prop.4.4, M is finitely setwise -injective

Recall that an R-module M is called semi-simple if each R- submodule N of M is a direct summand of M. A ring R is semi-simple if it is semi-simple R-module [10].Since every semi simple R-module is quasi-injective [11], thus finitely-pseudo-injective. Hence by Th.4.16, we have the following result.

Corollary 4.17 Every semi-simple finitely setwise ker-injective R-module is finitely setwise –injective.

From Th.4.9 and Th.4.16 we get the following corollary.

Corollary 4.18 The following statements are equivalent for an R-module M.

1-M$\oplus$E(M) is a finitely- pseudo-injective R-module.

2-M is finitely- pseudo-injective and finitely setwise ker-injective

Proof: (1) $\Rightarrow$(2) It follows from Th.4.9.

(2) $\Rightarrow$(1) By Th.4.16 M is finitely setwise –injective R-module, and hence by Th.4.9 M$\oplus$E(M) is finitely- pseudo-injective.

Remark 4.19 Direct sum of two finitely- pseudo-injective R-modules need not be finitely- pseudo-injective. For example; let p be a prime number, then Z_p and E(Z_p) are finitely- pseudo-injective Z-modules, but by Cor.4.12, $Z_p \oplus$E(Z_p) is not finitely- pseudo-injective Z-module.

The following proposition gives a condition under which direct sum of any two finitely- pseudo-injective R-modules is finitely- pseudo-injective.

Proposition 4.20 The following statements are equivalent .

1-Direct sum of any two finitely- pseudo-injective R-modules is finitely- pseudo-injective R-module.

2-Every finitely- pseudo-injective R-module is finitely setwise-injective.

Proof: (1) $\Rightarrow$(2) Let M be any finitely- pseudo-injective R-module. By hypothesis M$\oplus$E(M) is a finitely- pseudo-injective R-module. Then by Th.4.9, we have that M is finitely setwise -injective R-module. (2) $\Rightarrow$(1) Let M and N be any two finitely-pseudo-injective R-modules. Then by hypothesis M and N are finitely setwise-injective R-modules, and by Lemma 4.7 M$\oplus$N is finitely setwise-injective. This implies that M$\oplus$N is a finitely pseudo-injective R-module (Th.4.16).

As an immediate consequence of Prop. 4.20, we have the following corollary.

Corollary 4.21 Let the direct sum of any two finitely- pseudo-injective R-modules is finitely-pseudo-injective. Then

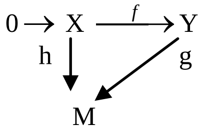
1-every finitely-quasi-injective R-module is finitely setwise-injective.

2-every semi-simple R-module is finitely setwise-injective.

3-every finitely generated semi-simple R-module is injective.

Proof: (3) follows from (2) and propositions 4.20,4.5 and 1.9.

Recall that an R-module M is finitely injective if every diagram of R-modules of the form

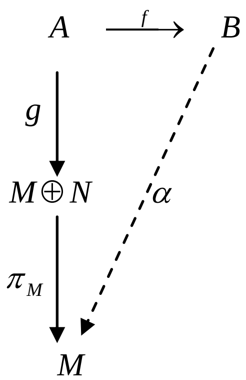


where X is finitely generated, and the row is exact can be embedded in commutative diagram [14].

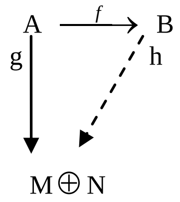
Before we give the last proposition of this section, we need to introduce the following lemma.

Lemma 4.22 If M and N are finitely- injective R-modules, then M$\oplus$N is a finitely-injective R-module.

Proof: Let f: A$\to$B be an R-monomorphism, where A is a finitely generated R-module and B is any R-module, and let g: A$\to$M$\oplus$N be an R-homomorphism. Consider the following diagram



where π_M : M$\oplus$N $\to$M is the canonical projection. Since M is finitely-injective , there exists an R-homomorphism α : B$\to$M such that α of=π_M og .Similarly, there exists an R-homomorphism β: B$\to$N such that β of=π_N og, where π_N: M$\oplus$N$\to$N is the canonical projection. Now consider the following diagram



Define h:B$\to$M$\oplus$N by h(b)=(α (b), β (b)). Let a $\in$ A, then h of(a)=(α of(a), β of(a))=(π_M og(a), π_N og(a))=g(a).Therefore M$\oplus$N is a finitely-injective R-module.

Corollary 4.23 If M is a finitely-injective R-module, then M$\oplus$E(M) is a finitely-injective R-module.

Proposition 4.24 The following statements are equivalent .

1. Every finitely-injective R-module is finitely setwise-injective.
2. Every finitely-injective R-module is finitely-quasi-injective.
3. Every finitely-injective R-module is finitely-pseudo-injective.

Proof: (1) $\Rightarrow$(2) By Lemma 4.8.

(2) $\Rightarrow$(3) trivial.

(3) $\Rightarrow$(1) Let M be a finitely-injective R-module. By Cor.4.23, M$\oplus$ E(M) is a finitely-injective R-module, and hence by hypothesis M$\oplus$E(M) is a finitely-pseudo-injective R-module. Then by Th. 4.9, M is finitely setwise-injective.

§5:Characterizations of Rings by means of Finitely-pseudo-injective R-modules

In this section, we introduce some new characterizations of strongly regular rings and we also present some new characterizations of semi-simple artinian rings by finitely-pseudo-injectivity property.

An R-module M is called strongly regular if every finitely generated R-submodule of M is a direct summand of M ,and a ring R is called strongly regular if every finitely generated ideal of R is a direct summand of R [14].

Proposition 5.1 If every finitely generated R-submodule of an R-module M is finitely-pseudo-M-injective, then M is strongly regular.

Proof: Follows from Prop1.15.

Theorem 5.2 The following statements are equivalent for a ring R.

1-R is a strongly regular ring.

2-Every R-module is finitely-pseudo-R-injective.

3-Every ideal of R is a finitely-pseudo-R-injective R-module .

4-Every finitely generated ideal of R is a finitely-pseudo-R-injective R-module.

Proof: (1) $\Rightarrow$(2) Let M be an R-module and let A be a finitely generated ideal of R. Suppose that f: $A\rightarrow M$ be a monomorphism. Since R is strongly regular, then A is a direct summand of R, that is, $R=A\oplus B$, where B is an ideal of R. Let i: $A\rightarrow R$ is the inclusion homomorphism. Define g: $R\rightarrow M$ such that $g(r)=g(a+b)=\{\begin{smallmatrix} f(a), if & r\notin B \\ o, if & r\in B\end{smallmatrix}$,where $a\in A$ and $b\in B$. It is clear that g is well defined. Let r_1 ,r_2 $\in R$. Thus $r_1=a_1+b_2$ and $r_2=a_2+b_2$,where a_1 ,a_2 $\in A$ and b_1 ,b_2 $\in B$. If $r_1+r_2\in B$, then then $a_1+a_2=b-(b_1+b_2)\in A\cap B=0$, for some $b\in B$, and hence $a_1+a_2=0$.From this, we have

$g(r_1+r_2)=\{\begin{smallmatrix} f(a_1+a_{21}), if & r_1+r_2\notin B \\ o, if & r_1+r_2\in B\end{smallmatrix}$ .Thus $g(r_1+r_2)=g(r_1)+g(r_2)$.Notice that $g(r_1\ r_2)=g(a_1a_2+b_1b_2)$. If $r_1\ r_2\notin B$, then $g(r_1r_2)=f(a_1a_2)=a_1f(a_2)=a_1f(a_2)+f(0)=a_1f(a_2)+f(b_1a_2)=(a_1+b_1)f(a_2)=r_1\ g(r_2)$.If $r_1\ r_2\in B$, then $a_1\ a_2=0$ Thus $g(r_1\ r_2)=0=f(a_1\ a_2)=a_1\ f(a_2)=(a_1+b_1)f(a_2)=r_1\ g(r_2)$ .From the above argument g is R-homomorphism. Then $(g\circ i)(a)=f(a)$ .Therefore M is finitely-pseudo-injective.

(2) $\Rightarrow$(3) and (3) $\Rightarrow$(4) are direct.

(4) $\Rightarrow$(1) By Prop. 1.15.

Theorem 5.3 The following statements are equivalent for a ring R.

1. R is semi-simple artinian.
2. Every R-module is finitely -pseudo-injective.
3. Every finitely generated R-module is finitely -pseudo-injective and a direct sum of any two finitely-pseudo-injective R-modules is finitely -pseudo-injective.

Proof: (1) $\Rightarrow$(2) Let M be an R-module. Since R is a semi-simple artinian ring, so M is semi-simple and hence M is quasi-injective . Thus M is finitely -pseudo-injective R-module.

(2) $\Rightarrow$(3) trivial

(3) $\Rightarrow$(1) Let M be a finitely generated R-module. By hypothesis (3) , M is finitely -pseudo-injective. By Prop.4.20, M is finitely setwise- injective . Then by Th.4.9, $M\oplus E(M)$ is finitely -pseudo-injective R-module. Thus by Cor.4.10 M is injective. Therefore R is a semi-simple artinian ring by [9].

Corollary 5.4 If the direct sum of any two finitely-pseudo-injective R-modules is finitely-pseudo-injective, then R is a regular ring.

Proof: Let M be a simple R-module. Then M is quasi-injective. Thus by Cor.4.21 M is a finitely setwise-injective R-module. Consequently, $M\oplus E(M)$ is finitely-pseudo-injective R-module Th.4.9 .This implies that M is afinitely-pseudo-$M\oplus E(M)$-injective R-module (Prop.1.13). Thus M is finitely –E(M)-injective(Prop.1.11).By Prop.1.9, M is injective. This prove that R is a regular ring[10].The following theorem gives other characterizations of semi- simple artinian ring which is a generalization of Osofsky's theorem [12] by finitely-pseudo-injectivity.

Theorem 5.5 The following statements are equivalent for a ring R.

1-R is semi- simple artinian ring.

2-For each R-module M, if N_1 and N_2 are finitely-pseudo-injective R-submodules of M then $N_1\cap N_2$ is a finitely-pseudo-injective R-module.

3-For each R-module M, if N_1 and N_2 are finitely quasi-injective R-submodules of M, then $N_1\cap N_2$ is a finitely-pseudo-injective R-module.

4-For each R-module M, if N_1 and N_2 are quasi-injective R-submodules of M, then $N_1\cap N_2$ is a finitely-pseudo-injective R-module.

5-For each R-module M, if N_1 and N_2 are injective R-submodules of M, then $N_1\cap N_2$ is a finitely-pseudo-injective R-module.

Proof: (1) $\Rightarrow$(2) follows from that every module over semi- simple artinian ring is semi-simple artinian.

(2) $\Rightarrow$(3) , (3) $\Rightarrow$(4) and (4) $\Rightarrow$(5) are direct.

(5) $\Rightarrow$(1) Let M be an R-module, and E=E(M) . Let $Q=E\oplus E$ and $K=\{(x,x)\in Q: x\in M\}$ and let $\overline{Q}=Q/K$. Put $M_1=\{y+K\in\overline{Q}: y\in E\oplus(0)\}$ and $M_2=\{y+K\in\overline{Q}: y\in(0)\oplus E\}$ then $\overline{Q}=M_1+M_2$. Let $\overline{h}=h+K\in\overline{Q}$,where $h\in Q$ Thus $\overline{h}=((h_1,0)+(0,h_2))+K=((h_1,0)+K)+((0,h_2)+K)$,where $h_1\in E_1$ and $h_2\in E_2$. Since $(h_1,0)+K\in M_1$ and $(0,h_2)+K\in M_2$, thus $\overline{h}\in M_1+M_2$. Consequently, $\overline{Q}\subseteq M_1+M_2$. It is easily proved that M_1 and M_2 are R-submodules of $\overline{Q}$. Thus M_1+M_2 is also R-submodule of $\overline{Q}$. Thus $M_1+M_2\subseteq\overline{Q}$ .From preceding argument, we have $\overline{Q}=M_1+M_2$ .Define $\alpha_1: E\rightarrow M_1$ by $\alpha_1(y)=(y,0)+K$, for all $y\in E$ and $\alpha_2: E\rightarrow M_2$ by $\alpha_2(y)=(0,y)+K$, for all $y\in E$. It is easily proved that α_1 and α_2 are R-isomorphisms. Since E is an injective R-module, therefore M_λ is injective R-submodule of $\overline{Q}$ for $\lambda=1,2$ [6]. Thus by (5), we have $M_1\cap M_2$ is a finitely-pseudo- injective R-module. Define f:$M\rightarrow M_1\cap M_2$ by f(m)=(m,0)+K, for all $m\in M$. It is easily seen that $M_1\cap M_2=\{y+K\in\overline{Q} : y\in M\oplus(0)\}$ and f is an R- isomorphism. Thus M is a finitely-pseudo-injective R-module by Prop.1.16. Hence every R-module is finitely-pseudo-injective, and this implies that R is a semi-simple artinian ring by Th.5.3.

References

[1] Abbas, M. S., On fully stable modules, Ph.D., thesis, Univ. of Baghdad (1990).
[2] Alamela, S., on quasi-injective modules over Noetherian rings, J. of the Idian. Math. Soc.,39 (1975) ,121-130.
[3] Anderson, F. W. and Fuller, K. R. , Rings and categories of modules, Managing Editors P.R. Halmos, Univ. of Delhi (1973)
[4] Barnard, A. , multiplication modules, J. of Algebra,71 (1981),174-178.
[5] Burton, D. M. ,Abstract and linear algebra, Univ. of New York., (1971).
[6] Faith, C. and Utumi, V. ,quasi-injective modules and their Endomorphisms rings, Archiv. Math., 15 (1964), 166-174.
[7] Goodearl, K. R. ,Rings theory, Marcel Dekker, New-York, (1976).
[8] Hunger Ford,T. W. ,Algebra, Springer-Verlage, New-York, (1974).
[9] Kash, F. ,Modules and Rings, Academic Press, London, New-York,(1982).
[10] Lam, T. Y. ,Lecture on modules and rings, Springer-Verlag, New-York (1998).
[11] Osotsky, B. L. ,Rings all whose finitely generated modules are injective, Pas. J. Math., 14 (1964), 645-650.
[12] Ramamurth, V. S. and Rangaswam, K. M. , On finitely injective modules, J. Austrual, Math. Soc., 16 (1973), 239-248.
[13] Sharp, D. W. and Vamos, P. , Injective modules, Cambridge, Univ.Press, London,(1972).
[14] Wongwal, S. ,On the endomorphism rings of a semi-injective modules, Acta, Math. Univ. Comenianae, LXXI, (2002),27-33.

المقاسات الاغمارية–N – الكاذبة المنتهية

علي سبع مجباس[١] ، هيبة كريم محمد علي[٢] ، نغم علي حسين[٢]

[١] *قسم الرياضيات، كلية علوم الحاسبات و الرياضيات، جامعة تكريت.*

[٢] *قسم الرياضيات، كلية التربية ، جامعة تكريت*

(تاريخ الاستلام: ١٣ / ٧ / ٢٠٠٩ ---- **تاريخ القبول:** ١٣ / ١٢ / ٢٠١٠)

الملخص

لتكن R حلقة ابدالية بمحايد و M مقاسا أحديا على R . مفهومي المقاسات الاغمارية–N– المنتهية و المقاسات الاغمارية–N– الكاذبة عممت في هذا البحث إلى المقاسات الاغمارية–N– الكاذبة المنتهية.أعطينا جملة من المكافئات والخواص للمقاسات الاغمارية– N – الكاذبة المنتهية.العلاقة بين صف المقاسات الاغمارية الكاذبة المنتهية وصفوف أخرى من المقاسات درست.وكذلك قدمنا مميزات جديدة للحلقات الارتينية شبه البسيطة و الحلقات المنتظمة بقوة باستخدام الصفة الاغمارية الكاذبة المنتهية . وفضلا عن ذلك،درسنا حلقات التشاكلات للمقاسات الاغمارية الكاذبة المنتهية.