Rings Over Which Certain Modules Are YJ-Injective

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Abstract

In this paper we continue to study the concept of YJ-injectivity which was first introduced by Ming in 1985. Furthermore, we give some characterizations and properties for it. Also, we give a sufficient condition for right weakly π-regular ring to be strongly π-regular. Finally, we give other results of SYJ-rings and connect it with other types of rings such as weakly π-regular, reduced ring and strongly regular ring.

Introduction

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. J(R) denote the Jacobson radical of R.A rightR-module M is called right principally injective (briefly,right P-injective) (Ming,1974)if for any principal right ideal aR of R and any right R-homomorphism of aR into M extends to one of R into M.R is called right P-injective if the right R-module R_R is P-injective.For any non empty subset X of a ring R,the right(left)annihilator of X will be denoted by r(X) (ℓ(X)),respectively.The annihilator ideal of I will be denoted and defined by,ann(I)={$a\in$I:ax=xa=0,for every $x\in$I}.A ring R is called π-regular (McCoy,1939)if for any $a\in$R,there exists a positive integer n and an element b of R such that $a^n=a^n b\, a^n$.A ring R is called strongly regular (Luh,1964)if for each $a\in$ R,there exists $b\in$R such that $a=a^2b$.It should be noted that in a strongly regular ring R,$a = a^2b$ if and only if $a = ba^2$. A ring R is called right (left) weakly regular (Ramamurthi,1973) if for each $a \in$ R, $a\in aRaR$($a\in RaRa$).R is called weakly regular if it is both right and left weakly regular.A ring R is called strongly π-regular(Azumaya,1954)if for every $a\in$R,there exists a positive integer n, depending on a and an element $b\in$R such that $a^n = a^{n+1}$b. or equivalently,R is strongly π-regular if and only if $a^nR=a^{2n}R$.It is easy to see that R is strongly π-regular if and only if $Ra^n=Ra^{2n}$.A ring R is called right(left)weakly π-regular(Gupta,1977) if $a^n \in a^nRa^nR$($a^n \in Ra^nRa^n$),for every $a\in$R and a positive integer n. R is called reduced if it has no non-zero nilpotent element.Recall that R is semi-prime ring if it contains no non-zero nilpotent ideal.An ideal I of a ring R is nilideal (Faith,1976)provided that every element of R is nilpotent. A ring R is a 2-primal ring (Birkenmeier,1994) if P(R)=N(R),where P(R)is the prime radical of R and N(R) is the set of all nilpotent elements.Recall that R is called right(left)duo ring (Brown,1973)if every right(left)ideal of R is two-sided. R is called weakly right duo(Chen,1999)if for any $a\in R$,there exists a positive integer n such that $a^n R$ is two-sided. Following(Yu,1964), a ring R is called right quasi-duo if every maximal right ideal of R is two-sided.

Characterizations and Properties of YJ-Injectivity

In this section we continue to study the concept of YJ-injectivity , which was first introduced by Ming in 1985. Furthermore , we give some characterizations and properties for it.Also,we give a sufficient condition for right weakly π-regular ring to be strongly π-regular.We start this section with the following definition.

Definition 1:

A right R-module M is called YJ-injective if for any $0\neq a \in R$,there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of $a^n R$ into M extends to one of R into M. Recall that R is called right (left) YJ-injective if the right (left) R-module R_R ($_R R$) is YJ-injective. The following lemma is a relation between right weakly π-regular ring and semi-prime ring.

Lemma 2:

If R is a right weakly π-regular ring, then it is semi-prime.

Proof:

Let R be a right weakly π-regular. Then, for each $x \in R$,there exists a positive integer n such that $x^n R = x^n R x^n R$.It means that every principal right ideal $I = x^n R$ is idempotent.If I is nilpotent, there exists a positive integer m such that $I^m=(0)$.Now,$(0)=I^m=I^{m-2}.I^2=I^{m-2}.I=I^{m-1}$,repeat this process(n-1) times,we obtain I = (0). Hence R is a semi-prime ring.

Lemma 3 (Abdul-Aziz,1998**):**

If R is a reduced ring, then $r(a^n)=r(a^{2n})$,for every $a\in R$ and a positive integer n. The following result is a characterization of YJ-injective right R-module in terms of left annihilator.

Theorem 4 (Ming,1985,Lemma 3**):**

The following statements are equivalent:

(1) R is YJ-injective right R-module ;

(2) For any $0 \neq a \in R$, there exists a positive integer n such that Ra^n is a non-zero left annihilator.

Lemma 5 (Abdul-Aziz, 1998**):**

Let R be a semi-prime, 2-primal ring. Then, R is a reduced ring. The following result contains a sufficient condition for right weakly π-regular ring to be strongly π-regular.

Theorem 6:

Let R be a right weakly π-regular, 2-primal ring. If R is a right YJ-injective ring, then R is strongly π-regular.

Proof:

Let R be a right weakly π-regular ring,then by Lemma 2.2,R is a semi-prime ring. Since R is a 2-prime ring, then by Lemma 2.5,R is reduced ring.Therefore, by Theorem 2.4, for any $0 \neq a \in R$,there exists a positive integer n such that Ra^{2n} is a non zero left annihilator. Since R is a reduced ring,then by Lemma 2.3, $r(a^n)=r(a^{2n})$.Hence,$Ra^n \subseteq \ell(r(Ra^n))=\ell(r(a^n))=\ell(r(a^{2n}))=\ell(r(Ra^{2n}))= Ra^{2n}$. Therefore, $a^n = d\,a^{2n}$,for some $d \in R$.Whence R is strongly π-regular.

The next lemma is due to Huaping Yu.

Lemma 7 (Abdul-Aziz,1998**):**

If R is a right quasi-duo ring, then R/J(R) is a reduced ring.

Theorem 8:

Let R be a right quasi-duo ring such that R/J(R) is a right YJ-injective. Then, R/J(R) is a strongly π-regular ring.

Proof:

Let $B=R/J(R), b \in B, b=a+J(R), a \in R$, $f: b^n B \to B$ a right B-homomorphism, for some positive integer n. Then , $f : (a^n R + J(R)) / J(R) \to R / J(R)$ and $f(a^n+J(R))= d+J(R)$, for some $d \in R$. Define a right R-homomorphism $g: a^n R \to R/J(R)$ by $g(a^n c)=d\,c+J(R)$,for all $c \in R$. Then, g is a well-defined right R-homomorphism. Since R/J(R) is a right YJ-injective, there exists $u \in R$ such that $g(a^n c)=u\,a\,c + J(R)$,for all $c \in R$. Therefore, $f(a^n c + J(R))= f(a^n + J(R))(c+ J(R))=(d +J(R))(c+J(R))=d\,c+J(R)=g(a^n c)=u\,a\,c + J(R)= (u+J(R))(ac+J(R))$, for all $c \in R$. This proves that B=R/J(R) is a right YJ-injective ring.Since R is a right quasi-duo ring, then by Lemma 2.7,B is a reduced ring.Therefore, by(Ming,1983),B is a strongly regular and hence R is strongly π-regular ring. The following result is a relation between π-regular ring with every principal right ideal $a^n R$ is YJ-injective.

Theorem 9:

Let R be a ring with every principal right ideal $a^n R$ is YJ-injective, for every a $\in$ R and a positive integer n, then R is π-regular ring.

Proof:

Let R be a ring.Since $a^n R$ is YJ-injective, then for every $a\in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from $a^n R$ into $a^n R$ extends to R into $a^n R$. Define $f: a^n R \rightarrow a^n R$ by $f(a^n x)= a^n x$, for every $x\in R$. Since $a^n R$ is YJ-injective, there exists $b\in R$ such that $f(a^n x)=a^n b a^n x$.Thus , it follows that $a^n = a^n b a^n$ .Whence R is π-regular ring.

Theorem 10:

Let R be a ring, if for any maximal right ideal N, NR is YJ-injective and ann(R) = ann(N), then N = NR.

Proof:

Let N be a maximal right ideal and let d be any element in N. Define a right R-homomorphism $f: a^n R \rightarrow NR$ by $f(a^n x)=d x$, for all $x\in R$ and a positive integer n. Then, f is a well-defined. Indeed; let x_1, x_2 be any two elements in R with $a^n x_1=a^n x_2$,then$(x_1-x_2)\in r(a^n)=r(d)$.So, $d x_1=d x_2$. Therefore, $f(a^n x_1)=d x_1=d x_2=f(a^n x_2)$.Since NR is YJ-injective,there exists c in NR such that $f(a^n x)=c(a^n x)$,for all $x\in R$ and a positive integer n. Thus, $d=f(a^n)=c a^n$ .Therefore, $d=c a^n$ .Hence $N\subseteq NR$ and since $NR\subseteq N$. Whence, N=NR.

Theorem 11:

The following statements are equivalent:

(1) R is either a right YJ-injective local ring whose J(R) is a nilideal or strongly π-regular;

(2) Every non-nil left ideal of R is YJ-injective right R-module.

Proof:

(1)$\Rightarrow$(2). If R is a right YJ-injective, local ring whose J(R) is nilideal, then the only non-nil left ideal is R. Therefore, (1) implies (2).

(2)$\Rightarrow$(1).Assume(2).First suppose that R contains a maximal left ideal K which is a nilideal.Then,$K\subseteq J$ which implies J=K is the unique maximal left (and hence right) ideal of R. Therefore, R is a local ring such that J(R) is nil and also R is right YJ-injective.Now,suppose that every maximal left ideal is non-nil.Then,every maximal left ideal of R is right YJ-injective, yielding $R b^n + \ell(b^n) = R$,for any $b\in R$ and a positive integer n.In that case,R is strongly π-regular.Thus,(2)implies(1).In similar method in(Nicholson & Yousif,1994) for right P-injective rings we obtain the proof of the following lemma.

Lemma 12:

If R is right YJ-injective and A, $B_1,\ldots, B_n$ are two-sided ideals of R,then $A\cap(B_1\oplus\ldots\oplus B_n)=(A\cap B_1)\oplus\ldots\oplus(A\cap B_n)$.

Theorem 13 (Puniniski, Wisbauer & Yousif,1995):

If $A\oplus(B\cap A_1)$is a direct summand of R_R and so A+B is also a direct summand of R_R.

Remark 14 (Puniniski, Wisbauer & Yousif,1995):

For a sub module $A\subseteq M$,the notion $A\subseteq^{\oplus}M$ will mean that A is a direct summand of M.

Theorem 15:

Let R be a right YJ-injective right duo ring. If A and B are right ideals of R with $A\subseteq^{\oplus}R_R$ and $B\subseteq^{\oplus}R_R$, then $(A\cap B)\subseteq^{\oplus}R_R$ and $(A+B)\subseteq^{\oplus}R_R$.

Proof:

We can write $R=A\oplus A_1=B\oplus B_1$,for some right ideals A_1 and B_1 of R. Then,by Lemma 2.12,$B=B\cap(A\oplus A_1)=(B\cap A)\oplus(B\cap A_1)$.Hence
$R=(B\cap A)\oplus(B\cap A_1)\oplus B_1$ and so$(A \cap B) \subseteq^{\oplus} R_R$. Also,

$A+B=A+((B\cap A)\oplus(B\cap A_1))$

$\quad=(A+(B\cap A_1))\oplus(B\cap A_1)$

$A+B=A \oplus (B\cap A_1)$.

Since both A and $(B\cap A)$ are direct summands of R_R, it follows from Theorem 2.13,$A\oplus(B\cap A_1)$is a direct summand of R_R and so A+B is also a direct summand of R_R. Recall that R is called quasi-Frobenius ring, abbreviated QF-ring(Faith,1976),provided that R is a left and right Artinian and R is injective as a right R-module. The following result in (Faith, 1976) is characterizations of QF-rings.

Theorem 16:

The following statements on a ring R are equivalent:

(a) R is QF-ring;

(b) R_R is injective and Noetherian;

(c) R_R is injective and Artinian;

(d) R_R is injective and ${}_RR$ is Noetherian.

Theorem 17:

If R is a commutative ring whose YJ-injective modules are injective and flat, then R is QF-ring.

Proof:

Since every direct sum of YJ-injective R-modules is YJ-injective(Ming, 1985),and every YJ-injective R-module is,by hypothesis injective,then any direct sum of injective R-modules is injective which implies that R is a Noetherian ring(Faith,1999).Since every injective R-module is flat,then R must be P-injective ring by(Jain,1993).Therefore,R is QF-ring by a result of H. H. Storrer(Storrer,1969).

Corollary 18:

A commutative ring R is a principal ideal QF-ring if and only if every finitely generated ideal of R is principal and every YJ-injective R-module is injective and flat.

Proof:

Follows from Theorem 17.

Definition 19 (Faith,1973):

Let R be a ring and let C be an R- module. Then , C is projective if and only if the following property holds : Given any R- module epimorphism f:A→B and homomorphism g:C→B,there exists h:C→A with g=f o h. i.e.; the diagram is commutative.

Theorem 20:

Let R be a weakly left duo ring. If R/ a^n R is YJ-injective and a^n R is projective, for every a∈R and a positive integer n.Then, R is strongly π-regular ring.

Proof:

Let a be an element of R. Define a right R-homomorphism f:R/a^nR→ a^nR/a^{2n}R by f(y+a^nR)=a^ny+a^{2n}R,for all y∈R and a positive integer n. Since a^nR is projective, there exists right R-homomorphism g:a^nR→R/ a^nR such that f (g(a^nx))=a^nx+a^{2n}R,for all x∈R.But R/a^nR is YJ-injective,there exists a non zero element c∈R such that

g(a^nx)=(c+a^nR)a^nx.Then,

a^nx+a^{2n}R= f(g(a^nx))

= f((c+a^nR)a^nx)

= f((a^nx+a^nR)) (since a^nx∈a^nR

=a^nc a^nx+a^{2n}R.

Since R is a weakly left duo ring, then c a^n ∈R a^n=a^nR implies that c a^n=a^nt,for some t∈R.So,a^nx+a^{2n}R=a^{2n}t x+a^{2n}R,yields a^nR = a^{2n}R. Thus, a^n = a^{2n}d, for some d ∈ R. Hence R is strongly π- regular ring.

Rings In Which Every Simple R-Module is YJ-Injective

In this section we give other results of SYJ-rings. Also,connect it with other types of rings such as weakly π-regular,reduced and strongly regular ring.We start this section with the following definition.

Definition 1 (Abdul-Aziz,1998):

A ring R is said to be right SYJ-ring if every simple right R-module is YJ-injective.

Theorem 2:

Let R be SYJ-ring. Then,

(1) Any reduced right ideal of R is idempotent;

(2) $R = R c^n R$, for any non-zero divisor c of R and a positive integer n.

Proof:

(1) Let P be a reduced right ideal of R. For any $b \in P$, $\ell (b^n) \subseteq r (b^n)$ and if $R b^n R + r (b^n) \neq R$, for some positive integer n. Let M be a maximal right ideal containing $R b^n R + r (b^n)$. If R / M is YJ-injective, then the right R-homomorphism $g: b^n R \rightarrow R / M$ defined by $g (b^n a) = a + M$, for all $a \in R$ yields $1+M=g(b^n)=d b^n +M$, for some $d \in R$, whence $1 \in M$,a contradiction. Thus, $R b^n R + r (b^n)=R$, for any $b \in R$, which completes the proof.

(2) If $R c^n R \neq R$, let M be a maximal right ideal containing $R c^n R$. Since $\ell (c^n) = r (c^n) = 0$, the proof of (1) shows that R / M is YJ-injective leads a contradiction. This proves that $R c^n R = R$.

Theorem 3:

If R is a weakly right duo SYJ-rings, then R is weakly π-regular ring.

Proof:

Suppose that $a^n R \neq (a^n R)^2$, for every $a \in R$ and every positive integer n. Then, there exists $y \in a^n R$, but $y \notin (a^n R)^2$. Since $(R y^n)^2 \subseteq R y^n \subseteq y^n R$, which means $R y^n R y^n \subseteq J \subseteq y^n R$, there exists a maximal right ideal M such that $J \subseteq M$. Then, $y^n R / M$ is YJ-injective, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of $y^n R$ into $y^n R / M$ extends to one of R into $y^n R / M$. Define $f : y^n R \rightarrow y^n R / M$ by $f (y^n t) = y^n t + M$, for every $t \in R$. Since f is a well-defined, there exists $c \in R$ such that $f (y^n t) = (y^n c + M) y^n t$, so $(y^n - y^n c y^n) \in M$. Since $y^n c y^n \in R y^n R y^n \subseteq M$. Therefore, $y^n \in M$. This implies $M = y^n R$, a contradiction, whence $a^n R = (a^n R)^2$. Therefore, R is weakly π-regular ring.

Lemma 4 (Yu,1995):

If R is right quasi-duo, then R / J(R) is a reduced ring.

Proposition 5 (Ming,1996):

Let R be a right SYJ-ring. Then, J(R) = (0).

Theorem 6:

If R is a right quasi-duo right SYJ-ring. Then, R is strongly regular ring.

Proof:

Let R be a right quasi-duo ring,then by Lemma 3.4,R/J(R) is a reduced ring. Since R is SYJ-ring,then by Proposition 3.5,J(R)=(0).Hence R is a reduced ring.Suppose that $aR+r(a)\neq R$,there exists a maximal right ideal M of R containing $aR+r(a)$.Since R/M is YJ-injective,there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of a^n R into R/M extends to one of R into R/M.Let $f:a^n R\rightarrow R/M$ be defined by $f(a^n t)=t+M$, for every $t\in R$.Since R is reduced,then f is a well-defined right R-homomorphism. Thus,there exists $c\in R$ such that $1+M=f(a^n)=ca^n+M$. But,$ca^n\in M$ and so $1\in M$, which is a contradiction. Therefore,a R+ r(a)=R.So, R is strongly regular.

Proposition .7 (Nam, Kim & Kim,1995):

Let R be a right SYJ-ring. Then, R is a semi-prime ring.

Theorem 8 (Nam, Kim & Kim,1995):

Let R be a 2-primal ring such that every simple right R-module is YJ-injective. Then, R is a reduced weakly regular ring.The following result is a slightly improvement of above theorem.

Theorem 9:

Let R be a 2-primal SYJ-ring.Then,R is a reduced strongly regular rings.

Proof:

Let R be SYJ-rings, then by Proposition 3.7, R is a semi-prime ring and since R is a 2-primal ring, then by Lemma 2.5, R is reduced. It remains to show that $xR+r(x)=R$,for any $x\in R$.Now,suppose that there exists $y\in R$ such that $yR+r(y)\neq R$.Then,there exists a maximal right ideal K of R containing $yR+r(y)$.Since R/K is YJ-injective,there exists a positive integer n such that $y^n\neq 0$ and any right R-homomorphism of y^n R into R/K extends to one of R into R/K. Now,let $f:y^nR\rightarrow R/K$ be defined by $f(y^n t)=t+K$,for every $t\in R$.Since R is reduced,then f is a well-defined right R-homomorphism.Now, since R/K is YJ-injective,there exists $c\in R$ such that $1+K=f(y^n)=cy^n+K$, so $(1-cy^n)\in K$ and hence $1\in K$ which is a contradiction.Therefore,$xR+r(x)=R$, for any $x\in R$. Hence R is a strongly regular rings.

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الحلقات التي تصبح مقاسات محددة عليها مقاسات مغمورة من نمط YJ

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الخلاصة

في هذا البحث نستمر في دراسة الأغمار من النمط-YJ ،كما عرف أولا من قبل Ming في ١٩٨٥.اضافة الى ذلك،نعطي المميزات والصفات لها.كذلك نعطي شرط الكافئ لحلقة منتظمة بضعف من النمط -π لتصبح حلقة منتظمة بقوة من النمط -π.أخيرا ،نعطي بعض النتائج الأضافية للحلقات من النمط -SYJ وربطها مع أنماط أخرى من الحلقات كحلقات منتظمة بضعف من النمط-π ، حلقة مختزلة و الحلقة منتظمة بقوة.