The Group Classification of One Class of Nonlinear Wave Equations

Esam R.Faiq
Technical Inst.of Kirkuk-Computer Center

Abstract

We perform group classification of one class of nonlinear wave equations with two independent variables and one dependent variable. It is shown that there is one, six nonlinear wave equations admitting (invariant) under one–and two–dimensional Lie algebras, respectively.

Introduction

The problem of group classification (determining the arbitrary functions are known as the group classification problem (Bluman & Cole,1974)of differential equations is one of the central problems of modern symmetry analysis of differential equations(Lahno & Magda,2003).Many papers on this problem of such equations have been published:

$$u_t = u_{xxx} + F(t,x,u,u_x,u_{xx}) \qquad\qquad (Gungor\ et.al, 2004)$$

$$u_t = u_{xx} + F(t,x,u,u_x) \qquad\qquad (Lahno\ \&\ Magda, 2003)$$

$$u_t = F(t,x,u,u_x)u_{xx} + G(t,u,u_x) \qquad\qquad (Basarab\ et.al, 2001)$$

$$u_t = F(t,x,u,u_x)u_{xx} + G(t,x,u,u_x) \qquad\qquad (Basarab\ et.al, 2000)$$

$$u_{tx} = g(t,x)u_x + f(t,x,u), g_x \neq 0, f_{uu} \neq 0 \qquad\qquad (Ibragimov, 2004)$$

$$u_{tx} = f(t,x,u)\ , f_{uu} \neq 0 \qquad\qquad (Ibragimov, 2004)$$

In this article, we consider a class of nonlinear wave equation

$$u_{tt} = -\lambda u_{xx} + F(u,u_x),$$

where λ is an arbitrary real constant, $u = u(t,x)\ and\ F$ is an arbitrary nonlinear smooth function. The approach used in the present article is that presented in (Zahdanov&Lahno,1999)(modified to be applicable for group classification of the equation under study), being a synthesis of the standard Lie algorithm for finding symmetries and the use of canonical forms for partial differential generators obtained with the equivalence group of the equation at hand.

Invariance of partial differential equations

This section applies infinitesimal transformation (Lie group of transformations) to study second order partial differential equation. Similarly for the case of ordinary differential equations, it will show that infinitesimal criterion for invariance (invariance condition) of partial differential equation leads to an algorithm to determine infinitesimal generators admitted by given partial differential equation.

Let us consider scalar second order partial differential equation:

$$F(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx}) = O \quad , \qquad \qquad …(1)$$

In two independent variables t, x and one dependent variable u. Second extension space is defined to be $(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})$, representing variables and derivatives up to order two.

In terms of coordinates $t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx}$ equation (1) becomes an algebraic equation which defines a hyper surface in $(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})$ –space. Note that Jacobian condition guarantees that the differential equation can be (in principle) written in a solved form, that is, $u,u_t,u_x,u_{tt},u_{tx},u_{xx}$ can be isolated on the left hand side, also note that a differential equation written in solved form automatically satisfies the Jacobian condition, and hence equation must be written in solved from whenever possible(Lisle, 1992).

Definition(Humi &Miller,1988):

A second order partial differential equation is said to be <u>invariant</u> with respect to a one–parameter Lie group, if its second extension leaves the equation unchanged.

Theorem 1:

(Infinitesimal criterion for invariance of a second order partial differential equation)(Bluman& Cole,1974;Bluman&Kumei, 1989&Ibragimov,2004)

Let

$$Q = \tau(t,x,u)\frac{\partial}{\partial t} + \zeta(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \qquad …(2)$$

be the infinitesimal generator of the one–parameter Lie group.

$$\bar{t} = T(t,x,u;\varepsilon) ,$$
$$\bar{x} = X(t,x,u;\varepsilon) , \qquad\qquad … (3)$$
$$\bar{u} = U(t,x,u;\varepsilon) .$$

Let

$$Q^{(2)} = \tau(t,x,u)\frac{\partial}{\partial t} + \zeta(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} + \eta_{(t)}(t,x,u,u_t,u_x)\frac{\partial}{\partial u_t} + \eta_{(x)}(t,x,u,u_t,u_x)\frac{\partial}{\partial u_x}$$

$$+ \eta_{(tt)}(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\frac{\partial}{\partial u_{tt}} + \eta_{(tx)}(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\frac{\partial}{\partial u_{tx}}$$

$$+ \eta_{(xx)}(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\frac{\partial}{\partial u_{xx}} \quad , \qquad \qquad ...(4)$$

be the corresponding second extended infinitesimal generator of (2), where

$$\eta_{(t)} = \eta_t + \eta_u u_t - \tau_t u_t - \zeta_t u_x - \tau_u (u_t)^2 - \zeta_u u_t u_x \ ,$$

$$\eta_{(x)} = \eta_x + \eta_u u_x - \zeta_x u_x - \tau_x u_t - \zeta_u (u_x)^2 - \tau_u u_t u_x \ ,$$

$$\eta_{(tt)} = \eta_{tt} + 2\eta_{tu} u_t - \tau_{tt} u_t - \zeta_{tt} u_x + \eta_u u_{tt} - 2\tau_t u_{tt} - 2\zeta_t u_{tx} + \eta_{uu}(u_t)^2 - 2\tau_{tu}(u_t)^2 - 2\zeta_{tu} u_t u_x$$
$$- \tau_{uu}(u_t)^3 - \zeta_{uu}(u_t)^2 u_x - 3\tau_u u_t u_{tt} - \zeta_u u_x u_{tt} - 2\zeta_u u_t u_{tx} \ ,$$

$$\eta_{(tx)} = \eta_{tx} + \eta_{tu} u_x - \zeta_{tx} u_x + \eta_{xu} u_t - \tau_{tx} u_t - \zeta_t u_{xx} + \eta_u u_{tx} - \tau_t u_{tx} - \zeta_x u_{tx} - \tau_x u_{tt} - \zeta_{tu}(u_x)^2$$
$$+ \eta_{uu} u_t u_x - \tau_{tu} u_t u_x - \zeta_{xu} u_t u_x - \tau_{xu}(u_t)^2 - \zeta_{uu} u_t(u_x)^2 - \tau_{uu}(u_t)^2 u_x - 2\zeta_u u_x u_{tx}$$
$$- 2\tau_u u_t u_{tx} - \tau_u u_x u_{tt} - \zeta_u u_t u_{xx} \ ,$$

$$\eta_{(xx)} = \eta_{xx} + 2\eta_{xu} u_x - \zeta_{xx} u_x - \tau_{xx} u_t + \eta_u u_{xx} - 2\zeta_x u_{xx} - 2\tau_x u_{tx} + \eta_{uu}(u_x)^2 - 2\zeta_{xu}(u_x)^2$$

$$- 2\tau_{xu} u_t u_x - \zeta_{uu}(u_x)^3 - \tau_{uu} u_t(u_x)^2 - 3\zeta_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{tx} \ . \qquad \qquad ...(5)$$

Then the one–parameter Lie group (2)-(4) is admitted by the partial
differential equation (1) if and only

$$Q^{(2)}F(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})\big|_{F=0} = 0 \qquad \qquad ...(6)$$

Symmetry groups

A one–parameter Lie group admitted by a partial differential equation is also called a <u>symmetry group</u> of the differential equation. Also the infinitesimal generator admitted by a differential equation is called an <u>infinitesimal symmetry</u> (admitted generator) of the equation.

Symmetry groups of differential equations converts any solution into a solution or equivalently, the symmetry transformations is just permute the solutions among themselves. Sometimes these solutions are unchanged under the symmetry groups; such solutions are called <u>invariant solutions</u>.

Finding symmetry groups are equivalent to the determination of its infinitesimal transformations. The set of all infinitesimal generators admitted by a differential equation forms a Lie algebra of infinitesimal

generators, called the <u>admitted Lie algebra</u>, if it is finite–dimensional, then the symmetry group of the differential equation is a Lie group of transformations.Theorem 1 provides an algorithm for finding the symmetry group of partial differential equations, one can let the coefficients τ , ζ and η of the infinitesimal generator Q of suppositive one–parameter symmetry group of the differential equation be unknown functions of t, x and u. The coefficients $\eta_{(t)}, \eta_{(x)}, \eta_{(tt)}, \eta_{(tx)}$ *and* $\eta_{(xx)}$ belonging to the extended infinitesimal generator $Q^{(2)}$ will be clear expressions including the partial derivatives of the coefficients τ, ζ and η with respect to both independent variables t, x and dependent variable u .

The infinitesimal criterion for invariance (6) thus contains t, x, u and the derivatives of u with respect to t and x, as well as τ, ζ, η and their partial derivatives with respect to t, x and u **.**

After substituting for the derivatives which occur on the left hand side of the differential equation since the second extension $Q^{(2)}$ only hold on solutions of the differential equation, we can then equate the coefficients of the remaining free partial derivatives of u to zero. This yields an overdetermined system of partial differential equations for the coefficients functions τ, ζ and η (called <u>determining equations</u>), since in general there are more than $n+1$ such determining equations. Note that if a partial differential equation is not a polynomial in its components one can still split it up into a system of linear homogeneous partial differential equations for (τ,ζ,η) using the independence of some values of the components .

Because the set of determining equations is an overdetermine system, two cases will arise: first their only solution is the trivial solution $(\tau,\zeta,\eta)=(0,0,0)$, Second if the general solution of the determining equations is non trivial, we have two cases : the general solution contains a finite number or an infinite number of arbitrary constants . The first associated with a finite–parameter Lie group, while the later associated with an infinite–parameter Lie group. Note that if the general solution of the determining equations contains arbitrary functions of t, x and u, then the associated <u>Lie group</u> also called an <u>infinite</u>.

Lie$\overset{'}{\circ}$ s algorithm for construction the symmetries of second order partial differential equations)(Bluman& Cole, 1974; Bluman& Kumei , 1989)

Since a Lie group of differential equation (satisfying the Jacobian condition) is a symmetry group of that differential equation if and only if

the invariance condition is satisfied for every infinitesimal of the Lie group, this leads to an algorithmic construction of the symmetry group of a differential equation. The key observation is that the invariance condition contains extension variables $u_t, u_x, u_{tt}, u_{tx}, u_{xx}$,which appear through the extension formulas $\eta_{(t)}, \eta_{(x)}, \eta_{(tt)}, \eta_{(tx)}, \eta_{(xx)}$, and in the differential equation itself. In both cases their occurrence is explicitly known. Hence invariance (symmetry, infinitesimal) condition can be split up by powers of these extension variables, yielding a system of determining equations for the infinitesimals τ , ζ , η.

The details of this algorithm are as follows:-

1- Write the differential equation in the solved form, that is, isolate derivatives on the left hand side.

2- Let τ, ζ and η be arbitrary functions of (t, x, u) .write the general infinitesimal generator

$$Q = \tau(t, x, u)\frac{\partial}{\partial t} + \zeta(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} .$$

3- Extend the infinitesimal generator acting on $(u_t, u_x, u_{tt}, u_{tx}, u_{xx})$. This adds to Q in terms

$$\cdots + \eta_{(t)}\frac{\partial}{\partial u_t} + \eta_{(x)}\frac{\partial}{\partial u_x} + \eta_{(tt)}\frac{\partial}{\partial u_{tt}} + \eta_{(tx)}\frac{\partial}{\partial u_{tx}} + \eta_{(xx)}\frac{\partial}{\partial u_{xx}} .$$

4- Apply the extended infinitesimal generator $Q^{(2)}$ to the function F which defines the differential equation.

5- Substitute for the derivatives which occur on the left hand side of the differential equation. Set the resulting expression to zero. This yields invariance conditions for an infinitesimal symmetry.

At this stage, the invariance conditions are linear homogeneous partial differential equations for the infinitesimals τ, ζ, η. The coefficients in these invariance conditions are known functions of t, x, u and derivatives from the differential equation in hand, that is, $u_t, u_x, u_{tt}, u_{tx}, u_{xx}$. Provided that these derivatives occur polynomially in the original differential equation, they occur polynomially in the invariance conditions, in an explicitly known manner. In this case, one is able to split up the invariance conditions according to powers of these derivatives into a finite number of determining equations for the infinitesimals τ, ζ, η .Usually this system of

determining equations is overdetermined, consisting of more equations than unknowns.

6-Split up invariance conditions by powers of the derivatives $u_t, u_x, u_{tt}, u_{tx}, u_{xx}$ to give determining equations for the infinitesimal symmetry group.

7- Solve the determining equations for the infinitesimals τ, ζ, η.

8- For each infinitesimal generator in the algebra of symmetry generators, integrate the initial value problem to yield a set of one–parameter subgroups of the symmetry group.

9- Compose these subgroups to give the symmetry group.

Steps of method

Zhdanov et.al approach for group classification of the class of partial differential equations consists of the following steps:

1- Use the usual Lie algorithm to find the general form of the infinitesimal generator which generates the symmetry group of the equation under study. As a result we obtain the determining equations, which connect the coefficients (infinitesimals) of the infinitesimal generators with the arbitrary function of the equation. It is possible that some of the determining equations do not contain the arbitrary elements; therefore they can be integrated immediately. Others, that is, the determining equations which explicitly depend on the arbitrary functions and their derivatives are called <u>classifying equations</u>. The main difficulty of group classification is the need to solve classifying equations with respect to the coefficients of the infinitesimal generator and arbitrary elements simultaneously. The corresponding invariant equations for each of the Lie algebras are obtained by solving these classifying equations.

2- Construct the equivalence group, which sets an equivalence relation (<u>two elements</u> of the equivalence group are called <u>equivalent</u> if they are transformed one into another with a transformation from the equivalence group).

3- Find realization of one–, and two–dimensional Lie algebras up to the equivalence relation above. To this end we use the classification of low dimensional abstract Lie algebras up to two dimensions. Inserting the so obtained infinitesimal generators into classifying equation, then we select those realizations corresponding to the Jacobi identity to be satisfied and that can be symmetry Lie algebras of the differential equation under study.

Note that <u>two realizations</u> are called <u>equivalent</u> if they are transformed into each other by the action of the equivalence group(Zahdanov&Lahno,1999).

The following notation: $A_{k.i} = <Q^1, Q^2, ..., Q^k>$, denotes a Lie algebra of dimension k, $Q^j (j=1, 2, ..., k)$ are its basis elements, and the index i denotes the number of the class to which the given Lie algebra belongs (Basarab et.al, 2000).

General analysis of symmetry properties of PDES

To classify the nonlinear wave equation

$$u_{tt} = -\lambda u_{xx} + F(u, u_x) \qquad ...(7)$$

that admits Lie algebras of dimension up to two, we start from an equation admitting one–dimensional Lie algebras, then extending these Lie algebras to describe the admitted two–dimensional Lie algebras.

1-The Most General Infinitesimal Generator:

The first step of group classification of partial differential equation (7) is to find the general form of the infinitesimal generator of the Lie group admitted (invariant) by (7), which is according to the Lie algorithm is of the form:

$$Q = \tau(t,x,u)\frac{\partial}{\partial t} + \zeta(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \qquad ...(8)$$

where t, x are independent variables and $u = u(t,x)$ is the dependent variable. Note that τ, ζ, η are real-valued smooth functions. The criterion condition for equation (7) to be invariant with respect to (8) reads as

$$\eta_{(tt)} + \lambda \eta_{(xx)} - \eta_{(x)} F_{u_x} - \eta F_u = 0, \qquad ...(9)$$

Substituting the formulas from (5), and then replacing u_{tt} by $-\lambda u_{xx} + F(u,u_x)$ whenever it occurs (9), we have:

$$\{\eta_{tt} + \lambda \eta_{xx} + \eta_u F - \eta F_u - \eta_x F_{u_x} - 2\tau_t F + u_x[(\zeta_x - \eta_u)F_{u_x} - \zeta_{tt} + 2\lambda \eta_{xu} - \lambda \zeta_{xx} - \zeta_u F] + (u_x)^2[\zeta_u F_{u_x} - 2\lambda \zeta_{xu} + \lambda \eta_{uu}]$$

$$(u_x)^3[-\lambda \zeta_{uu}]\} + u_t\{ [\tau_x F_{u_x} + 2\eta_{tu} - \tau_{xu} - \lambda \tau_{xx} - 3\tau_u F] + u_x[-2\zeta_{tu} + \tau_u F_{u_x} - 2\lambda \tau_{xu}] + (u_x)^2[-\lambda \tau_{uu}]\} +$$

$$(u_t)^2\{-\zeta_{uu} u_x - 2\tau_{tu} + \eta_{uu}\} + (u_t)^3\{-\tau_{uu}\} + u_{tx}\{-2\zeta_t - 2\lambda \tau_x - 2\lambda \tau_u u_x\} + u_t u_{tx}\{-2\zeta_u\} + u_{xx}\{2\lambda \tau_t - 2\lambda \zeta_x - 2\lambda \zeta_u u_x\}$$

$$+ u_t u_{xx}\{2\lambda \tau_u\} = 0 \qquad ...(10)$$

Replacing τ, ζ and η by a, b and f, respectively, in (10) and splitting by u_t, u_{tx}, $u_t u_{tx}$, u_{xx} *and* $u_t u_{xx}$, we will be left with the following equations, which are so called the classifying equations:

$$a_x F_{u_x} + 2f_{tu} = 0$$

$$f_{tt} + \lambda f_{xx} - f F_u - f_x F_{u_x} + (f_u - 2a_t)F + [(b_x - f_u)F_{u_x} + 2\lambda f_{xu}]u_x = 0 \qquad \dots(11)$$

We then have the following result:

Theorem 2:

The infinitesimal generator of the symmetry group of the equation (7) has the following form:

$$Q = a(t,\, x)\frac{\partial}{\partial t} + b(t,\, x)\frac{\partial}{\partial x} + f(t,x,u)\frac{\partial}{\partial u}\, , \qquad \dots(12)$$

where $a(t,\, x)$, $b(t,\, x)$ and $f(t,x,u)$ are arbitrary smooth functions which satisfy the classification equations (11).

2-The Equivalence Group:

An <u>equivalence transformation</u> of a second order partial differential equation in two independent variables $t,\, x$ and in one dependent variable u is a change of variables

$$\bar{t} = T(t,\, x,\, u) \quad,\bar{x} = X(t,\, x,\, u) \quad,\bar{u} = U(t,\, x,\, u)$$

taking any equation of the class into an equation of the same class (generally with different arbitrary function)(Lisle, 1992).

There are two different ways for constructing the equivalence group, the direct method and the infinitesimal method (Ibragimov, 2004). But we will use the first one because it gives us enough information about the Jacobi conditions.

In order to construct the equivalence group of the class of partial differential equations (7), one has to select from the set of invertable changes of variables of the space V (Ibragimov, 2004):

$\bar{t} = T(t,x,u)$, $\bar{x} = X(t,x,u)$, $\bar{u} = U(t,x,u)$,

where
$$\frac{D(T,X,U)}{D(t,x,u)} = \begin{vmatrix} T_t & T_x & T_u \\ X_t & X_x & X_u \\ U_t & U_x & U_u \end{vmatrix} \neq 0 \quad, \qquad \dots(13\,)$$

be those changes of variables which don't alter the form of the class of partial differential equations (7).

Theorem 3:

The equivalence group of the class of partial differential equations (7) reads as $\quad \bar{t} = T(t),\, \bar{x} = X(t,x),\, \bar{u} = U(t,x,u),$ \qquad $\dots(14)$

Where $\dot{T} \neq 0$, $X_x \neq 0, U_u \neq 0,\quad X_t = X_u = 0,\ \dfrac{D(T,X,U)}{D(t,x,u)} \neq 0.$

Proof:

Let (13) be an invertible change of variables that transform equation (7) into another equation of the same form of (7), namely,

$$\bar{u}_{\bar{t}\bar{t}} = -\lambda\, \bar{u}_{\bar{x}\bar{x}} + \overline{F}\,(\bar{u},\bar{u}_{\bar{x}}) \qquad\qquad ...(15)$$

Computing u_x , u_t , u_{xx} *and* u_{tt} according to (13), as follows:

The total derivatives D_t *and* D_x are transformed by (13) to the operators $D_{\bar{t}}$ *and* $D_{\bar{x}}$, and

$$D_t = D_t(T)\,D_{\bar{t}} + D_t(X)\,D_{\bar{x}}$$

$$D_x = D_x(T)\,D_{\bar{t}} + D_x(X)\,D_{\bar{x}} \qquad\qquad ...(16)$$

Now

$$D_x(U) = D_x(T)\,\bar{u}_{\bar{t}} + D_x(X)\,\bar{u}_{\bar{x}} \ ,$$

or

$$U_x + u_x U_u = (T_x + u_x T_u)\,\bar{u}_{\bar{t}} + (X_x + u_x X_u)\,\bar{u}_{\bar{x}} \ ,$$

Yields

$$u_x = \frac{T_x\,\bar{u}_{\bar{t}} + X_x\,\bar{u}_{\bar{x}} - U_x}{U_u - T_u\,\bar{u}_{\bar{t}} - X_u\,\bar{u}_{\bar{x}}} \ , \qquad\qquad ...(17)$$

as the function F in (7) and $\overline{F}$ in (15) are arbitrary functions of the corresponding arguments, we must have

$$u_x = g\,(\bar{u},\bar{u}_{\bar{x}}) \ ,$$

for the same function g. This implies that $T_x = T_u = 0$ in (17) consequently,

$$T = T(t)\ ,\ \dot{T} = \frac{dT}{dt} \neq 0 \quad and \quad u_x = \frac{X_x\,\bar{u}_{\bar{x}} - U_x}{U_u - X_u\,\bar{u}_{\bar{x}}} \ . \qquad ...(18)$$

Next, making the change of variables (13), where $T = T(t)$, we have

$$D_t(U) = D_t(T)\,\bar{u}_{\bar{t}} + D_t(X)\,\bar{u}_{\bar{x}} \ ,$$

$$U_t + u_t U_u = (\dot{T} + u_t T_u)\,\bar{u}_{\bar{t}} + (X_t + u_t X_u)\,\bar{u}_{\bar{x}} \ ,$$

yeilds
$$u_t = \frac{\dot{T}\,\bar{u}_{\bar{t}} + X_t\,\bar{u}_{\bar{x}} - U_t}{U_u - X_u\,\bar{u}_{\bar{x}}} \ .$$

Computing u_{xx}, with account of the $T = T(t)$ &(10) will be reduced to

$$D_x = D_x(X)\,D_{\bar{x}} \qquad\qquad ...(19)$$

Applying (19) to (18), we get

$$u_{xx} = (X_x + u_x X_u) \left[\frac{X_x \bar{u}_{\bar{x}\bar{x}}(U_u - X_u \bar{u}_{\bar{x}}) - (-X_u \bar{u}_{\bar{x}\bar{x}})(X_x \bar{u}_{\bar{x}} - U_x)}{(U_u - X_u \bar{u}_{\bar{x}})^2} \right].$$

Simplifying this we have that

$$u_{xx} = \bar{u}_{\bar{x}\bar{x}} \left[(X_x)^2 (U_u - X_u \bar{u}_{\bar{x}})^{-1} + 2X_x X_u (X_x \bar{u}_{\bar{x}} - U_x)(U_u - X_u \bar{u}_{\bar{x}})^{-2} + (X_u)^2 (X_x \bar{u}_{\bar{x}} - U_x)^2 (U_u - X_u \bar{u}_{\bar{x}})^{-3} \right] \quad \ldots (20)$$

Similarly

$$u_{tt} = (\dot{T})^2 \bar{u}_{\bar{t}\bar{t}}(U_u - X_u \bar{u}_{\bar{x}})^{-1} + (X_t)^2 \bar{u}_{\bar{x}\bar{x}}(U_u - X_u \bar{u}_{\bar{x}})^{-1} + X_t X_u \bar{u}_{\bar{x}\bar{x}}(\dot{T}\bar{u}_{\bar{t}} + X_t \bar{u}_{\bar{x}} - U_t)(U_u - X_u \bar{u}_{\bar{x}})^{-2}$$

$$+ \dot{T} X_t X_u \bar{u}_{\bar{t}} \bar{u}_{\bar{x}\bar{x}}(U_u - X_u \bar{u}_{\bar{x}})^{-2} + \dot{T}(X_u)^2 \bar{u}_{\bar{t}} \bar{u}_{\bar{x}\bar{x}}(\dot{T}\bar{u}_{\bar{t}} + X_t \bar{u}_{\bar{x}} - U_t)(U_u - X_u \bar{u}_{\bar{x}})^{-3} + (X_t)^2 X_u \bar{u}_{\bar{x}} \bar{u}_{\bar{x}\bar{x}}$$

$$(U_u - X_u \bar{u}_{\bar{x}})^{-2} + X_t(X_u)^2 \bar{u}_{\bar{x}} \bar{u}_{\bar{x}\bar{x}}(\dot{T}\bar{u}_{\bar{t}} + X_t \bar{u}_{\bar{x}} - U_t)(U_u - X_u \bar{u}_{\bar{x}})^{-3} - X_t X_u U_u \bar{u}_{\bar{x}\bar{x}}(U_u - X_u \bar{u}_{\bar{x}})^{-2} -$$

$$(X_u)^2 U_u \bar{u}_{\bar{x}\bar{x}}(\dot{T}\bar{u}_{\bar{t}} + X_t \bar{u}_{\bar{x}} - U_t)(U_u - X_u \bar{u}_{\bar{x}})^{-3}. \quad \ldots (21)$$

Inserting u_{xx}, u_{tt} from (20), (21) respectively into (7), we arrive to the following differential equation

$$\bar{u}_{\bar{t}\bar{t}} = \bar{u}_{\bar{x}\bar{x}} \left\{ -\lambda \left[\frac{(X_x)^2 (X_u)^2 - 2X_x X_u U_x U_u - (X_x)^2 (X_u)^2 (\bar{u}_{\bar{x}})^2 + 2X_x(X_u)^2 U_x \bar{u}_{\bar{x}}}{(\dot{T})^2 (U_u - X_u \bar{u}_{\bar{x}})^2} \right. \right.$$

$$\left. + \frac{X_x(X_u)^2 \bar{u}_{\bar{x}} - (X_u)^2 U_x}{} \right] + \frac{2\dot{T}(X_u)^2 U_t \bar{u}_{\bar{t}} - (X_u)^2 (U_t)^2 + 2X_t X_u U_t U_u}{}$$

$$\left. \frac{-\dot{T}^2 (X_u)^2 (\bar{u}_{\bar{t}})^2 - (X_t)^2 (U_u)^2 - 2\dot{T}X_t X_u U_u \bar{u}_{\bar{t}}}{} \right\} + \frac{F(u, u_x)}{(\dot{T})^2 (U_u - X_u \bar{u}_{\bar{x}})^2} \quad ,$$

where

$$\overline{F}(\bar{u}, \bar{u}_{\bar{x}}) = \frac{F(u, u_x)}{(\dot{T})^2 (U_u - X_u \bar{u}_{\bar{x}})^2}.$$

Taking into consideration (9) yields the relation

$$\frac{(X_x)^2 (U_u)^2 - 2X_x X_u U_x U_u - (X_x)^2 (X_u)^2 (\bar{u}_{\bar{x}})^2 + 2X_x(X_u)^2 U_x \bar{u}_{\bar{x}} + X_x(X_u)^2 \bar{u}_{\bar{x}} - (X_u)^2 U_x}{(\dot{T})^2 (U_u - X_u \bar{u}_{\bar{x}})^2} -$$

$$\frac{2\dot{T}(X_u)^2 U_t \bar{u}_{\bar{t}} - (X_u)^2 (U_t)^2 + 2X_t X_u U_t U_u - (\dot{T})^2 (X_u)^2 (\bar{u}_{\bar{t}})^2 - (X_t)^2 (U_u)^2 - 2\dot{T}X_t X_u U_u \bar{u}_{\bar{t}}}{\lambda(\dot{T})^2 (U_u - X_u \bar{u}_{\bar{x}})^2} = 1$$

$$\lambda(X_x)^2 (U_u)^2 - 2\lambda X_x X_u U_x U_u - \lambda(X_x)^2 (X_u)^2 (\bar{u}_{\bar{x}})^2 + 2\lambda X_x(X_u)^2 U_x \bar{u}_{\bar{x}} + \lambda X_x(X_u)^2 \bar{u}_{\bar{x}}$$

$$- \lambda(X_u)^2 U_x - 2\dot{T}(X_u)^2 U_t \bar{u}_{\bar{t}} + (X_u)^2 (U_t)^2 - 2X_t X_u U_t U_u + (\dot{T})^2 (X_u)^2 (\bar{u}_{\bar{t}})^2 + (X_t)^2 (U_u)^2$$

$$+ 2\dot{T}X_t X_u U_u \bar{u}_{\bar{t}} = \lambda(\dot{T})^2 (U_u)^2 - 2\lambda(\dot{T})^2 X_u U_u \bar{u}_{\bar{x}} + \lambda(\dot{T})^2 (X_u)^2 (\bar{u}_{\bar{x}})^2 \quad \ldots (22)$$

As T, X and U don't depend on u_t, u_x one can split the left hand side of (22) by $\bar{u}_{\bar{t}}$, $\bar{u}_{\bar{x}}$ and equating the coefficients of the various monomials, we get the following equations

$$\lambda (X_x)^2 (U_u)^2 - 2\lambda\, X_x X_u U_x U_u - \lambda (X_u)^2 U_x + (X_u)^2 (U_t)^2$$
$$- 2X_t X_u U_t U_u + (X_t)^2 (U_u)^2 - \lambda(\dot{T})^2 (U_u)^2 \quad = 0$$
$$\lambda (X_x)^2 (X_u)^2 + \lambda (\dot{T})^2 (X_u)^2 \quad = 0 \ ,$$
$$2\lambda\, X_x (X_u)^2 U_x + \lambda\, X_x (X_u)^2 + 2\lambda(\dot{T})^2\, X_u U_u \ = 0 \ ,$$
$$2\dot{T} X_t X_u U_u - 2\dot{T}(X_u)^2 U_t \quad = 0 \ ,$$
$$(\dot{T})^2 (X_u)^2 \quad = 0 \ .$$

From above equations, we have
$$\bar{t} = T(t)\,, \ \bar{x} = X(t,x)\,, \ \bar{u} = U(t,x,u) \quad ,$$
and the theorem is proved.

Group classification of equation (Humi&Miller,1988)

   Here we classify equations of the form (7) that admit symmetry Lie algebras of dimensions one and two.We start from describing equations admitting one–dimensional Lie algebras, and then proceed to investigate those equations which are invariant with respect to two– dimensional Lie algebras. An intermediate problem which is being solved, while classifying invariant equations of the form (7), is describing all possible realizations of one and two–dimensional Lie algebras by infinitesimal generators (12) within the equivalence relation.

1-One-Dimensional Lie Algebras:

   All in equivalent partial differential equations (7) admitting one – dimensional symmetry Lie algebras having the basis elements of the form (11) are given by the following theorem:

Theorem 4:

   There are equivalence transformations (14) that reduce infinitesimal generator (11) to one of the following generators:

$$A_1^1 =< \frac{\partial}{\partial t} >, \ A_1^2 =< \frac{\partial}{\partial x} > \ and \ \ A_1^3 =< \frac{\partial}{\partial u} > \ . \qquad \qquad \dots(23)$$

Proof:

By theorem 2 let $Q = a(t,\ x)\dfrac{\partial}{\partial t} + b(t,\ x)\dfrac{\partial}{\partial x} + f(t,x,u)\dfrac{\partial}{\partial u}$. Making use of (14),

we have

$$Q \to \overline{Q} = a\dot{T}\,\frac{\partial}{\partial \bar{t}} + (a\,X_t + bX_x)\frac{\partial}{\partial \bar{x}} + (aU_t + bU_x + fU_u)\frac{\partial}{\partial \bar{u}}.$$

Since Q is non-trivial symmetry, this implies that a, b, f can not be all zero at the same time. So there are seven cases, in a sequel, we consider the

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following three cases, while the remaining four cases reduce to Case 1, Case 2 or Case 3.

Case 1:

If $a \neq 0$, and $b = f = 0$ in Q. Hence $\bar{Q} = a\dot{T} \dfrac{\partial}{\partial \bar{t}} + a X_t \dfrac{\partial}{\partial \bar{x}} + a U_t \dfrac{\partial}{\partial \bar{u}}$.

Then choosing in (14) the function T to be the solution of the equation $a\dot{T} = 1$, and the functions X, U to be the solutions of the partial differential

equations $a X_t$, $a U_t = 0$ respectively. Which reduces $\bar{Q}$ to $\bar{Q} = \dfrac{\partial}{\partial \bar{t}}$. This means

that we can, take $Q = \dfrac{\partial}{\partial t}$.

Case 2:

If $b \neq 0$, and $a = f = 0$ in Q. That is $\bar{Q} = b X_x \dfrac{\partial}{\partial \bar{x}} + b U_x \dfrac{\partial}{\partial \bar{u}}$.

In a similar to case 1, we arrive to $Q = \dfrac{\partial}{\partial x}$.

Case 3:

Now if $f \neq 0$, and $a = b = 0$ in Q, that is, $\bar{Q} = f U_u \dfrac{\partial}{\partial \bar{u}}$, then choosing In

(14) the function U to be a solution of the equation $f U_u = 1$. So one can

take, $Q = \dfrac{\partial}{\partial u}$. The Lie algebras (23) are in equivalent, since it is impossible

to find such functions T, X, U (equivalence transformations) that one of the infinitesimal generators $\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial u}$, can be transformed to another

one. Thus, there are three in equivalent one − dimensional Lie algebras. And the theorem is proved.

1.1- Nonlinear Wave Equations Invariant under One-Dimensional Lie Algebras

The corresponding invariant equations for each of the Lie algebras (23) from the class (7) are obtained by inserting the coefficient of these Lie algebras in the classifying equation (11), and then solve it for the arbitrary element F, this yields that the corresponding invariant equations from class (7) have the form.

$$A_1^3 = < \frac{\partial}{\partial u} > : u_{tt} = -\lambda u_{xx} + F(u_x).$$

2-Two-Dimensional Lie Algebras:

As we it is well known, there are abstract two–dimensional Lie algebras (Basarab etal, 2000) namely, the commutative Lie algebras

$$A_{2.1} = <Q^1, Q^2>, [Q^1, Q^2]=0 \quad ,$$

and the solvable one

$$A_{2.2} = <Q^1, Q^2>, [Q^1, Q^2]=Q^2 \quad .$$

So the problem of describing of partial differential equations (7) admitting two–dimensional Lie symmetry algebras contains as a sub problem the one of solving the commutation relations (above) within the class of (11) up

to the equivalence relation (14). Next, we should solve (12) for each realization obtained. Having done this we get the following theorem.

Theorem 5:

The list of in equivalent realizations of two–dimensional lie algebras with the infinitesimal generator (11) and defined within the equivalence transformation (14) is given by the following Lie algebras:

$$A_{2.1}^1 = <\frac{\partial}{\partial t} , g(x)\frac{\partial}{\partial t}> \quad , \quad A_{2.1}^2 = <\frac{\partial}{\partial t} , \frac{\partial}{\partial x}> \quad ,$$

$$A_{2.1}^3 = <\frac{\partial}{\partial t} , \frac{\partial}{\partial u}> \quad , \quad A_{2.1}^4 = <\frac{\partial}{\partial t} , g(x)\frac{\partial}{\partial t}+\frac{\partial}{\partial u}> ,$$

$$A_{2.1}^5 = <\frac{\partial}{\partial t} , g(x)\frac{\partial}{\partial t}+\frac{\partial}{\partial x}> , \quad A_{2.1}^6 = <\frac{\partial}{\partial x} , g(t)\frac{\partial}{\partial x}> \quad ,$$

$$A_{2.1}^7 = <\frac{\partial}{\partial x} , \frac{\partial}{\partial u}> \quad , \quad A_{2.1}^8 = <\frac{\partial}{\partial x} , g(t)\frac{\partial}{\partial x}+\frac{\partial}{\partial u}> ,$$

$$A_{2.1}^9 = <\frac{\partial}{\partial x} , \frac{\partial}{\partial t}+\frac{\partial}{\partial u}> \quad , \quad A_{2.1}^{10} = <\frac{\partial}{\partial u} , g(t, x)\frac{\partial}{\partial u}> \quad ,$$

$$A_{2.2}^1 = <-t\frac{\partial}{\partial t} , \frac{\partial}{\partial t}> \quad , \quad A_{2.2}^2 = <(g(x)-t)\frac{\partial}{\partial t} , \frac{\partial}{\partial t}> \quad ,$$

$$A_{2.2}^3 = <-t\frac{\partial}{\partial t}+\frac{\partial}{\partial x} , \frac{\partial}{\partial t}> , \quad A_{2.2}^4 = <-t\frac{\partial}{\partial t}+\frac{\partial}{\partial u} , \frac{\partial}{\partial t}> \quad ,$$

$$A_{2.2}^5 = <(g(x)-t)\frac{\partial}{\partial t}+\frac{\partial}{\partial u} , \frac{\partial}{\partial t}> , \quad A_{2.2}^6 = <(g(x)-t)\frac{\partial}{\partial t}+\frac{\partial}{\partial x} , \frac{\partial}{\partial t}>,$$

$$A_{2.2}^7 = <x\frac{\partial}{\partial x} , \frac{\partial}{\partial x}> \quad , \quad A_{2.2}^8 = <\frac{\partial}{\partial t}+x\frac{\partial}{\partial x} , \frac{\partial}{\partial x}> \quad ,$$

$$A_{2.2}^9 = <x\frac{\partial}{\partial x}+\frac{\partial}{\partial u} , \frac{\partial}{\partial x}> \quad , \quad A_{2.2}^{10} = <\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+\frac{\partial}{\partial u} , \frac{\partial}{\partial x}>,$$

$$A_{2.2}^{11} = <\frac{\partial}{\partial t}-u\frac{\partial}{\partial u} , \frac{\partial}{\partial u}> \quad , \quad A_{2.2}^{12} = <\frac{\partial}{\partial x}-u\frac{\partial}{\partial u} , \frac{\partial}{\partial u}>,$$

$$A_{2.2}^{13} = < -u\,\frac{\partial}{\partial u}, \frac{\partial}{\partial u} > \ .$$

Proof:

Consider first the case commutative two–dimensional Lie algebra $A_{2.1}$. By Theorem 4 we can choose one of its basis generators Q^1 , say to be equal to one of those given in (23). Now, if $Q^1 = \frac{\partial}{\partial t}$, let

$$Q^2 = a(t,\,x)\,\frac{\partial}{\partial t} + b(t,\,x)\frac{\partial}{\partial x} + f(t,x,u)\frac{\partial}{\partial u}$$, then according to $[Q^1,Q^2]=0$,

we have that $\quad a_t\,\frac{\partial}{\partial t} + b_t\,\frac{\partial}{\partial x} + f_t\,\frac{\partial}{\partial u} = 0$, therefore a b and f are independent

of t .So we take $Q^2 = a(x)\frac{\partial}{\partial t} + b(x)\frac{\partial}{\partial x} + f(x,u)\frac{\partial}{\partial u}$.

The next step is to find the canonical form (the simplified form) for Q^2 under the equivalence transformations (14). However, we must now use only those equivalence transformations (14) which preserve the form of

$Q^1 = \frac{\partial}{\partial t}$. Thus we require that $Q^1 \to \bar{Q}^1$ with $\bar{Q}^1 = \dot{T}\,\frac{\partial}{\partial t} + X_t\,\frac{\partial}{\partial x} + U_t\,\frac{\partial}{\partial u} = \frac{\partial}{\partial \bar{t}}$,

Which yields $\dot{T} = 1$, $X_t = 0$ and $U_t = 0$. Hence we take:
$$\bar{t} = t + c^1 \qquad , \ \bar{x} = X(x) \ , \ \bar{u} = U(x,u) . \qquad\qquad \dots(24)$$
Under this type of transformations we find :

$$Q^2 \to \bar{Q}^2 = a(x)\frac{\partial}{\partial \bar{t}} + b(x)X_x\,\frac{\partial}{\partial \bar{x}} + (bU_x + fU_u)\frac{\partial}{\partial \bar{u}} \ .$$

In a sequel, we use have the following five cases:

Case 1:

If $a \neq 0$ and $b = f = 0$ in Q^2 . Hence $\bar{Q}^2 = a(x)\frac{\partial}{\partial \bar{t}}$.

This reduces Q^2 to $Q^2 = g(x)\frac{\partial}{\partial t}$ .Thus we have the realizations $A_{2.1}^1$.

Case 2:

If $b \neq 0$ and $a = f = 0$ in Q^2 . That is $\bar{Q}^2 = bX_x\,\frac{\partial}{\partial \bar{x}} + bU_x\,\frac{\partial}{\partial \bar{u}}$. Now in (24),

we choose X to be a solution of the equation $bX_x = 1$ and U to be a solution of the equation $bU_x = 0$. That is $Q^2 = \frac{\partial}{\partial x}$,which yields the realization $A_{2.1}^2$.

Case 3:

If $f \neq 0$, and $a = b = 0,$ in Q^2 . Then $\bar{Q}^2 = f U_u \dfrac{\partial}{\partial \bar{u}}$,

choosing from (24) U to be a solution of equation $f U_u = 1$. Thus we get the realization $A_{2.1}^3$.

Case 4:

If $b = 0$, and $a \neq 0$, $f \neq 0$ in Q^2 , then $\bar{Q}^2 = a \dfrac{\partial}{\partial \bar{t}} + f U_u \dfrac{\partial}{\partial \bar{u}}$.

Thus yielding the realization $A_{2.1}^4$.

Case 5:

If $f = 0$, and $a \neq 0$, $b \neq 0$ in Q^2 , then $\bar{Q}^2 = a \dfrac{\partial}{\partial \bar{t}} + b X_x \dfrac{\partial}{\partial \bar{x}} + b U_x \dfrac{\partial}{\partial \bar{u}}$.

Which give us the realization $A_{2.1}^5$. Lets us now turn to the cases when $Q^1 = \dfrac{\partial}{\partial x}$, $Q^1 = \dfrac{\partial}{\partial u}$ .Treating in a similar way as in the case when $Q^1 = \dfrac{\partial}{\partial t}$, and excluding the trivial and repeated realization, we will be left with the realizations $A_{2.1}^i$, $i = 6,7,...,10$; are in equivalent is established by direct verification. Consider now the case of solvable two–dimensional Lie algebra $A_{2.2}$. Taking in to account the results of theorem 4 we analyze the three possible forms of the infinitesimal generator Q^2 given in (23). Lets us first turn to the case $A_1 = A_1^1 = Q^2 = \dfrac{\partial}{\partial t}$,

let $Q^1 = a(t, x)\dfrac{\partial}{\partial t} + b(t, x)\dfrac{\partial}{\partial x} + f(t,x,u)\dfrac{\partial}{\partial u}$,

using the commutation relation $[Q^1, Q^2] = Q^2$, and making the change of variables $\bar{t} = t + c^1$, $\bar{x} = X(x)$, $\bar{u} = U(x,u)$, which preserves the form of Q^2 . We get $\bar{Q}^1 = (a - \bar{t})\dfrac{\partial}{\partial \bar{t}} + b X_x \dfrac{\partial}{\partial \bar{x}} + (b U_x + f U_u)\dfrac{\partial}{\partial \bar{u}}$. Then consider the six cases when

$a = b = f = 0; a \neq 0, b = f = 0; b \neq 0, a = f = 0; f \neq 0, a = b = 0; b = 0, a \neq 0, f \neq 0$ *and* $f = 0, a \neq 0, b \neq 0$.

Thus we get the following realizations: $A_{2.2}^i, i = 1,2,...,6$, .For the case

$$A_1 = A_1^2 = Q^2 = \dfrac{\partial}{\partial x} ,$$

the following realizations are $A_{2.2}^i, i = 7,8,...,10$ .At last, for the case when

$$A_1 = A_1^3 = Q^2 = \frac{\partial}{\partial u},$$

gives rise to the realizations $A_{2.1}^i, i = 11, 12, 13$. It is clear that these realizations are inequivalent, and the theorem is proved.

2.1 Nonlinear Wave Equations Invariant under Two- Dimensional Lie Algebras:

Now we drive all nonlinear wave equation (7), that admit two–dimensional Lie algebras as symmetry Lie algebra. Doing this we have to insert the coefficients of the obtained realizations in (11), then solving the

later for the arbitrary functions F. To this end we have the following nonlinear wave equations corresponding to their realizations:

$$A_{2.1}^1: \quad u_{tt} = -\lambda\, u_{xx} + F(u),$$

$$A_{2.1}^4: \quad u_{tt} = -\lambda\, u_{xx} \quad,$$

$$A_{2.1}^{10}: \quad u_{tt} = -\lambda\, u_{xx} + (g_{tt} + \lambda\, g_{xx})(g_x)^{-1} u_x + c \;,$$

$$A_{2.2}^4: \quad u_{tt} = -\lambda\, u_{xx} + c\, e^{2u} \quad,$$

$$A_{2.2}^9: \quad u_{tt} = -\lambda\, u_{xx} + F(u_x e^u) \quad,$$

$$A_{2.2}^{11}: \quad u_{tt} = -\lambda\, u_{xx} + c^1 u_x \;.$$

<u>Conclusion</u>

We have derived the preliminary group classification for the nonlinear wave equation of the form (7). One of the evident conclusions is that the complete group classification (description of all possible forms of the functions F, G that (11) admits a non-trivial symmetry group) of equation (7) still remain open. We hope to return to it in a forthcoming paper.

However, the full solution of this problem needs more powerful algebraic techniques like Live-Maltsev theorem , properties of simple, semi–simple and solvable Lie algebras.

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التصنيف الرمزي لأحد أصناف المعادلات الموجية اللاخطية

عصام رفيق فائق
المعهد التقني ــ كركوك

الخلاصة

تم أنجاز التصنيف الزمري لأحد أصناف المعادلات الموجية اللاخطية بمتغير معتمـد واحـد ومتغيـــرين مستقلين. تبين وجود واحد ، ستة معادلات موجية لاخطية لا متغايرة تحت تأثير جبور لي ذات الأبعاد واحـد، اثنان على الترتيب .