On Finitely Annihilated Modules

Ali S. Mijbass*   Hibat K. Mohammadali**   and   Najlaa A.saeed***
*Science college-Tikrit University
** Education College-Tikrit University
*** Education College-Tikrit University

Abstract

Let R be a commutative ring with identity and M be a unitary R-module. An R-module M is called finitely annihilated if there exists a finitely generated R-submodule N of M such that ann(M)=ann(N).Our main purpose in this work is to study this property in some known classes of modules such as quasi-injective, multiplication and other modules. We prove that:

1-If M is a quasi-injective R-module, then M is finitely annihilated if and only if M is finendo.

2-If M is a multiplication R-module, then M is finitely annihilated if and only if M is finitely generated.

3-M is a faithful finitely annihilated R-module if and only if M is a compactly faithful R-module.

Introduction

Let R be a commutative ring with unity and let M be a unitary R-module .C. Faith called an R-module M is bounded if there exist an element $x \in M$ such that ann(M)=ann(x)(Faith,1970), and he studied some properties of these modules. Also other properties of bounded modules were studied in(Ameen,1992).John A. Beachy and William D.Blair gave a generalization to the bounded module concept (Beachy&Blair,1978). They called an R-module M is finitely annihilated if there exists a finite set $\{x_1, x_2,..., x_n\}$,Where $x_i \in M, \forall i = 1, 2, \cdots, n$, such that $ann(M) = ann(\{x_1, x_2, \cdots, x_n\})$. It is clear that every bounded R-module is finitely annihilated. In section one, we study some properties of finitely annihilated R-modules. In section two, we present sufficient conditions for quasi-injective modules to be finitely annihilated and we study some properties of quasi-injective R-modules satisfy finitely annihilated property.

We prove that if M is a finitely annihilated quasi-injective R-module, then M is finitely generated over End(M)(Th.2.1). Also we prove that if M is a quasi-injective R-module, then M is finitely annihilated if and only if M is finendo(Corollary 2.2). Section three is devoted to study finitely annihilated property in the class of multiplication R-modules. We look for necessary or sufficient conditions for multiplication R-modules to be a finitely annihilated R-module. We prove that if M is multiplication R-module, then M is finitely annihilated if and only if M is finitely generated(Prop.3.1).In section four , we study finitely annihilated property in other classes of modules such as quasi-Dedekind, compressible, F-regular and compactly faithful.

Some Basic Properties of Finitely Annihilated modules.

In (Beachy&Blair, 1978), an R-module M is called finitely annihilated if there exists a finite set $\{x_1, x_2, \cdots x_n\}$ in M such that, $ann(M) = ann(\{x_1, x_2, \cdots x_n\})$. In this section, we present an equivalent statement for this concept. Furthermore, we study some properties and give a characterization for this concept. The proof of the following remark is easy and hence is omitted.

Remark1.1:

Let M be an R-module. M is finitely annihilated if and only if $ann(M) = ann(N)$ for some finitely generated R-submodule N of M.

Examples and Remarks 1.2:

1-Every torsion free R-module, where R is an integral domain, is finitely annihilated.

2-Every finitely generated R-module is finitely annihilated. But the converse is not true. For example Q (the set of all rational numbers) as a Z-module is finitely annihilated but not finitely generated,

3- Z_{p^∞} as a Z-module is not finitely annihilated.

4-The homomorphic image of finitely annihilated R-module may not be finitely annihilated. For example a Z-module Q is finitely annihilated but $\frac{Q}{Z}$ is not finitely annihilated Z-module.

5- The direct summand of finitely annihilated R-module may not be finitely annihilated. For example, the Z-module $M = Z \oplus Z_{p^\infty}$ is finitely annihilated

but Z_{p^∞} is not finitely annihilated. The proof of the following proposition is straightforward and hence is omitted.

Proposition 1.3:

Let M be an R-module and let I be an ideal of R such that $I \subseteq ann(M)$. Then M is finitely annihilated R-module if and only if M is finitely annihilated $\dfrac{R}{I}$ -module. The following result is an immediate consequences of Prop.1.3.

Corollary 1.4:

Let M be an R-module. Then M is a finitely annihilated R-module if and only if M is a finitely annihilated $\dfrac{R}{ann(M)}$ -module

Proposition 1.5:-

Let M_1 and M_2 be two finitely annihilated R-modules. Then $M_1 \oplus M_2$ is a finitely annihilated R-module.

Proof:

Since M_1 is finitely annihilated, there exists a finitely generated R-submodule N_1 of M_1 such that $ann(M_1) = ann(N_1)$. Similarly, there exists a finitely generated R-submodule N_2 of M_2 such that $ann(M_2) = ann(N_2)$. It is clear that $ann(M_1 \oplus M_2) \subseteq ann(N_1 \oplus N_2)$. Now, let $r \in ann(N_1 \oplus N_2)$, then $r(x,y) = (0,0)$ for all $x \in N_1$ and for all $y \in N_2$, that is $(rx, ry) = (0,0)$. Hence $rx = 0$, for all $x \in N_1$, and $ry = 0$, for all $y \in N_2$. This implies that $r \in ann(N_1) \cap ann(N_2) = ann(M_1) \cap ann(M_2)$.

Whence $\quad\quad\quad r \in ann(M_1 \oplus M_2)$.

This prove that $\quad ann(N_1 \oplus N_2) \subseteq ann(M_1 \oplus M_2)$.

Therefore $\quad\quad ann(M_1 \oplus M_2) = ann(N_1 \oplus N_2)$,

proving that $M_1 \oplus M_2$ is finitely annihilated. The following result is an immediate consequence of Prop.1.5.

Corollary 1.6:

A finite direct sum of finitely annihilated R-modules is finitely annihilated. However, an infinite direct sum of finitely annihilated R-modules may not be a finitely annihilated R-module, as it is shown in the following example.

Example 1.7: Z_p as a Z-module is finitely annihilated for all prime p by 1.2(2), but $\oplus Z_p$ is not a finitely annihilated Z-module.

The following characterization is appeared in (Beachy&Blair, 1978)

Proposition 1.8:

M is finitely annihilated R-module if and only if $\dfrac{R}{ann\,(M\,)}$ is embedded in M^k , where k is a positive integer. The following remark is needed in the proof of next proposition.

Remark 1.9:

Let M be an R-module. If N is E-submodule of M , where E= End(M), then N is R-module of M.

Proof:

It is easy.

The following result is a consequence of Remark 1.9.

Proposition 1.10:

Let M be an R-module. If M is a finitely annihilated E-module, where E= End(M), then M is a finitely annihilated R-module.

Proof:

Since M is finitely annihilated E-module, then there exists a finitely generated E-submodule N of M such that $ann_E(M\,)=ann_E(N\,)$.Let $\{x_1,x_2,\cdots x_s\}$ be a set of generator of N, where $x_i \in N, i=1,2,\cdots,s$ .Thus $N=(\{x_1,x_2,\cdots x_s\})$. Let K be an R- submodule of M generated by $\{x_1,x_2,\cdots x_s\}$. We claim that $ann_R(M\,)=ann_R(K)$. Let $r \in ann(K)$, and define $f:M \to M$ such that f(m)=rm, for all m in M. Thus $f(x_i)=rx_i=0, \forall i=1,2,\cdots s$ .By Remark 1.9, N is an R-submodule of M. Let n $\in$ N,then $n=h_1(x_1)+h_2(x_2)+\cdots+h_s(x_s)$, where $h_i \in E, \forall i=1,2,\cdots,s$. Hence

$$f(n)=m=h_1(rx_1)+h_2(rx_2)+\cdots+h_s(rx_s)=h_1(0)+h_2(0)+\cdots+h_s(0)=0$$

and consequently f (N) =rN=0. Therefore $f \in ann_E(N\,)$.But $ann_E(M\,)=ann_E(N\,)$ so that $f \in ann_E(M\,)$. This means that f(M)=rM=0. Whence $r \in ann(M\,)$. It is clear that $ann(M\,)\subseteq ann(K)$.◘ Recall that an R-module M is said to be finendo if it is finitely generated over End(M) (Faith,1970).

Corollary 1.11:

Let M be an R-module. If M is finendo, then M is finitely annihilated.

Proof:

Since M is finendo, thus M is finitely generated over End(M).By 1.2(2),M is finitely annihilated E-module. Thus by Prop.1.10, M is finitely annihilated R-module. In the following proposition, we investigate the behavior of finitely annihilated property under localization.

Proposition 1.12:

If M is a finitely annihilated R- module, then M_S is a finitely annihilated R_S -module, where S is a multiplicatively closed set of R.

Proof:

Suppose that M is a finitely annihilated R-module, then there exists a finitely generated R- submodule N of M such that $ann\left(M\right)=ann\left(N\right)$.

Since N is finitely generated, then N_S is finitely generated and $\left(ann\left(N\right)\right)_S = ann_{R_S}\left(N_S\right)$.It is clear that $ann_{R_S}(M_S) \subseteq ann_{R_S}(N_S)$. Let

$\dfrac{r}{s} \in ann_{R_S}\left(N_S\right)=\left(ann\left(N\right)\right)_S$. Thus $r \in ann\left(N\right)=ann\left(M\right)$ and t $\in$ S. Let

$x \in M_S , x = \dfrac{m}{s} , m \in M , s \in S$ .Whence $\dfrac{r}{t} \cdot \dfrac{m}{s} = \dfrac{rm}{ts} = \dfrac{0}{ts}$. This implies

that $\dfrac{r}{t} \in ann_{R_S}(M_S)$. Therefore $ann_{R_S}\left(M_S\right)= ann_{R_S}\left(N_S\right)$ which proves that M_S is a finitely annihilated R_S -module.

Finitely Annihilated Modules and Quasi-Injective Modules.

An R-module M is said to be quasi-injective if for each R-submodule N of M and every R-homomorphism from N to M can be extended to an R-endomorphism of M (Faith, 1970). In this section, we look for conditions for quasi-injective modules to be finitely annihilated. We begin with the following theorem which gives a condition under which the converse of Corollary 1.11 is true.

Theorem2.1:

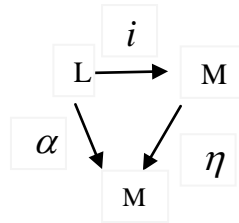If M is a quasi- injective finitely annihilated R-module, then M is finendo.

Proof:

By prop. 1.8, $0 \to \dfrac{R}{ann(M)} \xrightarrow{\ g\ } M^k$ is exact. We claim that

$Hom_R\left(M^k,M\right) \xrightarrow{Hom_R(g,I)} Hom_R\left(\dfrac{R}{ann(M)},M\right) \to 0$ is exact.

Let $\phi \in Hom_R\left(\dfrac{R}{ann(M)}, M\right)$.   Since   g   is   monomorphism,   then

$g(\dfrac{R}{ann(M)}) \cong L \subseteq M^k$,,where L is an R- submodule of M^k . Let

$$\alpha = j \circ \phi \circ g^{-1} : L \xrightarrow{g^{-1}} \dfrac{R}{ann(M)} \xrightarrow{\phi} M \xrightarrow{j} M^k$$

where j is the injection homomorphism. Consider the following diagram

$$\begin{array}{ccc} & i & \\ L & \longrightarrow & M \\ {\scriptstyle\alpha}\searrow & & \swarrow{\scriptstyle\eta} \\ & M & \end{array}$$

where i is the inclusion mapping. Since M is quasi-injective, then M^k is quasi-injective (Faith, 1970).Thus there exists a homomorphism $\eta : M^k \to M^k$ such that $\eta \circ i = \alpha$. That is $\eta | L = \alpha$. Let

$\gamma = \pi \circ \eta : M^k \xrightarrow{\eta} M^k \xrightarrow{\pi} M$, where $\pi : M^k \to M$ be the canonical projection. Thus

$Hom(g, I)\gamma = I \circ \gamma \circ g = I \circ \pi \circ \eta \circ g = I \circ \pi \circ \alpha \circ g = I \circ \pi \circ j \circ \phi \circ g^{-1} \circ g$

$= I \circ \pi \circ j \circ \phi \circ I = \phi$

Therefore $Hom(g, I)$ is   onto,   and   consequently

$Hom_R\left(M^k, M\right) \xrightarrow{Hom_R(g,I)} Hom_R\left(\dfrac{R}{ann(M)}, M\right) \to 0$

is exact. But $Hom_R(M^k, M) = [Hom_R(M, M)]^k$ (Kasch,1982)

,and $Hom_R(M^k, M) = Hom_{\frac{R}{ann(M)}}(M^k, M)$ (Kasch,1982).

And also $Hom(\dfrac{R}{ann(M)}, M) \cong M$ as $\dfrac{R}{ann(M)}$ – modules . Therefore

$\left[Hom_{\frac{R}{ann(M)}}(M, M)\right]^k \to M \to 0$ is exact. But $Hom_{\frac{R}{ann(M)}}(M, M) =$

$Hom_R(M, M) = End(M)$ (Kasch,1982), so $[End(M)]^k \to M \to 0$ is exact .

Put E=End(M), we have $E^k \to M \to 0$ is exact in E-module. Thus $\dfrac{E^k}{D} \cong M$,

where D is an E-submodule of E^k . Since E^k is finitely generated,

then $\dfrac{E^k}{D}$ is finitely generated. Therefore M is finitely generated as E-module. That is, M is finitely generated over End(M) and consequently M is finendo. ◘

The following result follows from Th.2.1 and Corollary.1.11.

Corollary 2.2:

Let M be a quasi-injective R-module. Then M is finitely annihilated if and only if M is finendo. An R-module M is called semi-simple if every R-submodule of M is a direct summand (Kasch, 1982). Since semi-simple R-module is quasi-injective (Faith, 1970), we have the following corollary.

Corollary 2.3:

Let M be a semi-simple R-module. Then M is finitely annihilated if and only if M is finendo. Recall that an R-module M is called Q-module if every R-submodule of M is quasi-injective (Mohammad, 2005).

Proposition 2.4:

Let M be Q-module. Then the following statements are equivalence:

1-Every R-submodule of M is finitely annihilated.

2-Every R-submodule of M is finendo.

3-Every R-submodule of M is finitely generated over End(M).

Proof:

(1)⇒(2) suppose that every R-submodule of M is finitely annihilated. Let N be an R-submodule of M. Thus N is a quasi-injective finitely annihilated R-submodule. By Th.2.1, N is finendo.

(2)⇒(3) Since $End(N) \subseteq End(M)$, then N is finitely generated over End(M).

(3)⇒(1) Assume L is an R-submodule of M which is finitely generated over End(M). Thus there exist $x_1, x_2, \cdots, x_n \in L$ such that for each $y \in L$, $y = f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n), f_i \in End(M), i = 1, 2, \cdots, n$. Let X be a finitely generated R-submodule of M generated by $x_1, x_2, \cdots, x_n$. It is clear that $X \subseteq L$. We claim that ann(L)=ann(X). Let $r \in ann(X)$, then $rx_i = 0, \forall i = 1, 2, \cdots, n$. Let $s \in L$. Then $s = g_1(x_1) + g_2(x_2) + \cdots + g_n(x_n)$, where $g_i \in End(M), \forall i = 1, 2, \cdots, n$. Thus $rs = g_1(rx_1) + g_2(rx_2) + \cdots + g_n(rx_n) = 0$. So $r \in ann(L)$. Therefore $ann(X) \subseteq ann(L)$. Since $ann(L) \subseteq ann(X)$, then $ann(L) = ann(X)$. Whence L is finitely annihilated◘.

Proposition 2.5:

Let M be an R-module and let $\dfrac{R}{ann(M)}$ is a semi-simple ring. Then M is a finitely annihilated R-module if and only if M is finendo.

Proof:

By Corollary1.4, M is a finitely annihilated $\dfrac{R}{ann(M)}$ -module. But $\dfrac{R}{ann(M)}$ is semi- simple, then M is an injective $\dfrac{R}{ann(M)}$ -module (Kasch, 1982),and hence M is a quasi-injective $\dfrac{R}{ann(M)}$ -module. Thus M is a quasi-injective R-module .Whence M is finendo (Th.2.1).The converse follows from Corollary.1.11. ◘. Recall that an R-module M is said to be fully stable if $ann_M(ann_R(x)) = x, \forall x \in M$ (Abass, 1990).

Proposition 2.6:

Let M be a fully stable quasi-injective R-module. Then M is finitely annihilated if and only if M is finitely generated.

Proof:

If M is finitely annihilated, then there exists a finitely generated R-submodule N of M such that ann(M) =ann(N).By (Abass,1990), M satisfies the double annihilator condition on finitely generated R-submodules. Hence $ann_M\left(ann_R(M)\right) = ann_M\left(ann_R(N)\right)$ and consequently M=N. Whence M is finitely generated.

The converse is clear. ◘.

Corollary 2.7:

Let M be a fully stable semi-simple R-module. Then M is finitely annihilated if and only if M is finitely generated.

Finitely Annihilated Property with Multiplication Modules.

An R- module M is said to be multiplication module if for every R-submodule N of M, there exists an idea I in R such that N=IM (Barnard,1981) .The following proposition shows that the converse of 1.2(2) is true in the class of multiplication modules.

Proposition 3.1:

Let M be a multiplication R-module. M is finitely annihilated if and only if M is finitely generated.

Proof:
Since M is a finitely annihilated R-module, then there exists a finitely generated R-submodule N of M such that Ann (M)=ann(N). By (Low&Smith, 1990), M is finitely generated. The converse follows from 1.2(2).The condition multiplication in Prop.3.1 can not be dropped. For example Q as Z-module is finitely annihilated, but Q is not finitely generated and not multiplication.

Corollary 3.2:
If M is a finitely annihilated multiplication R-module, then $\text{End(M)} \cong \dfrac{R}{ann(M)}$.

Proof:
It follows from Prop.3.1 and (Naoum, 1990).In the class of multiplication modules the converse of Prop.1.10 is true as the following proposition indicate that.

Proposition 3.3
Let M be a multiplication R-module and E=End(M). M is a finitely annihilated R-module if and only if M finitely annihilated E-module.

Proof:
If M is a finitely annihilated R-module, then by Corollary 1.4, M is a

finitely annihilated $\dfrac{R}{ann(M)}$ -module. But M is multiplication, then

$\text{End(M)} \cong \dfrac{R}{ann(M)}$ (Corollary 3.2). Thus M is a finitely annihilated E-

module .The converse follows from Prop.1.10. ◘ Recall that an R-module M is torsionless if and only if $\bigcap\limits_{f \in M^*} \ker f = (0)$, where $M^* = Hom(M,R)$

(Low&Smith, 1990), and the trace of an R-module M is $T(M) = \sum\limits_{f \in M^*} f(M)$ (Low&Smith, 1990). Now, we have the following proposition.

Proposition 3.4:
If M is a torsionless multiplication R-module and T(M) is finitely generated , then M is finitely annihilated.

Proof:
Since T(M) is finitely generated , then there exist $m_i \in M$ and $f_i \in M^*, 1 \le i \le n$, such that { $f_i(m_i):1 \le i \le n$ } generates T(M). Let N be the

R- submodule of M generated by the set $\{m_i : 1 \le i \le n\}$. We have to show that $ann(M) = ann(N)$. Let $r \in ann(N), m \in M$ and $f \in M^*$ then

$$f(rm) = rf(m) = r\sum_{i=1}^{n} \alpha_i f_i(m_i) = \sum_{i=1}^{n} \alpha_i f_i(rm_i) = (0),$$ where

$\alpha_i \in R, i = 1, 2, \cdots, n$. Thus $rm \in \bigcap_{f \in M^*} \ker f = (0)$, and hence rM=0. Therefore $r \in ann(M)$ and consequently $ann(N) \subseteq ann(M)$. It is clear that $ann(M) \subseteq ann(N)$. This completes the proof that M is finitely annihilated. Recall that an R- submodule N of an R-module M is dense in M if $\forall f \in M^*, f(N) = 0$, then f=0 (Naoum, 1990).

Proposition 3.5:
   If M is a torsionless multiplication R-module and contains a finitely generated dense R-submodule N, then M is finitely annihilated.

Proof:
   It is clear that $ann(M) \subseteq ann(N)$. Let $r \in ann(N)$, then $rf(N) = f(rN) = f(0) = 0, \forall f \in M^*$. But N is dense R-submodule, then $rf = 0, \forall f \in M^*$. That is $rM^* = 0$. This means that $r \in ann(M^*)$. But M is torsionless, so $ann(M) = ann(M^*)$ (Low&Smith,1990).Whence $r \in ann(M)$. Therefore $ann(N) \subseteq ann(M)$ .This shows that M is finitely annihilated. ◘ An R-module M is called non-singular if

$Z(M) = \{m \in M : ann(m) \text{ is essential in } R\} = 0$

where a nonzero R-submodule N of M is called essential if $N \cap K \ne (0)$ for each nonzero R-submodule K of M (Kasch,1982). Let L and D are R-submodules of M, then $(L : D) = \{r \in R : rD \subseteq L\}$.

Proposition 3.6:
   If M is a non-singular multiplication R-Module such that M contains an essential finitely generated R-submodule, then M is finitely annihilated.

Proof:
   Let N be an essential finitely generated R-submodule of M. It is clear that that $ann(M) \subseteq ann(N)$. Since N is essential in M. then for all $m \in M$,we have $(N : (m))$ is essential in R (Ahmad,1992).Let $r \in ann(N)$, then $rm(N : (m)) = (0)$.Thus $(N : (m)) \subseteq ann(rm)$.

But $\left(N:(m)\right)$ is essential in R, so $ann(rm)$ is essential in R. Since M is non-singular, then rm=0 for all m in M. that is rM=0, and hence $r \in ann(M)$. Therefore $ann(N) \subseteq ann(M)$. Whence M is finitely annihilated. ◘.

Corollary 3.7 :

If M is a multiplication R-module and contains a finitely generated essential R-submodule N with Z(N)=0, then M is finitely annihilated.

Proof:

It is enough to prove that M is non-singular, that is $Z(M) = 0$. Suppose that $Z(M) \neq 0$, then there exists a nonzero element $m \in M$ such that ann(m) is essential in R. But N is essential in M, then there exist r $\in$ R such that $0 \neq rm \in N$. Since $ann(m) \subseteq ann(rm)$, so $ann(rm)$ is essential in R. Thus $rm \in Z(N) = 0$. Therefore rm=0 which is a contradiction. Hence $Z(M) = 0$, that is , M is non-singular. By Prop.3.6, M is finitely annihilated. ◘.

Corollary3.8:

Let R be a ring such that $Z(R) = 0$. If M is multiplication torsionless R-module which contains a finitely generated essential R-submodule, then M is finitely annihilated.

Proof:

It follows from (Ahmad,1992)and Prop.3.6. ◘

The closure of an R-submodule N of M denoted by

$Cl(N) = \{m \in M : \left(N:(m)\right) is\, an\, essential\, ideal\, in\, R\}$. It is clear that

$N \subseteq Cl(N)$ and $Cl(0) = \{m \in M : \left(0:(m)\right) = ann(m) is\, essential\, in\, R\} = Z(M)$

(Goldie.1964).

Proposition3.9:

Let M be a non-singular R-module. If M contains a finitely generated R-submodule N such that $Cl(N) = M$, then M is finitely annihilated.

Proof:

Let $r \in ann(N)$. Then $rm\left(N:(m)\right) = 0$. Thus $\left(N:(m)\right) \subseteq ann(rm)$. But $\left(N:(m)\right)$ is essential in R, so $ann(rm)$ is essential in R. Whence $rm \in Z(M) = (0)$. That is rm=0 for all m in M. Thus $r \in ann(M)$ and consequently $ann(N) \subseteq ann(M)$. Since $ann(M) \subseteq ann(N)$, so $ann(N) = ann(M)$. This proves that M is finitely annihilated. ◘

We end this section by the following proposition.

Proposition3.10:

If M is a faithful multiplication module over integral domain, then M is finitely annihilated.

Poof:

By (Ahmad, 1992), M is torsion free and by 1.2(1), M is finitely annihilated. ◘

Finitely Annihilated Property with Some Types of Modules.

In this section, we study the relationship between finitely annihilated modules and other modules. An R-module M is said to be prime if $ann_R(M) = ann_R(N)$ for every non-zero submodule N of M (Beachy, 1976). It is clear that every prim R-module is finitely annihilated, but the converse is not true in general as the following example shows. $M = Z \oplus Z_n$ as Z-module is finitely annihilated module, but not a prim R-module, since $ann_Z(M) = (0)$ and $ann(0 \oplus Z_n) = nZ \neq (0)$. And an R-module M is said to be quasi-Dedekind if every nonzero R-submodule N of M is quasi-invertible, where an R-submodule N of M is called quasi-invertible if $Hom\left(\dfrac{M}{N}, M\right) = 0$ (Mijbass, 1997). It is known that every quasi-Dedekind module is prime (Mijbass, 1997) , and as an immediate consequence of this result the following proposition.

Proposition4.1:

If M is a quasi-Dedekind R-module, then M is finitely annihilated.

Recall that an R-module is said to be Dedekind module if every submodule of it is invertible and an R-module is called prufer module if every finitely generated submodule of it is invertible (Al-Alwan,1993). In particular, every Dedekind module and every prufer module is quasi-Dedekind (Mijbass,1997) . We get the following corollaries that are direct results from the Prop.4.1

Corollary 4.2:

if M is Dedekind R-module, then M is finitely annihilated module.

Corollary 4.3:

If M is prufer R- module, then M is finitely annihilated module.

An R-module is called compressible if every non-zero submodule of M contains an isomorphic copy of M (Desale & Nicholoson, 1981).

Proposition4.4:

If M is a compressible R-module, then M is finitely annihilated module.

Proof:

If M=0, then there is nothing to prove. Assume $M \neq 0$. Let N be a finitely generated non-zero R-submodule of M and let $r \in ann(N)$ where $r \in R$. Since M is compressible, then there exists a monomorphism f: M→N such that $M \cong f(M) \subseteq N$. Now, $0 = rf(M) = f(rM)$. But f is an R-monomorphism, so rM=0. Thus $r \in ann(M)$ and consequently $ann(N) \subseteq ann(M)$. Since $ann(M) \subseteq ann(N)$, then $ann(M) = ann(N)$. Whence M is finitely annihilated ∎.Recall that an R-submodule N of M is called prime if $rm \in N, r \in R$ and m ∈M, then either $m \in N$ or $r \in (N:M)$.

Corollary 4.5:

If M is a multiplication R-module which contains a finitely generated prime R-submodule N, then M is finitely annihilated.

Proof:

By (El-Baset&Smith, 1988), $\frac{M}{N}$ is compressible. Hence $\frac{M}{N}$ is finitely annihilated (Prop.4.4). Since M is multiplication, then $\frac{M}{N}$ is multiplication (El-Baset&Smith,1988). Thus $\frac{M}{N}$ is finitely generated R-module (Prop.3.1) and consequently there exists $x_1, x_2, \cdots, x_n \in M$, $such that$ $M = Rx_1 + Rx_2 + \cdots + Rx_n + N$. But N is finitely generated R-submodule of M, then M is finitely generated. Hence M is finitely annihilated by 1.2(2).

Corollary 4.6:

If M is a multiplication R-module such that $ann(M)$ is a prime ideal of R , then M is finitely annihilated.

Proof:

Since $ann(M) = ((0):M)$ is a prime ideal of R, then (0) is a prime R-submodule of M (El-Baset&Smith,1988). By Corollary 4.5, M is finitely annihilated.

Recall that an R-submodule N of M is called Pure If $IM \cap N = IN$ for every ideal I of R (Fieldhouse, 1969).

Proposition4.7:

If M is an R-module and contains a finitely generated essential pure R-submodule of M, then M is finitely annihilated.

Proof:

Let N be a finitely generated essential pure R-submodule of M. It is clear that $ann(M) \subseteq ann(N)$. Let $r \in ann(N)$, then $rN = (0)$. Since N is pure, then $N \cap rM = rN = (0)$. Since N is essential, then $rM = (0)$ which implies that $r \in ann(M)$. Therefore $ann(M) = ann(N)$. Whence M is finitely annihilated. An R-module M is F-regular if every R-submodule of M is pure (Fieldhouse, 1969). As an application of Prop.4.7, we have the following result.

Corollary4.8:

If M is a uniform F-regular R-module, then M is finitely annihilated.

An R-module M is divisible if $rM = M$ for every non-zero element r in R (Kasch, 1982).

Corollary4.9:

Let M be a uniform module over PID R such that every R-submodule of M is divisible. Then M is finitely annihilated.

Proof:

Let N be a finitely generated R-submodule of M. Thus N is essential and divisible. Whence N is pure .Therefore M is finitely annihilated (Prop.4.7). We give in the following proposition a condition under which finitely annihilated property equivalent to finitely generated property.

Proposition4.10:

Let M be R-module and satisfies the double annihilated property. Then M is finitely annihilated if and only if M is finitely generated.

Proof:

It is direct.

Now, we study the relation between the two concepts faithful and compactly faithful by using finitely annihilated property. An R-module M is said to be compactly faithful, provided that there is an embedded $0 \rightarrow R \rightarrow M^n$ for some finite integer n>0 (Faith, 1970). Any compact faithful module is faithful. The following theorem gives a necessary and sufficient condition for a faithful module to be finitely annihilated.

Teorem 4.11:

Let M be an R-module. Then M is faithful finitely annihilated if and only if M is compactly faithful.

Proof:

Suppose that M is faithful finitely annihilated. By Prop.1.8, $0 \to R \to M^k$ is exact for some integer k > 0. Thus M is compactly faithfulR-module. Conversely; Since M is compactly faithful , so $0 \to R \to M^n$ is exact for some integer n > 0 ,and hence M is faithful. By Prop.1.8, M is finitely annihilated.

Corollary4.12:

If M is a finitely annihilated semi-simple R-module, then $\dfrac{R}{ann(M)}$ is semi-simple ring.

Proof:

By Corollary 1.4, M is a finitely annihilated $\dfrac{R}{ann(M)}$ -module. But M is faithful as an $\dfrac{R}{ann(M)}$ -module, so M is compactly faithful (Th.4.11).Thus $\dfrac{R}{ann(M)}$ can be imbedded in M^k. Since M is semi-simple, so M^k is semi-simple. Whence $\dfrac{R}{ann(M)}$ is semi-simple.

References

- Abass M.S.,(1990): On fully sable Modules, Ph.D. Thesis, Univ. of Baghdad.
- Ahmad A.A.,(1992): on Submodules of Multiplication Modules. M.Sc. Thesis, Univ. of Baghdad.
- Ameen A.S.,(2002):Bounded Modules, M.Sc. thesis, Univ. of Baghdad.
- Al-Alwan F.H.,(1993): Dedekind modules and the problem of embeddability,Ph.D. Thesis , College of Science, University of Baghdad.
- Barnard A.,(1981):Multiplication Modules,J.Algebra 71,pp.174-178.
- Beachy J. A. and Blair D.W., (1978): Finitely Annihilated Modules orders in Artinian Rings, J. Algebra 6(1), pp.3-30.
- Beachy J. A.,(1976): Some Aspects of Non- Commutative Localization, Lecture Note in Math. Vol.545, Springer Verlage Heidelberg, New York.

- Desale G. and Nicholoson W.K.,(1981): Endoprmitive Rings, J. Algebra 70,pp.548-560.
- El-Bast Z.A. and Smith P.F.,(1988): Multiplication Modules, Comm. In Algebra 16, pp.755-779.
- Faith C.,(1970): Algebra II Ring Theory, Spring Verlag Berlin Heidelberg, New York.
- Fieldhouse D.J.,(1969):Pure Theories,Math.Ann.,Vol.184,pp.1-18.
- Goldie A.W.,(1964): Torsion Free Modules and Rings, J. Algebra 1, pp.268-287.
- Kasch F., (1982): Modules and Rings, Academic Press, London, New York.
- Low G.H.and Smith P.F.,(1990): Multiplication Modules and Ideals ,Comm. In Algebra 18,pp.4353-4375
- Lu c. P.,(1984) : Prime Submodules of Modules, Commtent Mathematics, Univ. Sancti Pauli 33, pp.61-69.
- Mijbass A.S.,(1997): Quasi-Dedekind Modules , Ph.D. Thesis, Univ. of Baghdad.
- Mohammad Ali H.K.,(2005) : Q- Modules, Ph.D. Thesis, Univ. of Tikrit.
- Naom A.G., (1990); On Ringof Endomorphism of Finitely Generated multiplication Modules,Periodica Mathematica hungarica, Vol. 21,pp.249-255.
- Naom A.G., (1990): Flat Modules and Multiplication modules, Periodica Mathematica Hungarica, Vol.21, pp.309-317.

مقاسات مقيدة التالف

الخلاصة

لتكن R حلقة أبدالية بمحايد وليكن M مقاسا أحاديا أيسرا على الحلقة R . نقول إن M مقاس مقيد التالف إذا تحقق وجود مقاس جزئي منتهي التولد مثل N من M بحيث إن ann(M)=ann(N). غرضنا الرئيس فـــي هذا البحث هو دراسة بعض المبرهنات المهمة عن المقاسات المقيدة التالف ونلقي نظرة كافية وضرورية علـــى المقاسات الشبه اغمارية ونضع الشرط الكافي للحصول على مقاس مقيد التالف. ندرس كذلك المقاسات التي تكون مقيد التالف على صنف من المقاسات الجدائية و أصناف أخرى من المقاسات، كما تحرينا على بعض الشـــروط الضرورية والكافية الواجب توافرها مع المقاسات الجدائية حتى يكون المقاس مقيد التالف. وهنا بعض النتائج التي توصلنا إليها:

١–إذا كان M مقاسا شبه أغماري على R فان M مقاس مقيد التالف إذا وفقط إذا كان M منتهي التولد.

٢– إذا كان M مقاسا جدائي على R فان M مقاس مقيد التالف إذا وفقط إذا كان M منتهي التولد.

٣– إذا كان M مقاسا على R فان M مقاس مخلص و مقيد التالف إذا وفقط إذا كان M مقاسا مخلصـــا بشـــكل مرصوص.