Diagnostics of Chaos by Gamma Test for Function of Two Dimension

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Abstract:
In this paper we have shown that Diagnostics of Chaos for Functions of Two Dimension by Gamma Test ,when the Gamma test is used for non-linear time series analysis and we illustates the relation ship between input /output data set of the from

(x_i , y_i) $(1\leq i\leq m)$, where $x\in R^m$ and without loss of generality $y\in R$ assume the data is described by an underlying model of the form.

$y= f(x)+ r =f(x_1,\dots, x_m) + r$

Where f is a smooth function with bounded first $(\nabla f(x))$ and second $(H\ f(x))$ order partial derivatives of f at the point x , defined by

$|H\ f(x)|<b_2$ and $|\nabla f(x)|<b_1$

with respect to x that there exist constants $b_1>0$ and $b_2>0$. we accounted the (Γ) test ($\gamma=A\delta+\Gamma$) and we conclusion that the two functions we used it in this paper of dimension two is chaotic.

1. Introduction
In the paper [1] we used the gamma test application" we using gamma test for analysis of chaos to functions of one dimension by algorithm for that test but in this paper we develop that algorithm to gamma test and applied it to functions of two dimension . the theoretical chapter we explained the characteristics of the test Gamma with some definitions of the topic with the writing and the algorithm flowchart either in the second quarter were taken Henon function with another function as examples of practical Gamma test.

The theoretical side
2- The Gamma test
Let x and x' be any two points in the input space C and r, r' the associated noise on the corresponding outputs. By [2]

$y= f(x_1,\dots, x_m) + r$ ……….. (1)

we have $y = f(x) + r$ and $y' = f(x') + r'$, and therefore

$$\frac{1}{2}(y-y')^2=\frac{1}{2}((r-r')+(f(x)-f(x')))^2 \quad \dots(2)$$

The continuity of f implies that

$$|f(x)-f(x')|\to 0 \quad \text{as} \quad |x-x'|\to 0 \quad \dots(3)$$

therefore, by (2) we obtain

$$\frac{1}{2}(y-y')^2\to\frac{1}{2}(r-r')^2 \quad \text{as} \quad |x-x'|\to 0 \quad \dots\dots(4)$$

Since the expectation of r is zero and r and r' are assumed to be independent and identically distributed we have

$$E\left(\frac{1}{2}(r-r')^2\right)=\sigma^2 \quad \dots\dots\dots\dots(5)$$

where σ^2 be the tanance

Hence, taking expectations of both sides in (4) it follows that

$$E\left(\frac{1}{2}(y-y')^2\right)\to\sigma^2 \quad as \quad |x-x'|\to 0 \dots.(6)$$

In the limit, if $x = x'$ then (6) implies $E\left(\frac{1}{2}(r-r')^2\right)=\sigma^2$.

So we look for points x' close to x and use the associated y' s in order to estimate σ^2.The Gamma test algorithm firstly constructs the kth ($1\leq k\leq p$) nearest neighbour lists $x_{N[i,k]}$ ($1\leq i\leq M$) of the input vectors x_i ($1\leq i\leq M$). Here, p is set before the algorithm begins, typically p ≈ 10. The construction of the near neighbour lists can be achieved in O(M log M) time using a kd-tree [1]. The algorithm then computes the following statistics

$$\delta_M(k)=\frac{1}{M}\sum_{i=1}^{M}|x_{N[i,k]}-x_i|^2 \quad \dots\dots(7)$$

where $|.|$ is the Euclidean distance, and

$$\gamma_M(k)=\frac{1}{2M}\sum_{i=1}^{M}|y_{N[i,k]}-y_i|^2 \quad \dots\dots\dots(8)$$

where $y_{N[i,k]}$ is the output value associated with $x_{N[i,k]}$. The relationship between $\gamma_M(k)$ and $\delta_M(k)$ ($1\leq k\leq p$) is approximately linear [3] for sufficiently small p. Therefore, regressing $\gamma_M(k)$ on $\delta_M(k)$ ($1\leq k\leq p$) gives an estimate for the variance of the noise, σ^2, at the intercept $\delta=0$. This estimate is denoted by Γ. The main result of [3] is that Γ converges in probability to σ^2 as $M\to\infty$.

3- Increasing Embedding
An increasing embedding technique based on the Gamma test can perform the same task of identifying the optimal embedding dimension.

Suppose we have a time series consisting of *M* points $x_1,\dots, x_M$. For each embedding dimension *m* ranging from 1 to some predefined maximum, we construct the set of *m*-1 dimensional delay vectors defined by:

$Z_i=(x_i,\dots,x_{i+m-1})$…………….(9)

We use the Gamma test to measure how smoothly the vectors ***zi*** determine the "next" point x_{i+m} of the time series. The value of *m* for which the Gamma statistic is closest to zero is taken to be the optimal embedding dimension of the time-series.[4]

4-Class of models f

The class of possible models f : $R^m \rightarrow R$ is restricted to the class of continuous functions having bounded first and second partial derivatives over the input space $C \subset R^m$. In particular the Gamma test cannot readily be applied to problems involving categorical data.

More formally, let $\nabla f(x) = (\partial f / \partial x_1, \ldots \partial f / \partial x_m)$ be the gradient vector and $Hf(x) = (\partial^2 f / \partial x_i \partial x_j)$ denote the Hessian matrix of second order partial derivatives. The Gamma test analysis requires that f be well behaved and continuous, the partial derivatives exist, and that ∇f and Hf are bounded over a convex, closed bounded set $C \subset R^m$.[5]

5-Assumptions

We elect to state our assumptions as clearly as possible. The class of models f .

The domain of possible models f : C → R is restricted to the class of continuous functions having bounded first and second-order partial derivatives over the input space C $\subset$ R^m . In particular, we remark that the Gamma test is not directly applicable to problems involving categorical data.

Let ∇ f(*x*) and Hf (*x*) denote the first and second partial derivatives of f at the point *x*, defined by $\nabla f(x) = \left(\frac{\partial f}{\partial x_i}\right)_{i=1}^{m}$ and $H f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1}^{m}$ ….(10)

where x_i and x_j are the ith and jth components of *x*, respectively. Let

H(C) denote the convex hull of C. Clearly, f can be extended to a continuous function having bounded first and second partial derivatives over H (C), and we require that there exist constants $b_1 > 0$ and $b_2 > 0$ such that, for all $H(C) \in x$

$$| \nabla f(x) | < b_1 \text{ and } |H f(x)| < b_2 \quad \text{…….(11)}$$

These conditions are required in order to apply the second mean-value theorem to the unknown function f.[3]

6-The noise distribution

Let Ψ denote the probability density function of the random noise variable *r* and let E_Ψ denote an expectation with respect to Ψ . As we have seen, we may assume, without loss of generality, that E_Ψ (r) =0 , and since we aim to estimate var(*r*), we also need that E_Ψ $(r^2) < \infty$. For technical reasons, we also require that the third and fourth moments are also finite, E_Ψ $(r^3) < \infty$ and E_Ψ $(r^4) < \infty$ We impose two further onditions on the noise distribution as follows. Firstly, two noise values ri and rj associated with two different outputs y_i and y_j (i ≠ j) must be independent of each other, so that

E_Ψ $(r_i r_j)$ = E_Ψ (r_i) E_Ψ (r_j) =0 for i ≠ j . Secondly, we assume that Ψ is fixed, i.e. that the noise is homogeneous over the input space. In the case of non-homogeneous noise, the Gamma test returns an estimate for the average noise variance .[3]

7-The Gamma test algorithm

The Gamma test works by exploiting the continuity of the unknown function f. Consider what happens when two data points x_i, x_j are close together. Then since f is smooth we should expect that $f(x_i)$ and $f(x_j)$ are also close together. If they are not then it can only be because of the addition of noise to each of these values. The Gamma test (or near neighbour technique) is based on the statistic

$$\gamma = \frac{1}{2M}\sum_{i=1}^{M}(y_i' - y_i)^2 \quad \text{……………..(1٢)}$$

Where y_i' is the y value corresponding to the first near neighbour of x_i. Given data samples (x_i, y_i),where $x_i = (x_{i1}, \ldots\ldots, x_{im})$, $1 \le i \le m$, let N[i, k] be the list of (equidistant) kth (1 ≤ k≤ p) nearest neighbours to x(i). We write

$$\delta_M(k) = \frac{1}{M}\sum_{i=1}^{M}\frac{1}{L(N[i,k]}\sum_{j\in N[i,k]}\left|x_i - x_j\right|^2 \quad \text{……(13)}$$

$$\delta_M(k) = \frac{1}{M}\sum_{i=1}^{M}\left|x_i - x_{(N[i,k])}\right|^2 \quad \text{…….(14)}$$

Where L(N[i,k]) lis the length of the list N[i,k] Here, inthe second sum, we use a convenient abuse of notation in which $x_{N[i,k]}$ stands for any one of the (equidistant) kth (1 ≤ k≤ p) nearest neighbours of x_i, since the distance is the same for all. Thus δ_k is the mean square distance to the kth (1 ≤ k≤ p) nearest neighbour. We also write

$$\gamma_M(k) = \frac{1}{2M}\sum_{i=1}^{M}\frac{1}{L(N[i,k]}\sum_{j\in N[i,k]}(y_j - y_i)^2 \quad \text{……..(15)}$$

where the y observations are subject to statistical noise assumed

independent of x and having bounded variance .Under reasonable conditions one can show that [6]

$$\gamma_M(k) = \mathrm{var}(r) + A(M,k)\,\delta_{M(k)} + o(\delta_M(k)) + O(1/M^{1/2-k}) \quad \text{…. (16)}$$

with probability greater than 1- $O(1/M^{2k})$ as M→∞ , where A(M, k) is defined by

$$A(M,k) = \frac{E_\phi(((x_{N[i,k]} - x_i).\nabla f(x_i))^2)}{2E_\phi(|x_{N[i,k]} - x_i|^2} \quad \text{….. (17)}$$

and satisfies the condition

$$0 \le A(M, k) \le \frac{1}{2}b_1^2 < \infty \quad \text{……. (18)}$$

where b_1 is the upper bound on the gradient of f over C as defined in(11).

The Gamma-test algorithm is shown in algorithm 8.

8-Algorithm. The Gamma test.

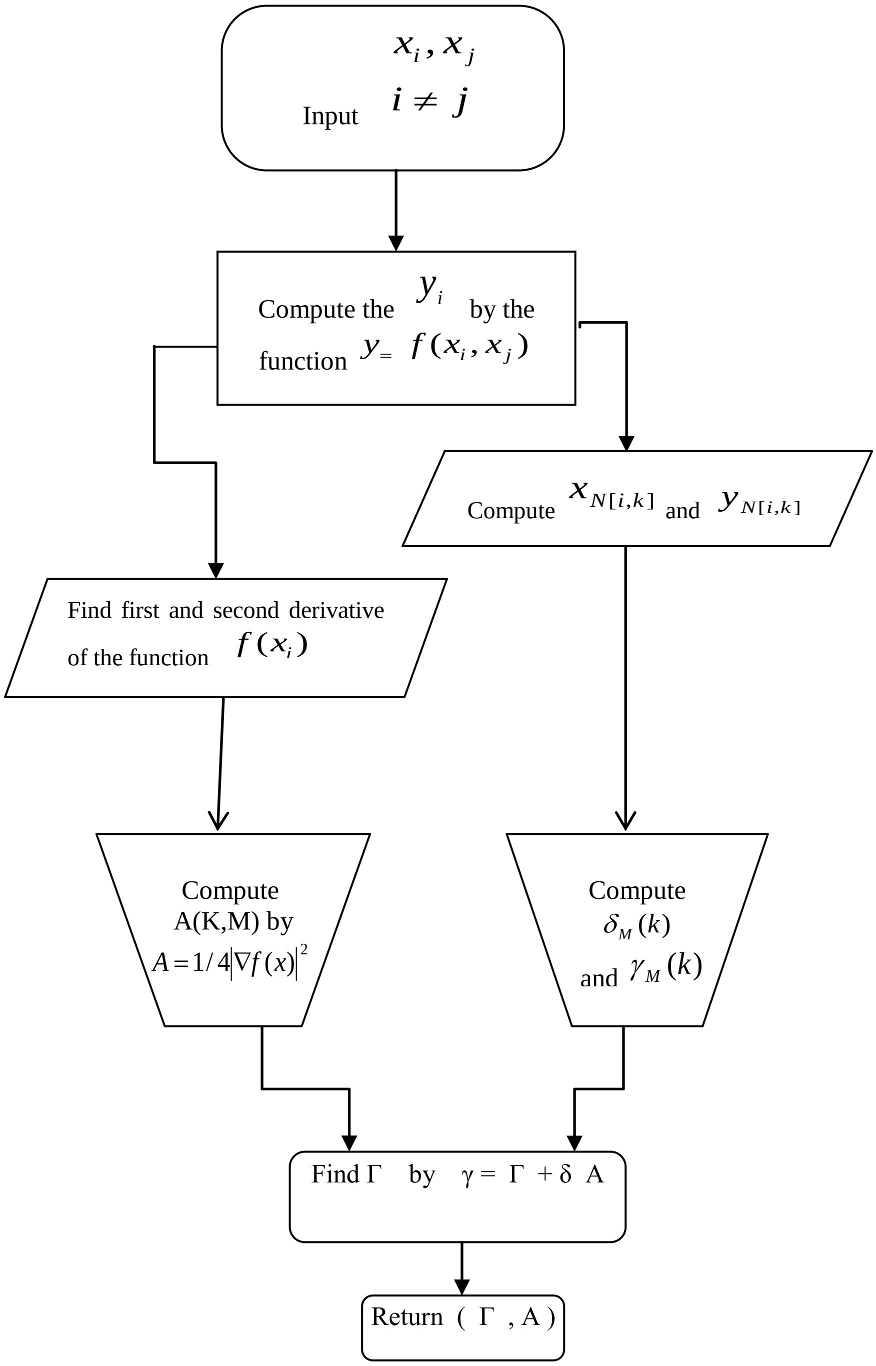
Procedure Gamma Test (data),[6] {data is an array of points (x_i, x_j) (1 ≤ i ≤ M) where x Є R^m and y Є R }

for i = 1 to M do

{compute nearest-neighbour list for each data point x_i -this can be done in O(M log M) time }
for k = 1 to p do
{ typically p ≈ 10 }
compute N[i, k], where $x_{N[i,k]}$ is thkth nearest neighbour of x_i
end for
end for

{ if multiple outputs, do the remainder for each output }
for k = 1 to p do
compute $\delta_M(k)$ as in (14)
compute $\gamma_M(k)$ as in (15)
end for
($\delta_M(k)$, $\gamma_M(k)$), $1 \le k \le p$ } obtaining
(say) $\gamma = \Gamma + A\delta$
return (Γ , A)

flow chart



Applied side:-

The Henon time series

Example(1):the henon time series is generated iteratively using the equation

$Z_t = f(Z_{t-1}, Z_{t-2}) = -z_{t-1}^2 + bZ_{t-2} + a$ ……..(1)

Where $Z_0 = -0.25$, $Z_1 = 0.25$, $a = 1.4$ *and* $b = 0.3$ point Z_t (these values are known to generate a chaotic time- series in figure (1) shows the 100 points of this time series . the points of the map are sampled from the attractor of the map which can be extracted from the time series data and visualised by simply plotting the inputs to the function against the output as shown in Figure(2) at the bottom of the diagram (in the 3-dimensional representation) , the relationship between the output (Z_t) and the input variables (Z_{t-1}) and (Z_{t-2}) .

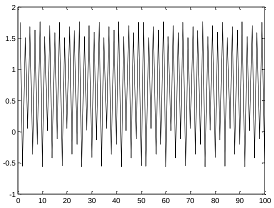


Figure (1):- first 100 pints of the clean

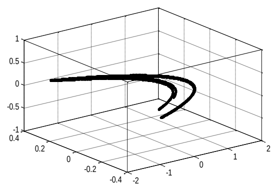


Figure (2):-the henon map with no add

plot which visually indicates the noise level . superimposed on this we plot **γ(k)** and **δ_x (k)** defined in (14) and (15) ,and perform a linear regression through these points . the intercept with the axis at δ =0 gives the estimate for the variance of the noise.

Figures (3) shows a Gamma scatter plot for the smooth function ($f(w,v) = w^2 + bv + a$ where w correspond to (Z_{t-1}) and v to (Z_{t-2})) with no added noise. As expected , for a noise- free function we have **γ→0** As δ→0 **.**

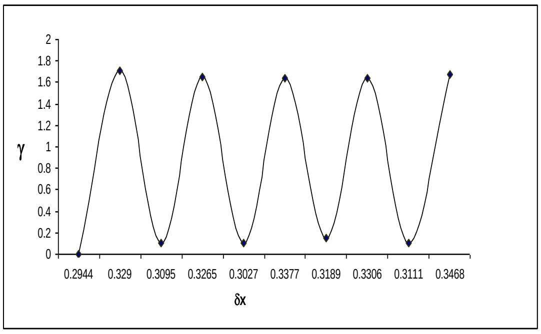


Figure (3) :Gamma scatter plot for the smooth function

The regression slope A(M,K):-

We select the function $f : IR^2 \rightarrow IR$ take example to be $f(x,y) = x^2 + y^2$,so that $\nabla f = (2x, 2y)$ and generate values of A(M , K) according to (17) , for the Henon map ,however ,we see that the A(M , k)for K in the range $1 \le k \le 10$, the A(M,K) is approximately equal to $\frac{1}{4}\varepsilon(|\nabla f|^2)$ over the input space $[0,1]^2$, table (1) list the feature space search results .The estimated variance of Γ=1.42778091 in the table (1) where Γ represents for The Henon time series with no noise of the variance and A is the gradient of the regression line fit. We see in[5] for example the lyapunov exponents from the henon time series are (1.25).

Table 1 :Best results from a complete feature

	A	*Γ*
١	3.2761	-0.96448384
٢	0.8720945	1.42778091
٣	1.87991521	-0.481033757
٤	0.02474329	1.640421316
٥	3.03637595	-0.811711
٦	0.56025225	1.449902815
٧	2.415599808	-0.617134779
٨	0.06997083	1.615067643
٩	3.242232384	-0.901258495
١٠	0.824137152	1.392289236

Irregular Embeddings

For a more complex time series , once the embedding dimension has been fixed the Gamma test can be used to select an appropriate irregular embedding using a full embedding search – in much the same way that we selected features in the Figure (4) dimensional example giver earlier.

Although it might appear that choosing an optimal subset of the input variables should not be critical , since in principle all the information required is present if the embedding dimension has been correctly estimated , nevertheless in practice correct choice of an irregular embedding ,the optimal embedding dimension selected by both methods uses the first two lag in the embedding ,which is what we would hope for given time series (1) .

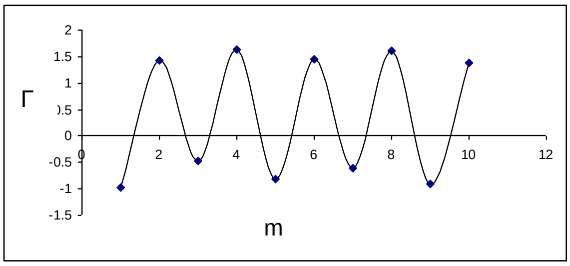
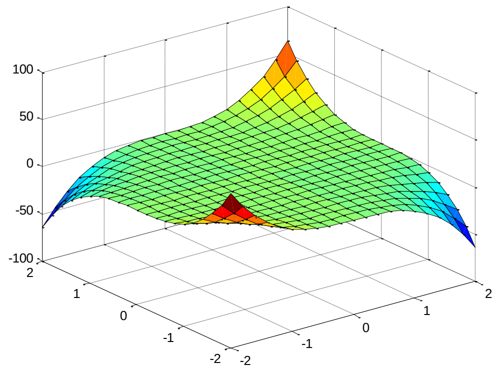


Figure (4):- The graph shows that an embedding dimension m=2 , Γ=1.42778091

Example(2):- consider the following

$f(x,y) = (x * y)^3$ …………(2)

Here x and y are randomly generated numbers from a uniform distribution in the range [-2,2] of length M=100.



Figure(5): the high frequency sin data over [-2,2]

plot which visually indicates the noise level. superimposed on this we plot **γ(k)** and **δ_x (k)** defined in (14) and (15) ,and perform a linear regression through these points . the intercept with the axis at δ =0 gives the estimate for the variance of the noise .Figures 6 shows a Gamma scatter plot for the smooth function (2) with no added noise. As expected , for a noise- free function we have **γ→0As δ→0 .**

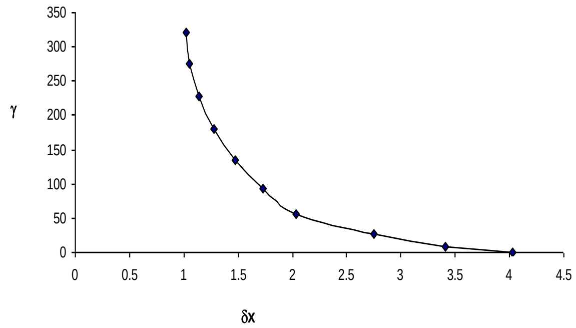


Figure6 :Gamma scatter plot for the smooth function with no added noise

Table2:- where Γ represents for the function with no noise of the variance and A is the gradient of the regression line fit.

	A	*Γ*
١	5184	-20858.9
٢	3439.172	-11729.2
٣	2231.542	-6109.05
٤	1412.591	-2818.22
٥	869.7309	-1409.37
٦	518.9854	-631.102
٧	298.8473	-202.455
٨	165.1855	38.9007
٩	87.07129	182.8528
١٠	43.40767	276.8819

Table (2) :Best results from a complete Irregular Embeddings

For a more complex time series, once the embedding dimension has been fixed the Gamma test can be used to select an appropriate irregular embedding using a full embedding search – in much the same way that we selected features in the Figure 7 .

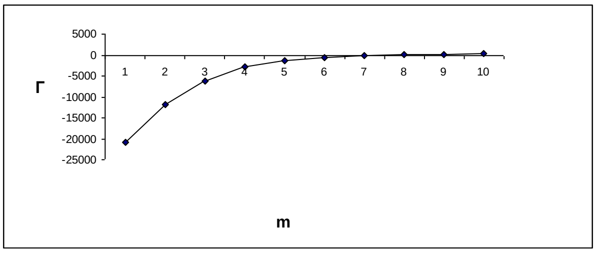


Figure7: False nearest neighbour embedding for function with no added noise . the graph shows that an embedding dimension m=5 ,Γ=-1409.37

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تشخيص الفوضى لدوال ذات بعدين باستخدام اختبار كاما

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الملخص :

في هذا البحث تم توضيح تشخيص الفوضى للدوال ذات البعدين باستخدام اختبار كاما، حيث ان هذا الاختبار يستخدم لتحليل السلاسل الزمنية الغير خطية كذلك تم توضيح العلاقة بين مجموعات العناصر الداخلة والخارجة بالشكل $((x_i , y_i)\ (1 \le i \le m))$ عندما $y \in R$ و $x \in R^m$ وباستخدام نموذج بالصيغة او الشكل التالي: −

$y= f(x)+ r =f(x_1,\dots\dots, x_m) + r$

عندما F هي دالة ملساء مقيدة بالمشتقة الجزئية الأولى $(\nabla f(x))$ والثانية بالنسبة $(H\ f(x))$ لـ x ومعرفة بالشكل

$|\nabla f(x)| < b_1$ and $|H\ f(x)| < b_2$

حيث ان $b_1 > 0$ and $b_2 > 0$ هما ثابتان . بعد ذلك قمنا بحساب اختبار (Γ) $(\gamma = A\delta + \Gamma)$ حيث استنتجنا ان الدالتين ذات البعدين التي استخدمناها في هذا البحث هي دالتين فوضويتين.