On Smarandache Semigroups

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Abstract

In this work we study some type of Smarandache semigroups and Smarandache subgroups of a semigroup such as Smarandache cyclic semigroups, Smarandache p-Sylow subgroups and Smarandache normal subgroups. In addition we introduce the concept of Smarandache ideal of a semigroup and study its relation with Smarandache normal subgroup.

Introduction

A semigroup S called a Smarandache semigroup if there is a proper subset of S which is a subgroup of S (Raual, 1998), (by a subgroup A of S we mean a subset A of S which is a group under the same operation of S). It is known that if e is an idempotent of a semigroup S then $G_e = \{a \in S \mid a = ae$ and $e = a_1 a = a a_1$ for some $a_1 \in S\}$ equal to S or it is the maximal subgroup of S having e as its identity (Mario, 1973).

Many Smarandache concepts introduced by Kandasamy,V. W. and many open research problems are given(Kandasamy, 2002). A Smarandache semigroup S called Smarandache cyclic semigroup if every subgroup of S is cyclic (Kandasamy, 2002). If S be a finite Smarandache semigroup, P a prime which divides the order of S, then a subgroup of S of order p or p^t (t >1) called Smarandache p-Sylow subgroup. In this work we give complete answer of the following problems given in (Kandasamy, 2002).

1- Find condition on n, n a non prime so that Z_n, the semigroup under multiplication modulo n is a Smarandache cyclic semigroup.

2- Let $(Z_{2^n},.)$ be the semigroup of order 2^n . For n>3 arbitrarily large find the number of Smarandache 2-Sylow subgroup of Z_{2^n}.

In addition we introduce the concepts of Smarandache ideal, Smarandache prime ideal and study some of their properties and we give the relation between Smarandache ideals and Smarandache normal subgroups.

S1: Smarandache cyclic semigroups

In this Section we discuss Smarandache cyclic semigroups, and find the number of cyclic subgroups of $(Z_{p^n},.)$ for n>2.

Lemma1.1.

$(\mathbb{Z}_{p^n},.)$ p prime, has no nontrivial idempotent.

Proof: The proof is easy.

Theorem 1.2.

$(\mathbb{Z}_{p^n},.)$ p an odd prime, n>2, is a Smarandache cyclic semigroup.

Proof: Since $\varphi(p^n) = p^n - p^{n-1}$ the number of elements in $\mathbb{Z}_{p^n}$ which have inverses form a group under multiplication, and then $\mathbb{Z}_{p^n}$ have a subset which is a group of order $p^n - p^{n-1}$. This subgroup is the largest subgroup with 1 as its identity. Since there exists an element $a \in S$ which is a primitive root of p^n (Kenneth, 2004), $a^{p^n - p^{n-1}} \equiv 1 (\mathrm{mod}\, p^n)$ and a generates S, thus S is cyclic. Hence all subgroups of $\mathbb{Z}_{p^n}$ are cyclic, and $\mathbb{Z}_{p^n}$ is a Smarandache cyclic semigroup.

Lemma 1.3.

Let (G,.) be a semigroup with identity 1and $S=\{x \in G: x^2=1\}$. Then (S,.) is a cyclic group if and only if S contains at most two elements.

Proof: The proof is easy.

Proposition 1.4.

1- The semigroup $(Z_{2^k},.)$, k>2 is a Smarandache semigroup which is not a Smarandache cyclic semigroup.

2- The semigroup $(Z_{2^k p},.)$, k≥2, p an odd prime, is a Smarandache semigroup which is not a Smarandache cyclic semigroup.

Proof: 1- Since $(2^{k-1}-1)^2 = (2^{k-1}+1)^2 = 1, (2^{k-1}-1)(2^{k-1}+1) = 2^k - 1$, and $(2^k - 1)^2 = 1$, then $S = \{1, (2^{k-1}-1), (2^{k-1}+1), (2^k - 1)\}$ is a subgroup of $(Z_{2^k},.)$ and by Lemma 1.3, S is not cyclic. Hence $(Z_{2^k},.)$ is not a Smarandache cyclic semigroup.

2- Similar to part 1.

Theorem1.5.

$(Z_{2p^n},.)$, p odd prime is a Smarandache cyclic semigroup.

Proof: First we show that Z_{2p^n} has two maximal subgroups of order $\varphi(2p^n)$. It is known that there exists a number a belonging to $\varphi(2p^n)(\mathrm{mod}\,2p^n)$, so $\varphi(2p^n)\equiv 1(\mathrm{mod}\,2p^n)$, and a generates a group (G_1) of order $\varphi(2p^n)$ with 1 as its identity. Since $\varphi(2p^n)\equiv 1+2kp^n)$ for some $k\geq 1$, then $\varphi(2p^n)+p^n\equiv(p^n+1)+2kp^n$. Therefore $\varphi(2p^n)+p^n\equiv(p^n+1)(\mathrm{mod}\,2p^n)$. We claim that $a+p^n$ generates a group of order $\varphi(2p^n)$ and $1+p^n$ is its identity element. $(p^n)^2\equiv(p^n)(\mathrm{mod}\,2p^n)$ and $(1+p^n)^2\equiv(1+p^n)(\mathrm{mod}\,2p^n)$, hence $(a+p^n)^2\equiv(a^2+p^n)(\mathrm{mod}\,2p^n)$ and $(a+p^n)^3\equiv(a^3+p^n)(\mathrm{mod}\,2p^n)$. If a is even, then $ap^n=p^n$, consequently $(a+p^n)^3\equiv(a^3+p^n)(\mathrm{mod}\,2p^n)$. If a is odd, then $ap^n\equiv p^n(\mathrm{mod}\,2p^n)$ which implies that $(a+p^n)^3\equiv(a^3+p^n)(\mathrm{mod}\,2p^n)$. Continuing in this manner we get $(a+p^n)^{\varphi(p^n)}\equiv 1+p^n(\mathrm{mod}\,2p^n)$, and $(a+p^n)^{\varphi(p^n)+1}\equiv a+p^n(\mathrm{mod}\,2p^n)$. This means that $(a+p^n)$generates a subgroup of order $\varphi(2p^n)$, and since $(a^l+p^n)(1+p^n)=a^l+p^n$, for each $1\leq l\leq\varphi(p^n)$ then $(1+p^n)$ is the identity element of the group generated by a+p^n which is cyclic (the group G_{1+p^n}) . Note that $\{p^n\}$ is a subgroup of Z_{2p^n}. Since the maximal subgroups are cyclic, Z_{2p^n} is a Smarandache cyclic semigroup.

Proposition 1.6.

$(Z_{p^nq^m},.)$, where p,q are odd primes, is a non cyclic Smarandache semigroup.

Proof: Since the congruence $x^2=1\,(\mathrm{mod}\,p^nq^m)$ has exactly 4 solutions(Kenneth,2004,p.152), the set $S=\{x;x^2=1\}$ contains four elements and by Lemma 1.3, S is a non cyclic subgroup of $Z_{p^nq^m}$.Then $Z_{p^nq^m}$ is not a Smarandache cyclic semigroups.

The direct product of two Smarandache cyclic semigroups need not be a Smarandache cyclic semigroup in general.

Example 1.7.

($\mathbb{Z}_5$,.) and ($\mathbb{Z}_7$,.) are Smarandache cyclic semigroups but $Z_5\times Z_7$ is not a Smarandache cyclic semigroup since $G=\{(x,y):0\neq x\in Z_5\ and\ 0\neq y\in Z_7\}$ is a non cyclic group.

Now, we give a condition under which the direct product of a finite number of Smarandache cyclic semigroups is Smarandache cyclic.

Theorem 1.8.

Let $(S_i,.)$, $i=1...n$ be finite Smarandache cyclic semigroups , such that for any maximal subgroups $G_1, G_2,...,G_n$ of $S_1, S_2,...,S_n$ respectively, order(G_i) and order(G_j) are relatively prime for each $i \neq j$. Then $S_1 \times S_2 \times ... \times S_n$ is a Smarandache cyclic semigroup.

Proof: Let G_i be a maximal subgroup of S_i for $1 \leq i \leq n$. Since G_i is a cyclic group, $G_i \cong Z_{p_i}$, $i=1,2,..n$, and since $(p_i,p_j)=1$ for each i,j , then $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n} \cong \mathbb{Z}_{p_1 p_2 \cdots p_n}$ which is a cyclic group and $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n} \cong G_1 \times G_2 \times \cdots \times G_n$ which is a subgroup of $S_1 \times S_2 \times ... \times S_n$, then $S_1 \times S_2 \times ... \times S_n$ is a Smarandache cyclic semigroup.

Proposition 1.9.

$S_{n\times n} = \{(a_{ij}), a_{ij} \in \mathbb{Z}_{2^k},\ k \geq 3\}$ under matrix multiplication is not a Smarandache cyclic semigroup.

Proof: Since

$$\left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & 1 & & \vdots \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \begin{pmatrix} 2^{k-1}-1 & 0 & \cdots & 0 \\ \vdots & 2^{k-1}-1 & & \vdots \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 2^{k-1}-1 \end{pmatrix}, \begin{pmatrix} 2^{k-1}+1 & 0 & \cdots & 0 \\ \vdots & 2^{k-1}+1 & & \vdots \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 2^{k-1}+1 \end{pmatrix}, \begin{pmatrix} 2^{k}-1 & 0 & \cdots & 0 \\ \vdots & 2^{k}-1 & & \vdots \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 2^{k}-1 \end{pmatrix} \right\}$$

is a non cyclic subgroup of $S_{n\times n}$, then $S_{n\times n}$ is not a Smarandache cyclic semigroup.

Theorem 1.10.

Consider the multiplicative semigroup *($\mathbb{Z}_n G$,.) of the group ring $\mathbb{Z}_n G$, $n \geq 3$,* and G is a cyclic group of order m . Then

1- If $n=2^k$ for some $k > 2$, then the Smarandache semigroup *($\mathbb{Z}_n G$,.)* is not cyclic

2- If m is an even number then the Smarandache semigroup ($\mathbb{Z}_n G$,.) is not cyclic.

Proof: 1- By Proposition 1.4, ($\mathbb{Z}_{2^k}$,.) has a non cyclic subgroup which is a subgroup of ($\mathbb{Z}_{2^k} G$,.).

2- Suppose G is generated by g. Since m is even, $g^{\frac{m}{2}} \in \mathbb{Z}_n G$ and(n-1) $g^{\frac{m}{2}} \in \mathbb{Z}_n G$. Moreover $\left(g^{\frac{m}{2}}\right)^2 = 1$, ((n-1) $g^{\frac{m}{2}})^2=1$, so $\{1, g^{\frac{m}{2}}, (n-1) g^{\frac{m}{2}}, n-1\}$ is a non cyclic subgroup of ($\mathbb{Z}_n G$,.).

S2: Smarandache p-Sylow subgroups

In this Section we study Smarandache p- Sylow subgroups of a semigroup, and we find the number of p- Sylow subgroups in $(\mathbb{Z}_{2^n},.)$.

Theorem 2.1.

The semigroup $(\mathbb{Z}_{2^n},.)$ *n>2*, has three Smarandache 2-Sylow subgroups of order two.

Proof: The congruence $x^2 \equiv 1\,(\bmod 2^n)$ has exactly 4 solutions (Kenneth,(2004),p.152), namely 1, $2^n - 1$, $2^{n-1}+1$, $2^{n-1}-1$. Then $A_1 = \{1, 2^n - 1\}, A_2 = \{1, 2^{n-1}-1\}$ and $A_3 = \{1, 2^{n-1}+1\}$ are Smarandache 2-Sylow subgroups of order two. Hence $\mathbb{Z}_{2^n}$ has three Smarandache 2-Sylow subgroups of order 2.

Theorem 2.2.

The semigroup $(\mathbb{Z}_{2^n},.)$, n>3 has three Smarandache 2-Sylow subgroups of order four.

Proof: Since $\mathbb{Z}_{2^n}$ has four elements each one is its own inverse (Kenneth, 2004) namely, 1, $2^n - 1$, $2^{n-1}+1$, $2^{n-1}-1$.Then

$A_1 = \{1, 2^n - 1, 2^{n-1}+1, 2^{n-1}-1\}$ is a Smarandache 2-Sylow subgroup of order 4. Since only one of the four solutions which is $2^{n-1}+1$ is a solution of the congruence $y \equiv 1\ (\bmod 8)$, then the congruence $x^2 \equiv 2^{n-1}+1\ (\bmod 2^n)$ has four solutions (Edmund, 1966) they are

$$x_1 = 2^{n-2} - 1,\ \ x_2 = 2^n - 2^{n-2} + 1,\ \ x_3 = 2^{n-2} + 1$$

$$\textit{and}\ \ x_4 = 2^n - 2^{n-2} - 1.$$

Now $x_1^2 = (2^{n-2}-1)^2 = 2^{n-1} + 1 \bmod (2^n)$.

$x_1^3 = (2^{n-1}+1)(2^{n-2}-1) = x_4 \bmod (2^{n)})$,

and $x_1^4 = (2^{n-1}+1)^2 = 1 \bmod (2^n)$. Hence $A_2 = \{1, x_1, x_4, 2^{n-1}+1\}$ is a Smarandache 2-Sylow subgroup of order 4 generated by x_4 and also generated by x_1. Let us compute x_2^2, x_2^3, x_2^4,

$$x_2^2 = 2^{n\ -1} + 1 \bmod (2^n),$$

$$x_2^3 = 2^{2n\ -1} - 2^{2n\ -3} + 2^{n-\ 1} + 2^n - \ 2^{n\ 2} + 1 = x_3 \bmod (2^n),$$

$x_2^4 = (2^{n-1}+1)^2 = 1 \bmod (2^n)$. Hence $A_3 = \{1, x_2, x_3, 2^{n-1}+1\}$ is a Smarandache 2-Sylow subgroup of order 4 generated by x_2 and also it is generated by x_3. Hence $\mathbb{Z}_{2^n}$ has three Smarandache 2-Sylow subgroups of order four namely A_1, A_2 and A_3.

Theorem 2.3.

The semigroup $(\mathbb{Z}_{2^n},.)$, n>4 has three Smarandache 2-Sylow subgroups of order 8.

Proof: Similar to the proof of Theorem 2.2.

Theorem 2.4.

The semigroup $(\mathbb{Z}_{2^n},.)$, n>5 has three Smarandache 2-Sylow subgroups of order 16.

Proof: As we have seen in the last theorem that $\mathbb{Z}_{2^n}$ has eight elements of order 8 which are

$y_1 = 2^{n-3} + 1,\ y_2 = 2^{n-2} - 2^{n-3} + 1,\ y_3 = 2^{n-1} + 2^{n-3} + 1,\ y_4 = 2^n - 2^{n-3} - 1$

$z_1 = 2^{n-3} - 1,\ z_2 = 2^{n-1} - 2^{n-3} + 1,\ z_3 = 2^n - 2^{n-3} + 1,\ and\ z_4 = 2^{n-1} + 2^{n-3} - 1.$

Since $y_1 \equiv 1\ (\mathrm{mod} 8), y_3 \equiv 1\ (\mathrm{mod} 8), z_2 \equiv 1\ (\mathrm{mod} 8)$ and $z_3 \equiv 1\ (\mathrm{mod} 8)$. As before each of the following congruence has four solutions

$$x^2 = y_1\ (\mathrm{mod} 2^n) \qquad (1)$$

$$x^2 = y_3\ (\mathrm{mod} 2^n) \qquad (2)$$

$$x^2 = z_2\ (\mathrm{mod} 2^n) \qquad (3)$$

$$x^2 = z_3\ (\mathrm{mod} 2^n). \qquad (4)$$

So there are 16 elements of $\mathbb{Z}_{2^n}$ of order 16 which are

$A_1 = 2^{n-4} - 1,\ A_2 = 2^n - 2^{n-4} + 1,\ A_3 = 2^{n-1} + 2^{n-4} - 1,\ A_4 = 2^{n-1} - 2^{n-4} + 1$

$B_1 = 2^{n-2} - 2^{n-4} + 1,\ B_2 = 2^{n-2} + 2^{n-4} - 1,\ B_3 = 2^n - 2^{n-2} - 2^{n-4} + 1,$

$B_4 = 2^n - 2^{n-3} - 2^{n-4} - 1, C_1 = 2^{n-4} + 1,\ C_2 = 2^{n-1} + 2^{n-4} + 1, C_3 = 2^n - 2^{n-4} - 1,$

$C_4 = 2^{n-1} - 2^{n-4} - 1, D_1 = 2^{n-2} - 2^{n-4} - 1,\ D_2 = 2^{n-2} + 2^{n-4} + 1$

$D_3 = 2^n - 2^{n-2} - 2^{n-4} - 1,\ and\ D_4 = 2^n - 2^{n-3} - 2^{n-4} + 1.$ Then $E_1=\{C_1, y_3, B_3, x_1, D_2, z_3, A_2, w_1, C_2, y_1, x_1, B_1, D_4, z_4, A_4, 1\}$

where $w_1=2^{n-1}+1$, is a cyclic group generated by any one of the elements $C_1, B_3, D_2, A_2, C_2, B_1, D_4$, and A_4. Hence E_1 is a Smarandache 2-Sylow subgroup of order 16. $E_2=\{A_1, z_2, D_3, x_2, B_2, C_3, y_1, w_1, A_3, z_3, D_1, x_1, B_4, y_3, C_4, 1\}$ is a cyclic group of order 16 generated by any one of elements $A_1, D_3, B_2, C_3, A_3, D_1, B_4$, and C_4. Since by the last theorem

$\{y_1, x_1, z_2, w_1, y_3, x_2, z_3, 1\}$ and $\{y_2, x_1, z_1, w_1, y_4, x_2, z_4, 1\}$ and $A_3 = \{x_1, x_4, w_1, 1, x_2, x_3, x_1x_2, x_1x_3\}$ are subgroups of order 8 then

$E_3 = \{y_1, x_1, z_2, w_1, y_3, x_2, z_3, 1, y_2, z_1, y_4, z_4, y_1y_2, y_1z_1, y_1y_4, y_1z_4\} =$

$\{y_1, x_1, z_2, w_1, y_3, x_2, z_3, 1, x_3, x_4, x_1x_3, x_1x_4, y_1x_3, y_1x_4, y_1x_1x_3, y_1x_1x_4\} =$

$= \{y_2, x_1, z_1, w_1, y_4, x_2, z_4, 1, x_3, x_4, x_1x_3, x_1x_4, y_2x_3, y_2x_4, y_2x_1x_3, y_2x_1x_4\},$

Is a Smarandache 2- Sylow subgroup of order 16. Then $\mathbb{Z}_2{}^n$ has three Smarandache 2- Sylow subgroups of order 16.

Combining the previous theorems, we get the following result.

Theorem 2.5.

$(\mathbb{Z}_2{}^n,.)$ n>1, has (3n-5) Smarandache 2- Sylow subgroups

Proof: It is well known that $\mathbb{Z}^*_m$, the set of all invertible elements in $\mathbb{Z}_m$, the ring of the integer modulo m contains φ (m) elements, so $\mathbb{Z}_2{}^n$ has $\varphi(2^n) = 2^{n-1}$ invertible elements. Hence the semigroup $(\mathbb{Z}_2{}^n,.)$ Contains a subgroup of order 2^n-1 which is the largest subgroup with 1 as its identity namely G_1. By Theorem 2.1 for large n, $\mathbb{Z}_2{}^n$ has 3-Sylow subgroup of order 2 and by Theorems 2.2, 2.3 $(\mathbb{Z}_2{}^n,.)$ Has three subgroup of order 8 and three subgroup of order 16. Continuing in this manner we get that $\mathbb{Z}_2{}^n$ contains three subgroup of order 2^k for each $1 \le k \le n-2$.Hence the number of Sylow subgroup equal to $3(n-2)+1=3n-5$.

Example 2.6.

The Smarandache semigroup $(\mathbb{Z}_{64},.)$, has the following 2-Sylow subgroups, $\mathcal{A}_1 = \{1,63\}, \mathcal{A}_2 = \{1,31\}, \mathcal{A}_3 = \{1,33\}, \mathcal{A}_4 = \{1,63,31,33\}$, of order 2.
It has three Smarandache 2- Sylow subgroups of order 4, three Smarandache 2- Sylow subgroups of order 8, three Smarandache 2- Sylow subgroups of order 16 and one Smarandache 2-Sylow subgroup of order 32.

Theorem 2.7.

If $k \mid \varphi(2p^n)$, then $\mathbb{Z}_{2p^n}$ has two cyclic subgroups of order k.

Proof: Suppose $k \mid \varphi(2p^n)$. By Theorem 1.6, $\mathbb{Z}_{2p^n}$ has two maximal Subgroups of order $\varphi(2p^n)$ and since $k \mid \varphi(2p^n)$, each maximal subgroup has exactly one cyclic subgroup of order k (Neal & Thomas, 1977), then $\mathbb{Z}_{2p^n}$ has two cyclic subgroups of order k.

Corollary 2.8.

If $k^m \mid \varphi(2p^n)$ where k, p are prime numbers, then $\mathbb{Z}_{2p^n}$ has 2m Smarandache k-Sylow subgroups.

S3. Smarandache ideals and Smarandache normal subgroups

A non empty subset T of a semigroup S is a left ideal of S if s∈S, t∈*T* imply st∈T, T is a right ideal if s∈S, t∈T imply ts∈T, T is a two-sided ideal if it is both a left and right ideal(Mario, 1973, p.5). In this section we study Smarandache normal subgroups and we introduce the concepts of Smarandache ideal and Smarandache prime ideal of a semigroup and discuss the relation between Smarandache ideals and Smarandache normal subgroups.

Definition 3.1.

Let S be a semigroup and I an ideal of S. Then I is said to be a Smarandache ideal of S if I contains a proper subset which is a group.

Clearly every Smarandache ideal of a semigroup is an ideal of the semigroup but the converse need not be true, for example, ($\mathbb{Z}$,.) is a semigroup and I=3$\mathbb{Z}$ is an ideal of $\mathbb{Z}$ but not a Smarandache ideal because no subset of I is a subgroup.

Remark 3.2.

If I_1 and I_2 are Smarandache ideals of the semigroup S, then $I_1 \cap I_2$ need not be a Smarandache ideal, for example in($\mathbb{Z}_{20}$,.) take I_1= {0,2,4,6,8,10,12,14,16,18} and I_2={ 0,5, 10, 15}. I_1 and I_2 are Smarandache ideals but $I_1 \cap I_2$= {0, 10} is an ideal but not a Smarandache ideal of ($\mathbb{Z}_{20}$,.)

Theorem 3.3.

Let S be a Smarandache semigroup and I is a Smarandache ideal of S. Then I contain a maximal subgroup of S.

Proof: Let A be a subgroup of S with identity e. Then G_e is the maximal subgroup of S with e as its identity. Clearly A is a subgroup of G_e. If $G_e \not\subset I$, then there exists $x \in G_e$, $x \notin I$. Since I is an ideal, hence $x = x.e \in I$, contradiction. Therefore $G_e \subseteq I$ and I contains a maximal subgroup of S.

Definition 3.4.

Let S be a semigroup. A Smarandache ideal I of S is a Smarandache prime ideal if it is a prime ideal of S.

Example 3.5.

I= {2, 4, 6, 8, 10, 12, 14, 16, 18, 0} is a Smarandache prime ideal of the multiplicative semigroup Z_{20}.

Note that if *M* be a Smarandache maximal ideal of a semigroup S with identity, then *M* is a Smarandache prime ideal.

Proposition 3.6.

Let $S_1, S_2, \ldots, S_n$ be Smarandache semigroups, I_i be an ideal of S_i for each i. Then $I_1 \times I_2 \times \ldots \times I_n$ is a Smarandache ideal of $S_1 \times S_2 \times \ldots \times S_n$.

Proof: Suppose that I_i is a Smarandache ideal of S_i, we will show that $I_1 \times I_2 \times \ldots \times I_n$ is a Smarandache ideal of $S_1 \times S_2 \times \ldots \times S_n$. Let $(a_1, a_2, .. a_n)$ be an element of $I_1 \times I_2 \times \ldots \times I_n$ and $(b_1, b_2, .. b_n)$ be an element in $S_1 \times S_2 \times \ldots \times S_n$ then $(a_1, a_2, .. a_n).(b_1, b_2, .. b_n) = (a_1.b_1, a_2.b_2, \ldots, a_n.b_n) \in I_1 \times I_2 \times \ldots \times I_n$. Since I_i is an ideal, $a_i.b_i \in I_i$ for each $1 \le i \le n$. Hence $I_1 \times I_2 \times \ldots \times I_n$ is an ideal of $S_1 \times S_2 \times \ldots \times S_n$. Let A_i be a subgroup of I_i for each i, then (A_1, *A2... An*) Is a subgroup of $I_1 \times I_2 \times \ldots \times I_n$. Then $I_1 \times I_2 \times \ldots \times I_n$ is a Smarandache ideal of the semigroup $S_1 \times S_2 \times \ldots \times S_n$.

Definition 3.7 (Mario, 1973**).**

An element 0 of a semigroup S (if exists) called the zero of S if x0=0x=0 for each $x \in S$.

Definition 3.8(Kandasamy**,** 2002**)**.

A subgroup A of a Smarandache semigroup S is called a Smarandache normal subgroup of S if $xA \subseteq A$ and $Ax \subseteq A$ or xA= {0} and Ax= {0} for all $x \in S$ (0 is the zero of S)

Theorem 3.9.

Let S be a Smarandache semigroup with identity 1. If 1 is the identity of all subgroups of S, then S has no Smarandache normal subgroup

Proof: Suppose that A is a proper subgroup of S and $1 \in A$, let $0 \neq x \in S \backslash A$. Then $0 \neq x.1 = x \notin A$, which implies $xA \not\subset A$ and $xA \neq \{0\}$. Hence A is not a Smarandache normal subgroup of S.

Theorem 3.10.

Let S be a Smarandache semigroup. If A is a Smarandache normal subgroup of S, then A is a maximal subgroup of S.

Proof: Suppose A is a Smarandache normal subgroup of S contained in a subgroup $A' \neq S$ i.e $A \subset A'$.Then there is an element $x \in A' \backslash A$. This implies $0 \neq x = x.e \notin A$ where e is the identity of A, thus $xA \not\subset A$ and $xA \neq \{0\}$ contradiction.

Theorem 3.11.

Let S be a Smarandache semigroup with 0, and A be Smarandache normal subgroup of S. Then $A \cup \{0\}$ is a Smarandache ideal of S.

Proof: Since $xA\subseteq A$ or $xA=\{0\}$ for $x\in S$, then clearly $A\cup\{0\}$ is an ideal of S and A is a subgroup of $A\cup\{0\}$. There fore $A\cup\{0\}$ is a Smarandache ideal of S.

The converse of the last theorem need not be true in general for example, $I=\{2,4,6,8,10,12,14,16,18,0\}$ is a Smarandache ideal of $(Z_{20},.)$ but not a Smarandache normal subgroup.

Theorem 3.12.

The Smarandache semigroup $(\mathbb{Z}_{2p^n},.)$, has only one Smarandache normal subgroup which is trivial.

Proof: We show that no non trivial subgroup is normal. We saw (Theorem 1.5) that $\mathbb{Z}_{2p^n}$ has two maximal subgroups one of them is generated by a primitive root a of $2p^n$ and the other generated by $a+p^n$, and both of them are of order $\varphi\ (2p^n)\ =p^{n-1}(p-1)$. The subgroup generated by a cannot be normal, since it contains 1. It remains to prove that the subgroup generated by $a+p^n$ is not normal. Remember that $(1+p^n)$ is the identity of this Subgroup which usually denoted by G_{p^n+1}, and it is the maximal subgroup having $1+p^n$ as its identity. We claim that $2\in G_{p^n+1}$. First $2(1+p^n)=2\ (mod\ 2p^n)$. Next consider the congruence $2x=p^n+1\ (mod\ 2p^n)$, which has exactly two solutions (Edmund, 1966, p.62). So $2\in G_{p^n+1}$. Since $p\ (p^n+1)=p^n+p\ (mod2p^n)\neq p\ (mod\ 2p^n)$ hence $p\notin G_{p^n+1}$ moreover $2p\notin G_{p^n+1}$ and $2p\neq 0$, hence G_{p^n+1} is not a Smarandache normal subgroup of $\mathbb{Z}_{2p^n}$. So no non trivial subgroup is Smarandache normal subgroup.

Theorem 3.13.

$(\mathbb{Z}_{pq^n},.)$ p,q are odd prime numbers is a Smarandache semigroup which has a nontrivial Smarandache normal subgroup.

Proof: Let $S_1=\{q^n, 2qn\ldots (p-1)\ qn\}$. We claim that S_1 is a Smarandache normal subgroup. Its well known that $\mathbb{Z}_{pq^n}\cong\mathbb{Z}_p\times\mathbb{Z}_{q^n}$ as rings so $\mathbb{Z}_{pq^n}$ has a subring isomorphic to $\mathbb{Z}_p$, that is $F_1=\{0,q^n,2q^n,\ldots,(p-1)q^n\}$is a field with addition and multiplication mod pq^n. Hence S_1 is a group under multiplication. It remains to show that S_1 is a normal subgroup of $\mathbb{Z}_{pq^n}$. Let $x\notin S_1$. If $x=lq$ then $lqa=0$ for each $a\in S_1$. If $x\neq lp$ and $0<x<p$, then $xq^n\in S_1$. If $x\neq lp$ and $x>p$, then by Euclidean Algorithm $x=sp+r$ $0\leq r<p$, thus $xq^n=(sp+r)q^n=spq^n+q^n\in S_1$. Hence $xS_1\subseteq S_1$ or $xS_1=0$ Finaly if $x\in S_1$ then $xS_1\subseteq S_1$. This means that S_1 is normal. Similarly $\mathbb{Z}_{pq^n}$ has a subring isomorphic to $\mathbb{Z}_{q^n}$ which is $T=\{0,p,2p,\ldots,(q^n-1)p\}$ and T has a subfield namely $F_2=\{0,p,2p,\ldots(q-1)p,(q+1),\ldots(2q-1)p,(2q+1)p,\ldots(q^n-1)p\}$, then $S_2=\{p,2p,\ldots(q-1)p,(q+1),\ldots(2q-1)p,(2q+1)p,\ldots(q^n-1)p\}$ with multiplication is a group which is not a normal subgroup, since $q\in\mathbb{Z}_{pq^n}$, but $pq\neq 0$ and $pq\notin S_2$. There are three maximal subgroups S_1, $S2$ and S_3 where $S_3=\{a: (a,$

pq^n) =1}, S_1 is Smarandache normal subgroup, but S_2, S3 are not Smarandache normal subgroups.

Theorem 3.14.

Let n= $p_1p_2\ldots p_n$, where p_i are prime numbers. Then the semigroup $\mathbb{Z}_n$ has at least n Smarandache normal subgroup.

Proof: For each $1 \le j \le n, \mathbb{Z}_n \cong \mathbb{Z}_{p_1\cdots p_{j-1}p_{j+1}\cdots p_n} \times \mathbb{Z}_{p_j}$ as rings and $\mathbb{Z}_{p_1\cdots p_{j-1}p_{j+1}\cdots p_n} \cong \mathbb{Z}_{p_1\cdots p_{j-1}p_{j+1}\cdots p_n} \times \{0\}$ which is a subring of $\mathbb{Z}_{p_1\cdots p_n}$. put $k = p_i \cdots p_{j-1}p_{j+1} \cdots p_n$. Then $(\{0,k,2k,\cdots,(p_j-1)k\},+,.\} \cong (\mathbb{Z}_{p_j},+_{p_j},\cdot_{p_j})$ which is a field. Hence $s_j = \{k, 2k, \cdots, (p_j-1)k\}$ is a group under multiplication which is a subgroup of the semigroup ($\mathbb{Z}_n$,$\cdot_n$). Now if $x \in \mathbb{Z}_n$, and $x=tp_j$, $0<t<n$ then $xk=0$ so $xS=\{0\}$. If $0<x<p_j-1$, then $xS \subseteq S$, otherwise $x=tp_j+r$, $0<r<p$, $xk= xtp_j+rx \in S$. Hence S is a Smarandache normal subgroup. Then $\mathbb{Z}_n$ has at least n Smarandache normal subgroups.

References

- David, S. D. and Richard, M. F., (2004): Abstract Algebra, John Wiley and Sons, New York, 932p.
- Edmund, L., (1966): Elementary Number Theory, Chelsea Publishing Company, New York, 256p.
- Kandasamy,V.W., (2002): Smarandache Semigroups, American Research press Rehoboth, 93p.
- Kenneth, H.R., (2004): Elementary Number Theory and its Applications, ddison Wesley, Reading, 725p.
- Mario,p., (1973): Introduction to Semigroups, Bell and Howell Company, Ohio, 198p.
- Neal, H.M. and Thomas, R. B., (1977): Algebra: Groups, Rings and Other Topics. Allyn and Bacon, Boston, 657p.
- Raual, P., (1998): Smarandache Algebraic Structures, Bull of pure and applied Sciences, Delhi, Vol. 17E, No.1, pp. 119-121.

حول شبه الزمر السمرنداشية

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الخلاصة

في هذا البحث درسنا بعض انواع شبه الزمر السمرنداشية و الزمرالجزئية السمرنداشية لشبه زمرة ، مثل شبه الزمرة السمراندشية الدائرية والزمر الجزئية p–سايلو السمرنداشية و الزمر الجزئية السمرنداشية الناظمية. بالاضافة الى ذلك عرضنا مفهوم مثالية سمرنداشية لشبه زمرة ودرسنا العلاقة بينها و بين الزمـــرة الجزئيـــة السمرنداشية الناظمية.