Journal of Kirkuk University –Scientific Studies , vol.4, No.2,2009

Some Numerical Methods to Solve a System of Fredholm Integral Equations of the 2nd Kind with Symmetric Kernel

Nejmaddin A. Sulaiman
College of Science Education-University of Salahaddin

Abstract

In this paper the modification in successive approximation method is defined to solve a system of linear fredholm integral equations(SLFIE) of the 2^{nd} kind with symmetric kernel, also applied Aitken method depending on successive approximation method to the solve SLFIE of the 2^{nd} kind and the algorithm for the methods are suggested, efficiency of the methods is clear through solving some numerical examples, the results in tables (1,2,3 and 4) indicate the accuracy of the approachs.

Introduction

The theory and application of integral equations is an important subject in applied mathematics. The name integral equation was introduced by Du Bois-Reymond in 1888. A more general class of linear integral equations was discussed by Fredholm in 1900 and subsequently, named after him "Fredholm integral equation". He employed a similar approach to that introduced by Volterra in 1884 Volterra apparently first met with an integral equation in 1884 when he was dealing with problem of distribution of electric changes on a segment of the surface of a sphere (Al-Rowi,1995).

Recently,(Saeed, 2006)]used successive approximation methods for solving system of Volterra integral,integro differential equations and fredholm integral equations (Babolian & Vahidi,2004) , used the Adomian decomposition method for solving linear system of Fredholm integral equations of the 2^{nd} kinds (LSFIEs,2^{nd}). Also (Delves & Mohamed, 1985) were used Trapezoid method for solving 2×2 linear systems of Fredholm integral equations of the 2^{nd} kind, and (Jerri,1985) used iterative method to solve linear Fredholm and Volterra integral equations of the second kind & (Sheelan,2007) investigated several numerical methods to solve linear system of fredholm integral equations with degenerate kernel.

Types of integral equations

This paper is related to the integral equations,for the most parts,the integration according to a single real variable,for higher order integral equations the extension of the terminology and methods appear in straight forward. A preliminary classification of integral equations are introduced through the following definitions:-

Definition (1) (Jerri, 1985; Sulaiman& Hassan,2008)

An equation which contain variable (x) that is an integral over a domain of variable (t) of the product K(x,t) and unknown function f(x) is called integral equation.

The general form of non-linear integral equation can be expressed as:

$$h(x)y(x) = f(x) + \lambda \int_a^{b(x)} k(x,t,y(t))dt \qquad a \le x \le b \qquad (1)$$

where the known functions h(x),f(x) and the kernel k(x,t,y(t)) are known functions ($a \le t \le x \le b$) are assumed to be bounded and usually to be continuous, and λ is a scalar parameter.

Definition(2) (Delves & Mohamed,1985)

If the kernel k(x,t,y(t)) = k(x,t)y(t)

Then integral equation (1) is called linear integral equation may be written as follows:

$$h(x)y(x) = f(x) + \lambda \int_a^{b(x)} k(x,t)y(t)dt \qquad a \le x \le b \qquad (2)$$

Definition (3) (Mohammad, 2002 ; Porter & Stirling,1993)

The equation (2) is called Volterra integral equations in that the integration limit is indefinite, the classical form of the second kind volterra equation is

$$y(x) = f(x) + \lambda \int_a^{x} k(x,t)y(t)dt \qquad a \le x \le b \qquad (3)$$

and the corresponding first kind Volterra equation can be written as

$$f(x) = \lambda \int_a^{x} k(x,t)y(t)dt \qquad a \le x \le b \qquad (4)$$

Definition(4) (Mohammad, 2002 ; Porter & Stirling,1993)

The equation(2) is said to be fredholm integral equation if the integral having fixed finite limits of integration. Denote these limits by a and b, the integral equation of the form

$$f(x) = \lambda \int_a^{b} k(x,t)y(t)dt \qquad a \le x \le b \qquad (5)$$

is called fredholm integral equation of the first kind,and the equation

$$y(x) = f(x) + \lambda \int_a^{b} k(x,t)y(t)dt \qquad a \le x \le b \qquad (6)$$

is called fredholm integral equation of the second kind, the classification equations 1-6 is that which is standard in the literature. Such equations arise quite naturally from the theory of differential equations (Delves & Mohammad, 1985)

Definition(5)

Dual systems of equations introduce no significant new features or difficulties consider a system of equations of the form

$$y_i(x) = f_i(x) + \sum_{j=1}^{n} \int_a^{b(x)} k_{ij}(x,t,y_j(t))dt \quad i=1,\dots,n \quad a\le x\le b \qquad (7)$$

where the y_i , k_{ij} , i ,j=1,2…n are known functions , if b(x)=b and each k_{ij} is a linear function ,then the equation(7) is called linear system of fredholm integral equation may be written as:

$$y_i(x) = f_i(x) + \sum_{j=1}^{n} \int_a^{b} k_{ij}(x,t)y_j(t)dt \quad i=1,\dots,n \quad a\le x\le b \qquad (8)$$

Definition(6) (Jerri,1985)

The kernel k(x,t) of the integral equations is called symmetric if the condition k(x,t)=k(t,x) (a≤x , t≤b) is fulfilled.

Definition (7) (Saeed,2006)

Aitken method is based on the assumption that sequence $\{\hat{y}_n\}_{n=0}^{\infty}$ defined by $\hat{y}_{n+1} = \dfrac{y_{n+2}y_n - y_{n+1}{}^2}{y_{n+2} - 2y_{n+1} + y_n}$ converages more rapidly to y than original sequence $\{y_n\}_{n=0}^{\infty}$.

Theorem(1)

If K(x,t) is symmetric kernel and does not vanish at all points of X (X a set) where it is continuous, then all of iterated functions $K_2(x,t)$, $K_3(x,t)$, …are symmetric and none of them is identically zero.

Proof:- in (Maxime ,1971)

Theorem(2)

Let $\{y_n\}_{n=0}^{\infty}$ be a sequence of numbers that converges to a limit y then the new sequence $\hat{y}_{n+1} = \dfrac{y_{n+2}y_n - y_{n+1}{}^2}{y_{n+2} - 2y_{n+1} + y_n}$ converges to y faster than $\{y_n\}_{n=0}^{\infty}$

Proof: in (Kincaid & Cheney, 2002).

In this paper the modify in the successive approximation to solve the system of linear fredholm integral equation of the 2^{nd} kind with symmetric kernel, presented.

Modification in successive approximation method to solve a system of LFIE 2nd kind (MSAM)

The modified method is derived to use successfully for increasing the rate of convergence of successive approximation for solution of a system of linear fredholm integral equations of the 2nd kind with symmetric kernel, in equation (8) as follows :

let $y_i^0(x)=f_i(x)$

$$\left.\begin{aligned}
y_1^{p+1}(x)&=f_1(x)+\sum_{j=1}^{n}\int_a^b k_{1j}(x,t)y_j^p(t)dt\\
y_2^{p+1}(x)&=f_2(x)+\int_a^b k_{21}(x,t)y_1^{p+1}(t)dt+\sum_{j=2}^{n}\int_a^b k_{2j}(x,t)y_j^p(t)dt\\
&\vdots\\
y_{n-1}^{p+1}(x)&=f_{n-1}(x)+\sum_{j=1}^{n-1}\int_a^b k_{n-1j}(x,t)y_j^{p+1}(t)dt+\int_a^b k_{n-1n}(x,t)y_{n-1}^p(t)dt\\
y_n^{p+1}(x)&=f_n(x)+\sum_{j=1}^{n-1}\int_a^b k_{nj}(x,t)y_j^{p+1}(t)dt+\int_a^b k_{nn}(x,t)y_n^p(t)dt
\end{aligned}\right\}\quad (9)$$

in general equation (9) can be written as follows :

$$y_i^{p+1}(x)=f_i(x)+\sum_{j=1}^{i-1}\int_a^b k_{ij}(x,t)y_j^{p+1}(t)dt+\sum_{j=i}^{n}\int_a^b k_{ij}(x,t)y_j^p(t)dt \quad (10)$$

where i=1,2,…,n ; p=0,1,2,…for i=1 in the equation (10) neglect the first summation .The following algorithm can be summarized to solve eq.(10)

Algorithm

Step (1) :

Put the initial solutions $y_i^0(x)=f_i(x)$ for i=1, 2.3,…,n , in the equation(9) with symmetric kernel

Step (2) :

put $k_1(x,t)=k(x,t)$ which is symmetric kernel

Step (3) :

Assume p=0,substituting in equation (9),to obtain $y_i^1(x)$, i=1,2,3,…,n

Step (4) :

Assume p=1,substituting in equation (9),to obtain $y_i^2(x)$, i=1,2,3,…,n

Step (5) :

Repeating this process to obtain $y_i^{p+1}(x)$, i=1,2,3,…,n , and then $y_i^{p+1}(x) \to y_i(x)$ as $p\to\infty$ for i=1,2,3,…,n

Numerical Examples and results

Some numerical examples were tested in this section but only two of them (1) and (2) are shown for solving the linear system of fredholm integral equations of the second kind with symmetric kernel , the exact solution is used only to show the accuracy of the numerical solution obtained by methods,successive approximation and modified successive approximation.

Example (1):

Consider the following system of linear FIE of the 2nd with the symmetric kernel,

$$y_1(x) = 1 - \frac{x^2}{6} + \int_0^1 x^2 t^2 y_2(t)dt$$

$$y_2(x) = x^3 - \frac{x}{2} + \int_0^1 xt\, y_1(t)dt$$

the exact solution of the above system is:

$y_1(x) = 1$, $y_2(x) = x^3$

The successive approximation method (SAM),(Saeed, 2007) used to solve this example first, then solve it by modified successive approximation (MSAM) and also Applied Aitken method(AM) on SAM to solve the same example.The numerical results of this problem by using SAM ,MSAM and AM with exact solution are shown in tables (1) and (2) for $y_1(x)$,$y_2(x)$ respectively.

Table (1): comparison between methods SAM , MSAM and AM with exact solution, for example (1),for approximate value of $y_1(x)$ at x=0.8

p	0	1	2	3	4	5
Exact solution	1.0	1.0	1.0	1. 0	1. 0	1.0
SAM	$89.33*10^{-2}$	$92.00*10^{-2}$	$99.333*10^{-2}$	$99.500*10^{-2}$	$99.958*10^{-2}$	$99.969*10^{-2}$
MSAM	$89.333*10^{-2}$	$92.00*10^{-2}$	$99.500*10^{-2}$	$99.958*10^{-2}$	$99.968*10^{-2}$	$99.997*10^{-2}$
AM	$103.9*10^{-2}$	$101.83*10^{-2}$	$100.24*10^{-2}$	$999.969*10^{-2}$	$99.976*10^{-2}$	$99.999*10^{-2}$
e_S	$10.67*10^{-2}$	$8.00*10^{-2}$	$6.67*10^{-3}$	$5.00*10^{-3}$	$4.2*10^{-4}$	$3.1*10^{-4}$
e_M	$10.667*10^{-2}$	$8.000*10^{-2}$	$5.00*10^{-3}$	$4.200*10^{-3}$	$3.2*10^{-4}$	$3*10^{-5}$
e_A	$3.9*10^{-2}$	$1.83*10^{-2}$	$2.40*10^{-3}$	$3.1*10^{-4}$	$2.40*10^{-4}$	$1.10*10^{-5}$

Where

p is the number of iteration ; $e_i = | y_i(x) - y_i^p(x) |$, i=1,2,…

e_S error between exact solution and solution by SAM

e_M error between exact solution and solution by MSAM

e_A error between exact solution and solution by AM

Table (2):Comparison between methods SAM , MSAM and AM with exact solution for example(1) for $y_2(x)$ at x=0.8

p	0	1	2	3	4	5
Exact solution	$51.2*10^{-2}$	$51.2*10^{-2}$	$51.2*10^{-2}$	$51.2*10^{-2}$	$51.2*10^{-2}$	$51.2*10^{-2}$
SAM	$11.2*10^{-2}$	$47.867*10^{-2}$	$48.70*10^{-2}$	$50.992*10^{-2}$	$51.044*10^{-2}$	$51.187*10^{-2}$
MSAM	$11.2*10^{-2}$	$48.700*10^{-2}$	$50.9917*10^{-2}$	$51.044*10^{-2}$	$51.187*10^{-2}$	$51.190*10^{-2}$
AM	$48.680*10^{-2}$	$50.630*10^{-2}$	$51.045*10^{-2}$	$51.090*10^{-2}$	$51.195*10^{-2}$	$51.199*10^{-2}$
e_S	$40.00*10^{-2}$	$3.333*10^{-2}$	$2.500*10^{-2}$	$2.01*10^{-3}$	$1.560*10^{-3}$	$1.300*10^{-4}$
e_M	$400.00*10^{-2}$	$2.500*10^{-2}$	$2.080*10^{-3}$	$1.560*10^{-3}$	$1.300*10^{-4}$	$1.000*10^{-4}$
e_A	$2.519*10^{-2}$	$0.568*10^{-2}$	$1.550*10^{-3}$	$1.100*10^{-3}$	$1.00*10^{-4}$	$1.000*10^{-5}$

Example (2) :

Consider the following system of linear FIE of the 2nd with the symmetric kernel,

$$y_1(x) = \frac{4x^2}{5} - \frac{1}{4} + \int_0^1 x^2t^2y_1(t)dt + \frac{1}{2}\int_0^1 y_2(t)dt$$

$$y_2(x) = \frac{3x}{4} - \frac{1}{4} + \int_0^1 x\,t\,y_1(t)dt + \frac{1}{2}\int_0^1 y_2(t)dt$$

the exact solution of the above system is:

$y_1(x) = x^2$, $y_2(x) = x$

the successive approximation method (SAM),(Saeed, 2007) used to solve this example first, then solve it by modified successive approximation (MSAM) and also Appled Aitken method(AM) on SAM to solve the same example.The numerical results of this problem by using SAM, MSAM and AM with exact solution are shown in tables (3) and (4) for $y_1(x)$,$y_2(x)$ respectively.

Table (3): Comparison between methods SAM, MSAM and AM with exact solution, for example (2), for $y_1(x)$ at x=0.95.

p	0	1	2	3	4	5
Exact solution	$90.025*10^{-2}$	$90.025*10^{-2}$	$90.025*10^{-2}$	$90.025*10^{-2}$	$90.025*10^{-2}$	$90.025*10^{-2}$
SAM	$47.20*10^{-2}$	$60.369*10^{-2}$	$68.633*10^{-2}$	$74.553*10^{-2}$	$78.723*10^{-2}$	$81.867*10^{-2}$
MSAM	$47.20*10^{-2}$	$60.369*10^{-2}$	$69.871*10^{-2}$	$76.565*10^{-2}$	$81.083*10^{-2}$	$84.113*10^{-2}$
AM	$82.945*10^{-2}$	$85.549*10^{-2}$	$88.723*10^{-2}$	$90.393*10^{-2}$	$90.555*10^{-2}$	$90.460*10^{-2}$
e_S	$43.05*10^{-2}$	$29.881*10^{-2}$	$21.617*10^{-2}$	$15.697*10^{-2}$	$11.527*10^{-2}$	$8.383*10^{-2}$
e_M	$43.05*10^{-2}$	$29.881*10^{-2}$	$20.379*10^{-2}$	$13.685*10^{-2}$	$9.167*10^{-2}$	$6.137*10^{-2}$
e_A	$7.30*10^{-2}$	$4.71*10^{-2}$	$1.527*10^{-2}$	$0.143*10^{-2}$	$0.305*10^{-2}$	$0.210*10^{-2}$

Table (4): Comparison between methods SAM, MSAM and AM with exact solution, for example (2), for $y_2(x)$ at x=0.95.

p	0	1	2	3	4	5
Exact solution	$95.00*10^{-2}$	$95.00*10^{-2}$	$95.00*10^{-2}$	$95.00*10^{-2}$	$95.00*10^{-2}$	$95.00*10^{-2}$
SAM	$46.25*10^{-2}$	$59.625*10^{-2}$	$69.413*10^{-2}$	$76.409*10^{-2}$	$81.373*10^{-2}$	$85.069*10^{-2}$
MSAM	$46.25*10^{-2}$	$64.414*10^{-2}$	$74.501*10^{-2}$	$81.277*10^{-2}$	$85.815*10^{-2}$	$88.853*10^{-2}$
AM	$97.12*10^{-2}$	$93.10*10^{-2}$	$93.43*10^{-2}$	$95.869*10^{-2}$	$95.84*10^{-2}$	$95.63*10^{-2}$
e_S	$48.75*10^{-2}$	$35.375*10^{-2}$	$25.587*10^{-2}$	$18.591*10^{-2}$	$13.627*10^{-2}$	$9.931*10^{-2}$
e_M	$48.75*10^{-2}$	$30.586*10^{-2}$	$20499*10^{-2}$	$13.723*10^{-2}$	$9.185*10^{-2}$	$6.147*10^{-2}$
e_A	$2.12*10^{-2}$	$1.9*10^{-2}$	$1.57*10^{-2}$	$0.87*10^{-2}$	$0.841*10^{-2}$	$o.630*10^{-2}$

Conclusions

Introduced the successive approximation method and Aitken method on SAM to obtain solution of the second kind system of linear Fredholm integral equations with the symmetric kernel .Also numerical experiments show that the new modified successive approximation and Aitken method are converges faster and they are more efficient than successive approximation. The tables (1,2,3 and 4) give the numerical comparisons.

References

- AL-Rowi , S.N. ,(1995) : Numerical solution of first kind integral equation of convolution type;M.Sc. thesis,University of Technology 90 P
- Babolian, E., Biazar, J. and Vahidi, A. R. (2004) :The decomposition method applied to system of Fredholm integral equations of the second kind, Applied Mathematics and computation,Vol.148,pp.443-452.
- Delves , L.M. and Mohamed , J.L. ,(1985) :Computational Method for ntegral Equation ; Cambridge University Press 376 P.
- Hussain ,S.O., (2007) :Suitable numerical techniques for solving a system of linear FIE of the 2nd kind,M.Sc.Thesis, university of Sulaimani, 54p
- Jerri , A.J. , (1985) : Introduction to integral equations with applications , Marcel Dekken , Inc.New york 254 P.
- Kincaid, D.and Cheney,W.(2002):Numerical analysis mathematics of scientific computing ,3rd edition, wadsworth group, Brooks, 817 p.
- Maxime,B. A.,(1971): An introduction to study of integral equations, Hafnre publishing Com. New York 70 p.
- Porter , D. and Stirling , D.S.G. ,(1993):Integral Equation A Practical Treat from Spectral Theory to Applications ;Cambridge University Press, 372 P.
- Saeed , R. K. , (2006) :Computational Methods for Solving System of Linear Volterra Integral and Integro-Differential Equation ;Ph.D.Thesis,University of Salahaddin Hawler,college of Science 146 P.
- Saeed , R.K. ,(2007): Solution of a system of linear Fredholm Integral Equation of the 2nd kind by Iteration Method, Journal of university of Kirkuk- Scientific study Vol.2, PP. 57-68.
- Sulaiman N.A. and Hassan T.I., (2008): Successive approximation method for solving fredholm integral equations of the 1st kind with symmetric kernel,Journal of Education and Science University of Musol,Vol.21, pp. 149-159.

Journal of Kirkuk University –Scientific Studies , vol.4, No.2,2009

بعض الطرق العددية لحل نظام من المعادلات فريد هولم التكاملية الخطية من النوع الثاني مع النواة المتناظرة

نجم الدين عبداللة سليمان
كلية التربية العلمية ــ جامعة صلاح الدين/ اربيل

الخلاصة

في هذا البحث تم تعريف الطريقة التطويرية التقريبية التكرارية لحل نظام من المعادلات فريدهولم التكاملية الخطية من النوع الثاني مع نواة المتناظرة وأيضا تم تطبيق طريقة أيتكن المعتمد على الطريقة التكرارية لحل نظام من المعادلات فريدهولم التكاملية الخطية من النوع الثاني مع النواة المتناظرة ،تم اقتراح خوارزمية لطرق ،كفاءة الطرق واضحة من خلال حل بعض الامثلة العـــــــــددية ،النتائج في الجداول (١،٢،٣،٤) تشير الى دقة التقنيات.