

Semi Regular Lattice Polyhedra

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Abstract

The aim of this paper is to explain the probability of the existence of five regular and thirteen semi regular polyhedra; and we indicate that among these thirteen geometrical figures there are only three lattice polyhedra. Also in this work we present a proof of the existence of three regular lattice polygons.

Introduction

In 1985 two mathematicians, Peter Hilton and Jean Pederson have done a research on the folding regular star polygons. They have taken benefit from a practical method which is, folding paper, for proving the number of existence of regular polygons (Hilton & Pederson, 1985), and the way they have used is similar to the way which we used. In 1988, Bokowski and Wills described some fundamental ideas in the study of regular maps and their polyhedral realizations in the Euclidean 3-space (Bokowski & Wills, 1988). In collecting information, we faced some problems and the greatest problem seems to be the shortage of sources. Lattice's subject includes two aspects, geometric and algebraic. In our district we can find those sources which have been dealt with algebraic aspect. The alert reader will have surmised by now that we are prepared to go along way with this topic. But we restrain ourselves! We will focus our attention, and organize this article, to bring out the special features of this subject.

Definition [1]. (Diested,2005).

A polygon is a simple closed curve which comes in to existence by union of some intersecting straight parts in which each two parts cut each other, and it is regular if it is both equilateral and equiangular. The polygon is said to be convex if we take any gon, the polygon be in one side of it.

Definition [2]. (Fraleigh, 2003 ; Scott, 1987)

A point (x, y, z) in coordinates 3- space R^3 is called a lattice point if

x, y and z are all integers, and so a polygon in R^3 is called lattice polygon if all its vertices are lattice point.

Definition [3]. (Diested,2005; Scott, 1987)

A polyhedra is a solid which comes to existence by union of some intersecting planes in which any two planes cut each other, and it is regular if its faces are congruent regular polygons; and each vertex has the same number of faces surrounding it. A semiregular polyhedra has regular polygons as faces and all vertices congruent, but they admit a variety of such polygons in one solid. A polyhedra in R^3 is called lattice if all its vertices are lattice points.

Some information about the main subject

Before talking about our main subjects it is necessary to have some information about:

- 1) Those regular polygons which can be drawn and those which can not be drawn.
- 2) The angle between two gones of a regular polygon.

The regular polygons which the number of their gons indicated bellow are those which can be drawn :

2×2^n , 3×2^n and 5×2^n $n = 0,1,2,\dots$. For instance 3, 4, 5, 6, Also

Gauss proved that if a prime number $P = 2^{2^n} + 1$ then P -gons can be drawn. The prime numbers which are in this form are 3, 5, 17, 257, 65537, And the regular 7, 9, 11, 13, 14, 19, 23,... gons can not be drawn.

To find the angle btween two gons of regular polygon, we divide regular n-gons in to traingles by drawing diagonals from one of the vertex and the polygon will divide into ($n-2$) traingles. Hence the mesure of each angle

between two gons is $\alpha = \frac{180 \times (n - 2)}{n}$ for example $n = 5$ so

$$\alpha = \frac{180 \times (5 - 2)}{5} = 108^\circ.$$

Theorem [4]. (Scott, 1987)

A covex lattice n-gon is equiangular if and only if $n = 4$ or 8 .

Theorem [5]. (Scott, 1987)

A regular n-gon can be embeded is the three dimentional integral lattice if and only if $n = 3, 4$ or 6 .

Regular and Semi Regular Lattice Polyhedra

In coordinate 3– space R^3 there are five regular and thirteen semi regular polyhedra. To prove the existence of five regular polyhedra let β be the sum of all angles around a vertex of a regular polyhedra and we shall use n to indicate the type of the faces meeting at a vertex and m is the number of all faces around a vertex . (it is clear that $m \geq 3$)

If $n=3$ then $\alpha = 60^0$ (α is the angle between any two gons of n -gon) and if $m=3$ then $\beta = 3 \times 60^0 = 180^0 < 360^0$ is acceptable case and other cases when $\beta < 360^0$ are also acceptable (which are $m= 4, 5$) and do not acceptable when $\beta \geq 360^0$ which are the cases $m > 5$.

If $n= 4$ then $\alpha = 90^0$ the only acceptable case is $m=3$ because $\beta = 270^0 < 360^0$.

If $n=5$ then $\alpha = 108^0$ here also the only acceptable case is $m=3$ because $\beta = 324^0 < 360^0$. For other choice of $n \geq 6$ the cases are not acceptable and hence there are only five polyhedra [figure 1].



Figure 1 (The five regular polyhedra)

Now we want to prove that there are thirteen semiregular polyhedra First we take the all possible arrangements(orders) of n_1 -gon, n_2 -gon, ..., n_r -gon around a vertex. We note that the only regular polygons which are candidates to participate in our proof are 3-gon, 4-gon, 5-gon, 6-gon, 8-gon, and 10-gon. Then $A=$ the number of all arrangements is given by:

For $m = 3$ we have $A = \binom{6}{3} + 6 \times \binom{5}{1} = 50$

For $m = 4$ we have $A = \binom{6}{4} + 2 \times 6 \times \binom{5}{1} + 6 \times \binom{5}{1} \times \binom{4}{1} = 195$

For $m = 5$ we have $A = \binom{6}{5} + 2 \times 6 \times \binom{5}{1} + 2 \times 6 \times \binom{5}{1} \times \binom{4}{1} + 6 \times \binom{5}{1} \times \binom{4}{1} \times \binom{3}{1} = 666$

If $m=3$ we shall use (n_1, n_2, n_3) to indicate the faces meeting at a vertex where $n_1, n_2, n_3 = 3, 4, 5, 6, 8, 10$ then the all accept possible arrangements which $\beta < 360^0$ are:

(n_1, n_2, n_3)	β	(n_1, n_2, n_3)	β
(3,3,4)	210 ^o	(3,8,8)	330 ^o
(3,3,5)	228 ^o	(3,8,10)	339 ^o
(3,3,6)	240 ^o	(3,10, 10)	348 ^o
(3,3,8)	255 ^o	(4,4,5)	288 ^o
(3,3,10)	264 ^o	(4,4,6)	300 ^o
(3,4,4)	240 ^o	(4,4,8)	315 ^o
(3,4,5)	258 ^o	(4,4,10)	324 ^o
(3,4,6)	270 ^o	(4,5,5)	306 ^o
(3,4,8)	285 ^o	(4,5,6)	318 ^o
(3,4,10)	294 ^o	(4,5,8)	333 ^o
(3,5,5)	276 ^o	(4,5,10)	342 ^o
(3,5,6)	288 ^o	(4,6,6)	330 ^o
(3,5,8)	303 ^o	(4,6,8)	345 ^o
(3,5,10)	312 ^o	(4,6,10)	354 ^o
(3,6,6)	300 ^o	(5,5,6)	336 ^o
(3,6,8)	315 ^o	(5,5,8)	351 ^o
(3,6,10)	324 ^o	(5,6,6)	348 ^o

"Collection 1"

By a similar method we shall use (n_1, n_2, n_3, n_4) to indicate the faces meeting at a vertex for $m=4$, where $n_1, n_2, n_3, n_4 = 3,4,5,6,8,10$, then the accept arrangements for which $\beta < 360^o$ are:

(n_1, n_2, n_3, n_4)	β	(n_1, n_2, n_3, n_4)	β
(3,3,3,4)	270 ^o	(3,4,3,6)	330 ^o
(3,3,3,5)	285 ^o	(3,4,3,8)	345 ^o
(3,3,3,6)	300 ^o	(3,4,3, 10)	354 ^o
(3,3,3,8)	315 ^o	(3,4,4,4)	330 ^o
(3,3,3,10)	324 ^o	(3,5,3,5)	336 ^o
(3,4,3,4)	300 ^o	(3,5,3,6)	348 ^o
(3,4,3,5)	315 ^o	(4,3,4,5)	348 ^o

"Collection 2"

For $m=5$ the accept arrangements for which $\beta < 360^{\circ}$ are:

$(n_1, n_2, n_3, n_4, n_5)$	β
(3,3,3,3,4)	330°
(3,3,3,3,5)	348°

"Collection 3"

In collection (1) we refuse all cases which one of n_1, n_2 or n_3 equal to 3 and the other two are distinct with each other. [to refuse this cases we use a practical methode]. To explain this wording suppose that $n_1=3$ and $n_1 \neq n_3$ then the vertex A of that polyhedra which construct by n_1, n_2, n_3 has three polygons which at least two of them are distinct, and they are in the form 3-gon, n_2 -gon and n_3 -gon. [Figure 2]

To complete this polyhedra the vertex C needs a polygon with n_2 -gons. After drawing n_2 -gon for vertex C the vertex B takes two n_2 -gons, which is contradiction.

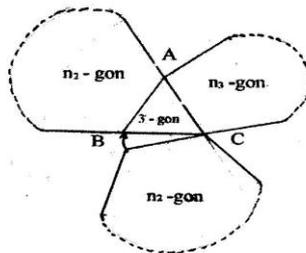


Figure 2

We note that some of the other cases of collection (1) can not construct a semiregular polyhedra also. For example, we take the cases (4, 5, 4) and (4, 5, 8) see [Figure 3] and [Figure 4].

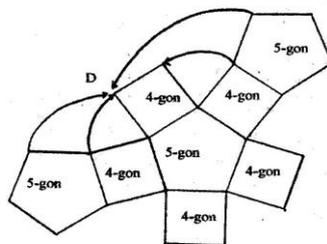


Figure 3

The vertex D takes the polygons in the form (5, 4, 5) which is contradiction.

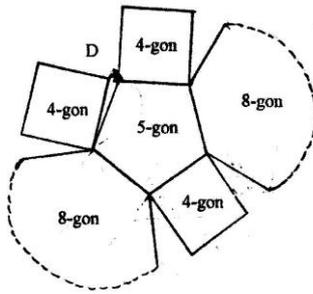


Figure 4

In this case the vertex D takes the polygons in the form (4, 5, 4,) which is contradiction In our list of collection (1) only (3, 6, 6), (3, 10, 10) , (4, 6, 6) , (4, 6, 8) , (4, 6,10) and (5, 6, 6) occur as a polyhedron in coordinate 3-space R^3 . In collection 2 also there are these contradicitons, and only in the cases (3, 4, 3, 4), (3, 5, 3, 5), (4, 3, 4, 5) and (3, 4, 4,4) do not occur this contradiction. If we choose a case from collection (2) arbitrary as (3, 4, 3, 5) see [Figure 5]

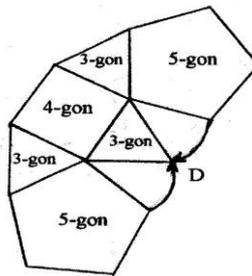


Figure 5

Here we have a conrtadiction also since the vertex D takes two 5-gon at once. In collection (3) there is no contradiction in surrounding polygons around any vertex.

Then we obtain that there are only seven semiregular polyhedra in collection (1) and four semiregular polyhedra in collection (2) and two semiregular polyhedra in collection (3). Hence there are thirteen semiregular polyhedra, which we shall use (n_1, n_2, \dots, n_r) where $n_1, n_2, \dots, n_r = 3, 4, 5, 6, 8, 10$ to indicate the faces meeting at a vertex are successively an n_1 -gon, an n_2 -gon, ..., and n_r -gon as we cycle a round a vertex. The thirteen semiregular polyhedra are shown in the Figure 6.

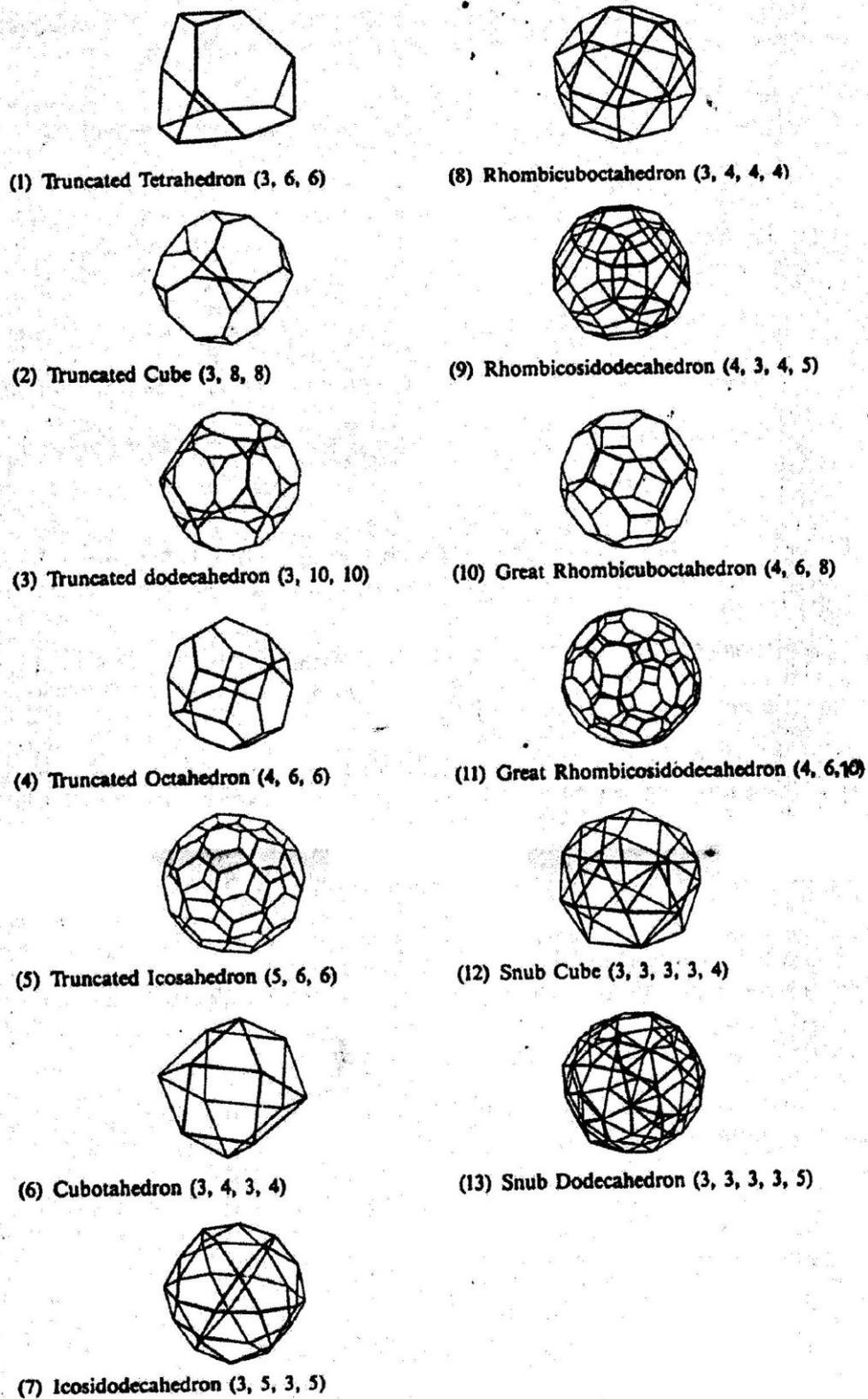


Figure 6 (The thirteen semiregular polyhedra)

Theorem [6]. (Scott, 1987)

Let there be thirteen semiregular polyhedra, then only the truncated tetrahedron, the truncated octahedron and cuboctahedra occur as lattice polyhedron.

Proof:

Since every face of a lattice polyhedron is a lattice polygon , by theorem (5) the only semiregular polyhedra in our list which are candidates to be lattice polyhedra are those with triangles, squares or hexagons as faces; that is, numbers (1), (4), (6), (8) and (12) [See figure 6].

Number (8) the rhombicuboctahedron is quickly excluded sicne the vertices along an edge of one of the encircling bands of squares determine a regular octagon by theorem (8), these points can not be embedded in the integral lattice.

Let us now show that number (12), the snub cube , can not occur as a lattice polyhedron. To do this we first observe that the volume of a tetrahedron with vertices $(X_i , Y_i , Z_i) (1 \leq i \leq 4)$ is given by 1/6 the modulus of :

$$\begin{vmatrix} X_1 & Y_1 & Z_1 & 1 \\ X_2 & Y_2 & Z_2 & 1 \\ X_3 & Y_3 & Z_3 & 1 \\ X_4 & Y_4 & Z_4 & 1 \end{vmatrix}$$

We see from this , that any lattice tetrahedra has rational volume. We show that if the snub cube can occur as a lattice polyhedra, then there exists a lattice tetrahedron T with irrational volume.

By symmetry, the six square faces of the snub cube lie one on each face of a cube ξ with the gaps filled by bands of eqilateral triangles. Experiment shows that this is impossible , if the sides of the squares are parallel to the edges or diagonals of the faces of ξ . From this observation and the regular nature of the snub cube, we deduce that its squares lie obliquely, centrally, and in the exactly congruent positions in the faces of the cube ξ [see figure 7]. Suppose ξ has side length 2a. Choose the origin at the centre O of the cube ξ , and let the axes be drawn parallel to the edges of ξ .

Now the vertices of the sqare face ABCD lying in the face $X = a$ of ξ .Will be represented for some b , c by $A(a, b, -c)$, $B(a, c, b)$, $C(a, -b, c)$, $D(a, -c, -b)$ [The coordinates of the other vertices of the snub cube are obtained by suitable permutation and change of sign].

If E is the vertex $(b, -c, a)$ then equating the lengths BE and BC of the eqilateral triangle BCD gives $C^2 = a(2b-a) \dots (1)$

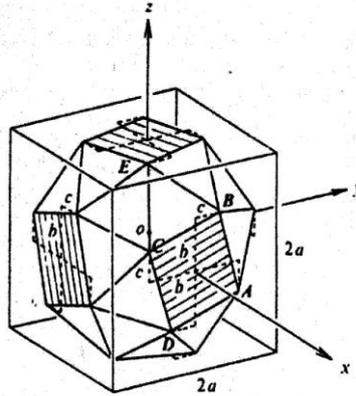


Figure 7

Similarly, equating the side lengths CE and BC gives $(a+b)C = a(a-b)$

Now substituting $C = a(a-b)/(a+b)$ in to (1) and setting :

$Y - 1 = b/a$, we obtain

$$Y^3 - 2y^2 + 2y - 2 = 0 \quad (2)$$

$$Y = 2(1+y^2)/(2+y^2) \quad (2a)$$

Consider now the tetrahedron T which has vertices A, B, C and the vertex $(-a,-b,c)$ of the snub cube. This tetrahedron has altitude $2a$, and its base the isosceles right – angled triangle ABC with AB and BC of equal length, say S. Let us assume that the snub cube can be placed as a lattice polyhedron. Then in particular, T will be a lattice tetrahedron. Since S gives the distance between two lattice points, we deduce (from Pythagora's theorem) that S^2 is an integer. Since the volume of T is $V(T)=2a(S^2/6)$ (3) and $V(T)$ is rational, we deduce that a is rational. Also by substitution $S^2=AB^2=2(b^2+c^2)$ from eq (3) we get

$$\begin{aligned} V(T) &= 2a(b^2+c^2)/3 \\ &= 2a(b^2+2ab-a^2)/3 \quad [\text{form (1)}] \\ &= 2a^3[(b/a)^2+2(b/a)-1]/3 \\ &= 2a^3(y^2-2)/3 \end{aligned}$$

Since $V(T)$ and a are rational, so is y^2 , and hence from (2a), so is y. But the polynomial in equation (2) is monic and has integer coefficients. Therefore, any rational root must be a divisor of the constant term -2 . since non of $\pm 1, \pm 2$ is a root of (2), we have a contradiction. This establishes that the snub cube can not occur as a lattice polyhedra. Finally, we must show that the three remaining semiregular polyhedra can occur as a lattice polyhedra. We observe that the truncated tetrahedron (1) is obtained by joining the points of trisection of the edges of the parent tetrahedron. Thus choosing the tetrahedra to have vertices $(0, 0, 0), (3, 3,$

0), (3, 0, 3) and (0, 3, 3) we obtain the truncated tetrahedron with every vertex at a lattice point. Similarly, the vertices of the truncated octahedron (3) are the points of trisection of the edges of the parent octahedra. Choosing the octahedron to have vertices $(\pm 3,0,0)$, $(0,\pm 3,0)$ and $(0,0, \pm 3)$ ensures that the truncated octahedron is a lattice polyhedron. The vertices of the cuboctahedra (6) are the midpoints of the edges of a cube. Hence choosing the cube to have vertices $(\pm 1,\pm 1,\pm 1)$ we obtain a corresponding lattice cuboctahedra. This completes the proof of Theorem [6].

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متعددة السطوح الشبكية شبه المنتظمة

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الخلاصة

الهدف من هذا البحث هو دراسة احتمالية وجود خمس متعددات سطوح منتظمة وثلاثة عشرة متعددات سطوح شبه المنتظمة وتبين انة ثلاثة فقط من هذه الاشكال الهندسية هي متعددات سطوح شبكية و بالاضافة الى ذلك برهنا وجود ثلاثة مضلعات شبكية منتظمة .