A Comparison between the Maximum Likelihood Method (MLM) and Least Square Method (LSM) To Estimate Parameters of Weibull-Gumbel Distribution by Using Simulation

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The Weibull-Gumbel distribution, which is resulting of the integration of two parameters Weibull Distribution, and Gumbel Distribution which where proposed distributions to expand the distribution of Weibull distribution by adding new parameters. In this research, a compares has been achieved between two methods of estimation to estimate the parameters of the proposed distribution by using simulation as a sampling technique for data from Weibull – Gumbel distribution with different samples sizes and parameters. The total standard deviations were used to compare accuracy of both estimation methods.

Keyword: *Weibull distribution, Gumbel distribution, Weibull-Gumbel distribution, Transformed-Transformer family*

(1)Introduction

In some applications the behavior probabilistic of the sample data are non-homogeneous and cannot use signal distribution to describe the random behavior of these phenomenon. Therefore, distributions that are combined from more than one single distribution have been used to describe this behavior. One way can be used to find new combined distributions is adding new parameters by integration, using the probability density function of the first distribution and the probability accumulative of second distribution. Increasing the number of parameters results in difficulty in estimating the parameters of the distributions. That is because the increase in number of parameters produces a system of nonlinear equations, which cannot be solved by using normal mathematical methods. Therefore, numerical methods are used; this requires skill in using numerical methods and knowledge in software. The maximum likelihood method and least square method are important methods of estimation of parameters, so, these methods where used to estimate parameters of proposed distribution, which is called Weibull Gamble distribution. nonlinear equations where solved using newton raphson method.

(2) Research Problem

The issue of finding accurate estimates is one of the problems of estimation. This problem increases as the number of parameters increases, and when estimation methods such as Sum of square method or maximum likelihood method are used, a system of nonlinear equations will result. The nonlinearity leads obtaining non-accurate estimations. The importance of this research is obtained by finding accurate parameters by using sum of square method and maximum likelihood method and comparing between the two methods.

(3)Objective Problem

This research aims to use simulation to compare the method of Sum of Squares whith maximum likelihood to estimate Weibull-Gamble distribution.

(4)Weibull-Gamble Distribution

Alzaatreh, Lee and Famoye[5,6] (2013) suggested a method for finding new probability distributions which is known as (T-X family) or (Transformed-Transformer family). This distribution is a development of other distributions by combining two of them. The distribution function (cdf) for (T-X family) is

$$F(x) = \int_0^{Q(x)} r(t)dt \qquad (1)$$

Where Q(x) take some forms as

$$Q(x) = log\left(\frac{G(x)}{1-G(x)}\right) \;\; Or \;\; Q(x) = -log(1-G(x))$$

Where G(x) is a (cdf) distribution function of (x) and r (t): is probability density function (pdf) of (t). The distribution function (cdf) of (T-X family) will be as follows

$$F(x) = R(Q(x)) \qquad (2)$$

While the probability density function of (t) is

$$f(x) = \frac{\partial R(Q(x))}{\partial(x)} \qquad (3)$$

Al-Aqtash et al. (2014) defined the four-parameter Gumbel-Weibull distribution (GWD) by using (T-X family) or (Transformed-Transformer family) method. They *combined Weibull distribution (W.D) with Gumbel distribution (G.D) considering one of extreme value distributions which the form of probability density function is*[1]

$$g(r) = \frac{1}{\sigma} e^{-\left(\frac{r-\mu}{\sigma}\right)} exp\left(e^{\left(\frac{r-\mu}{\sigma}\right)}\right) \qquad (4)$$

Where $-\infty < r, \mu < \infty; \quad \sigma > 0$

If G(x) is Weibull density function [4]*:*

$$G(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$$

Where $x; \alpha; \beta > 0$

Then, equation (1) will become

$$F(x) = \int_{-\infty}^{ln\left(\frac{1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}}{e^{-\left(\frac{x}{\beta}\right)^{\alpha}}}\right)} \frac{1}{\sigma} e^{\left(-\frac{(t-v)}{\sigma}\right)-e^{-\frac{(t-v)}{\sigma}}} dt \qquad (5)$$

Therefore, density function of new distribution, Weibull Gumbel distribution (WGD) will be as follows

$$F(x) = exp\left(\left(e^{\left(\frac{x}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}} - e^{\frac{\tau}{\sigma}}\right) \quad (6)$$

In addition, the (pdf) for Weibull Gumbel distribution (WGD) is:

$$f(x) = \frac{\alpha e^{\frac{\tau}{\sigma}}}{\beta\sigma}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{\left(\frac{x}{\beta}\right)^{\alpha}} exp\left(-e^{\frac{\tau}{\sigma}}\left(e^{\left(\frac{x}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}}\right)\left(e^{\left(\frac{x}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}-1} \quad (7)$$

(5)Some Properties of WGD

(5-1)Reliability function

Reliability[2] *is defined as the continuity and its ability to remain systems without malfunction or failure for period time. The Reliability function at time (t) is given by*

$$R(t) = pr(T > t) = 1 - exp\left(\left(e^{\left(\frac{x}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}} - e^{\frac{\tau}{\sigma}}\right) \qquad (8)$$

(5-2)Moments

The r^{th} *moment of WGD is*[5,6]*:*

$$\mu^{r} = \frac{\alpha e^{\frac{\tau}{\sigma}}}{\beta\sigma}\int_{-\infty}^{\infty} x^{r}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{\left(\frac{x}{\beta}\right)^{\alpha}} exp\left(-e^{\frac{\tau}{\sigma}}\left(e^{\left(\frac{x}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}}\right)\left(e^{\left(\frac{x}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}-1} dx \qquad (9)$$

(5-2-1)Expectation:

The expectation of WGD can be found by solving the following integration [5,6]*:*

$$\mu = \frac{\alpha e^{\frac{\tau}{\sigma}}}{\beta^{\alpha}\sigma}\int_{-\infty}^{\infty} x\, e^{\left(\frac{x}{\beta}\right)^{\alpha}} exp\left(-e^{\frac{\tau}{\sigma}}\left(e^{\left(\frac{x}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}}\right)\left(e^{\left(\frac{x}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}-1} dx \quad (10)$$

(5-2-2)Variance

The variance of WGD is [5,6]*:*

$$\sigma^2 = \frac{\alpha e^{\frac{\tau}{\sigma}}}{\beta^{\alpha}\sigma}\int_0^{\infty} x\, e^{\left(\frac{x}{\beta}\right)^{\alpha}} exp\left(-e^{\frac{\tau}{\sigma}}\left(e^{\left(\frac{x}{\beta}\right)^{\alpha}}-1\right)^{-\frac{1}{\sigma}}\right)\left(e^{\left(\frac{x}{\beta}\right)^{\alpha}}-1\right)^{-\frac{1}{\sigma}-1}dx - \left(\frac{\alpha e^{\frac{\tau}{\sigma}}}{\beta^{\alpha}\sigma}\int_0^{\infty} x^2\, e^{\left(\frac{x}{\beta}\right)^{\alpha}} exp\left(-e^{\frac{\tau}{\sigma}} e^{\left(\frac{x}{\beta}\right)^{\alpha}}-1)^{-\frac{1}{\sigma}}\right) e^{\left(\frac{x}{\beta}\right)^{\alpha}}-1)^{-\frac{1}{\sigma}-1}dx\right) \quad (11)$$

(6)Estimation of Parameters:

In this research, two methods of estimation were used to estimate WGD parameters. These methods are Maximum Likelihood Method (MLM) and Least Square Methods (LSM)

(6-1)Maximum Likelihood Method (MLM)

This method was suggested by Fisher in 1922, based on finding estimates for parameters by finding roots that maximize the maximum likelihood equation of sample data. To find WGD parameters by using (MLM), the log of maximum likelihood equation is found to be

$$\mathcal{L} = n\left(ln\,\alpha - \alpha\,ln\,\beta - ln\,\sigma + \frac{\mu}{\sigma}\right) + (\alpha-1)\sum_{i=1}^{n} ln(x_i) - \frac{1}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} - \left(1+\frac{1}{\sigma}\right)\sum_{i=1}^{n}\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}-1\right) + \frac{e^{\frac{\mu}{\sigma}}}{\sigma}\sum_{i=1}^{n} ln\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}-1\right) \quad (12)$$

By finding first derivative to eq (12) for parameters $(\alpha;\beta;\mu;\sigma)$ *we get*

$$\frac{\partial\mathcal{L}}{\partial\alpha} = n\left(\frac{1}{\alpha} - ln\,\beta\right) + \sum ln\,x_i + \sum\left(\frac{x_i}{\beta}\right)^{\alpha} ln\left(\frac{x_i}{\beta}\right) - \left(1+\frac{1}{\sigma}\right)\left(\sum\left(\frac{x_i}{\beta}\right)^{\alpha} ln\left(\frac{x_i}{\beta}\right) e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} \Big/ \left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}-1\right)\right) + \frac{e^{\frac{\mu}{\sigma}}}{\sigma}\left(\sum -\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}-1\right)^{-\frac{1}{\sigma}}\left(\frac{x_i}{\beta}\right)^{\alpha} ln\left(\frac{x_i}{\beta}\right) e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} \Big/ \left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}-1\right)\right) \quad (13)$$

$$\frac{\partial\mathcal{L}}{\partial\beta} = -\frac{n\alpha}{\beta} - \frac{\alpha}{\beta}\sum\left(\frac{x_i}{\beta}\right)^{\alpha} + \frac{\alpha\left(1+\frac{1}{\sigma}\right)}{\beta}\left(\sum\left(\frac{x_i}{\beta}\right)^{\alpha} e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} \Big/ \left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}-1\right)\right) - \frac{\alpha e^{\frac{\mu}{\sigma}}}{\sigma\beta}\left(\sum\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}-1\right)^{-\frac{1}{\sigma}}\left(\frac{x_i}{\beta}\right)^{\alpha} e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} \Big/ \left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}-1\right)\right) \quad (14)$$

$$\frac{\partial\mathcal{L}}{\partial\sigma} = -n\left(\frac{\mu}{\sigma^2} + \frac{1}{\sigma}\right) + \frac{1}{\sigma^2}\sum ln(e-1) - \frac{\mu e^{\frac{\mu}{\sigma}}}{\sigma^2}\sum\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}-1\right)^{-\frac{1}{\sigma}} + \frac{e^{\frac{\mu}{\sigma}}}{\sigma^2}\left(\sum\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}-1\right)^{-\frac{1}{\sigma}} ln\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}-1\right)\right) \quad (15)$$

$$\frac{\partial\mathcal{L}}{\partial\mu} = \frac{n}{\sigma} + \frac{e^{\frac{\mu}{\sigma}}}{\sigma}\left(\sum\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}-1\right)^{-\frac{1}{\sigma}}\right) \quad (16)$$

Equally, the equations (13-16) by zero and using Newton Raphson Method for solving nonlinear equations we will get estimate the parameters of WGD.

<u>(6-2)Least Square Method (LSM)</u>

It is one of the most important methods of estimation, based on minimizing the sum of square error for finding estimation for unknown parameters. This method based on finding correlate between two or more variables. To find parameters of WGD the steps below are followed

$$ols = \left(exp\left(-e^{\frac{\mu}{\sigma}}\left(e^{\left(\frac{x_i}{\beta}\right)} - 1\right)^{-\frac{1}{\sigma}}\right) - \frac{i}{n+1}\right)^2 \qquad (17)$$

$\frac{i}{n+1}$ *Non-parmetric quantity*

By deriving equation (17) for $(\alpha;\beta;\mu;\sigma)$ *we get:*

$$\frac{\partial ols}{\partial \alpha} = \frac{2e^{\frac{\mu}{\sigma}}}{\sigma}\sum_{i=1}^{n}\left(exp\left(-e^{\frac{\mu}{\sigma}}\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right)\right) - \frac{i}{n+1}\right)^2 \left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right)^{-1-\frac{1}{\sigma}} \left(\frac{x_i}{\beta}\right)^{\alpha} Ln\left(\frac{x_i}{\beta}\right) e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} \qquad (18)$$

$$\frac{\partial ols}{\partial \beta} = \frac{2\alpha e^{\frac{\mu}{\sigma}}}{\sigma}\sum_{i=1}^{n}\left(exp\left(-e^{\frac{\mu}{\sigma}}\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right)\right) - \frac{i}{n+1}\right)^2 \left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right)\left(\frac{x_i}{\beta}\right)^{\alpha} e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} \qquad (19)$$

$$\frac{\partial ols}{\partial \mu} = \frac{-2e^{\frac{\mu}{\sigma}}}{\sigma}\sum_{i=1}^{n}\left(exp\left(-e^{\frac{\mu}{\sigma}}\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}}\right) - \frac{i}{n+1}\right)^2 \left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}} \qquad (20)$$

$$\frac{\partial 0}{\partial \sigma} = \frac{2}{\sigma}\sum_{i=1}^{n}\left(exp\left(-e^{\frac{\mu}{\sigma}}\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}}\right) - \frac{i}{n+1}\right)^2 \left(\frac{\mu e^{\frac{\mu}{\sigma}}\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}}}{\sigma^2} - \frac{e^{\frac{\mu}{\sigma}}\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right)^{-\frac{1}{\sigma}} \ln\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right)}{\sigma^2}\right) \qquad (21)$$

Equalizing equations (18-21) to zero and using newton raphson method for solving nonlinear equations we will get estimate the parameters of WGD.

<u>(7)Newton Raphson Methods</u>

To solve the system of nonlinear equations, there are many of methods such as expectation mathematical method (EM), scoring method, and newton raphson method… ets. Newton raphson is the most important method for solving system of nonlinear equations. The general form of newton raphson method is [(3)]*:*

$$\left(\psi^{(t+1)}\right) = \left(\psi^{(t)}\right) + I^{-1}\left(\psi^{(t)}, x\right)s\left(x, \psi^{t}\right) \qquad (22)$$

Where

(x, ψ^{t})*: is a Vector (k*1) represents the first derivative of the system of nonlinear equations. Where (k) represent unknown parameters*

$I^{-1}\left(\psi^{(t)}, x\right)$*: Hussian Matrix is a matrix of (k*k) of second derivative of of the system of nonlinear equations.*

$(\psi^{(t)})$ *: Vector of (k*1), representing the value of estimated parameters in replicate (t).*

$(\psi^{(t+1)})$*: Vector of (k*1), representing the value of the estimate parameters in replicate (t+1).*

(8)Simulation

Simulation was used to compare between (MLM) and (LSM) for estimate (WGD) parameters simulation has been chosen because it is a technique, which allows us to choose different sampling sizes. To generate data from WGD distribution, the following form was used [(5)]

$$x_i = \beta . \left[ln \left\{ \left({u_i}/{e^{\frac{\mu}{\sigma}}} \right) + 1 \right\}^{-\sigma} \right]^{1/\alpha} \qquad 23$$

Where $u_i \sim u(0,1)$.

Simulation was carried out using Matlab program and was replicated (1000) times to obtain the required homogeneity. The following assumptions were taken.

1. *Four different sizes of samples (30, 50, 100, 200, and 500) were used.*
2. *number of parameters were suggested according to the relationships among them as in table(1):*

Table (1): Suggested parameters

parameter	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
α	0.5	4	3	4	100	2
β	0.5	3	6	3	5	100
μ	0.5	2	3.5	8	4	2
σ	0.5	0.5	1	0.5	2	6

(8-1) Comparison between Estimation Methods:

To know the best method, Total absolute prestige error was used for estimation WGD parameters. This criteria, measure the distance between the real and estimated values for many parameters. This criterion is unitless with the following general form:

$$TD = \sum_{i=1}^{k} \left| \frac{\hat{\theta}_i - \theta}{\theta} \right| \qquad (24)$$

(8-2)Result Analysis

Results of simulation are tabulated through table (2) to table (6) according to the cases mentioned in table (1) (see appendix)

Case 1: In this case sample size was to be (n=30). Results showed similarity between the (MLM) and the (LSM). (See table 2 in appendix)

Case 2: In this case sample size was to be assumed (n=50). Results showed similarity Between the MLM method and the LSM method with a small advantage for MLM. (See table 3 in appendix)

Case 3: In this case sample size was assumed to be (n=100). Results showed similarity between the MLM method and the LSM method with a small advantage for MLM. (See table 4 in appendix)
Case 4: In this case sample size was assumed to be (n=200. Results showed similarity between the MLM method and the LSM method with a small advantage for MLM in some cases. Whereas other cases LSM, method was better
Case 5: In this case, sample size wase assumed to be (n=500). The results showed a similarity between the MLM method and the LSM method with a small advantage of MLM in some cases. Unlike other cases

(9)Conclusions:

According to simulation results, the following conclusion are obtained:

1. *Results show similarity between the MLM and LSM estimates.*
2. *Results show small advantage of MLM at small size (n=30) sample and big sizes sample (n=500), but at sample sizes (n=100&n=200) LSM is better*

Results show that the estimation of the parameters of both methods of estimation

3. *Becomes more accurate as the size of the sample is bigger.*

(10)Recommendations:

Through what has been exposed, we recommend that:

1. *Interest in studying technique mixing probability distributions because of their flexibility in representing heterogeneous populations,*
2. *Increase in number of mixing probability distribution parameters leads to decrease the accuracy of estimation parameters. For this reason, studying new estimation methods that gives largest accuracy in estimation based on technology and software programs is recommended.*
3. *The accuracy of the parameters estimated in the extended distributions depends on the method of estimation in addition to the numerical methods used to solve the system of nonlinear equations so we recommend choosing the appropriate numerical method for nonlinear equations.*

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Appendix

Table (2): Simulation result for estimate parameters of pdf of WGD when sample size (n=30)

n	Case 1	Value	T.D		Case 4	T.D	T.D	
			MLM	LSM			MLM	LSM
30	α	0.5	0.3658	0.4988	α	4	0.5702	0.6449
	β	0.5			β	3		
	μ	0.5			μ	2		
	σ	0.5			σ	0.5		
	Case 2				Case 5			
	α	3	0.3102	0.3755	α	1	0.5722	0.5925
	β	6			β	12		
	μ	3.5			μ	12		
	σ	1			σ	0.5		
	Case 3				Case 6			
	α	100	0.3625	0.3625	α	2	0.3708	0.3872
	β	5			β	100		
	μ	4			μ	2		
	σ	2			σ	6		

Table (3): Simulation result for estimate parameters of pdf of WGD when sample size (n=50)

n	Case ٢	Value	T.D MLM	T.D LSM	Case ٣	value	T.D MLM	T.D LSM
50	α	0.5	0.3512	0.4847	α	4	0.3474	0.3700
	β	0.5			β	3		
	μ	0.5			μ	2		
	σ	0.5			σ	0.5		
	Case 3	Value	MLM	LSM	Case 3	Value	MLM	LSM
	α	3	0.3160	0.3019	α	1	0.3524	0.3175
	β	6			β	12		
	μ	3.5			μ	12		
	σ	1			σ	0.5		
	Case 3	Value	MLM	LSM	Case 3	Value	MLM	LSM
	α	100	0.3875	0.3875	α	2	0.3555	0.3212
	β	5			β	100		
	μ	4			μ	2		
	σ	2			σ	6		

Table (4): Simulation result for estimate parameters of pdf of WGD when sample size (n=100)

n	Case ١	Value	T.D MLM	T.D LSM	Case ٤	Value	T.D MLM	T.D LSM
100	α	0.5	0.2693	0.2483	α	4	0.2593	0.2683
	β	0.5			β	3		
	μ	0.5			μ	2		
	σ	0.5			σ	0.5		
	Case ٢	Value	MLM	LSM	Case ٥	Value	MLM	LSM
	α	3	0.2616	0.2035	α	1	0.2388	0.2175
	β	6			β	12		
	μ	3.5			μ	12		
	σ	1			σ	0.5		
	Case 3	Value	MLM	LSM	Case ٦	Value	MLM	LSM
	α	100	0.2182	0.2071	α	2	0.2463	0.2565
	β	5			β	100		
	μ	4			μ	2		
	σ	2			σ	6		

Table (٥): Simulation result for estimate parameters of pdf of WGD when sample size (n=200)

n	Case 1	Value	T.D		Case 4	Value	T.D	
			MLM	LSM			MLM	LSM
200	α	0.5	0.0461	0.0907	α	4	0.0584	0.0762
	β	0.5			β	3		
	μ	0.5			μ	2		
	σ	0.5			σ	0.5		
	Case 2	Value	MLM	LSM	Case 5	Value	MLM	LSM
	α	3	0.0375	0.0459	α	1	0.0137	0.0175
	β	6			β	12		
	μ	3.5			μ	12		
	σ	1			σ	0.5		
	Case 3	Value	MLM	LSM	Case 6	Value	MLM	LSM
	α	100	0.0875	0.0875	α	2	0.0558	0.0242
	β	5			β	100		
	μ	4			μ	2		
	σ	2			σ	6		

Table (٥): Simulation result for estimate parameters of pdf of WGD when sample size (n=500)

n	Case 1	Value	T.D		Case 4	Value	T.D	
			MLM	LSM			MLM	LSM
٥00	α	0.5	0.0061	0.0078	α	4	0.0069	0.0090
	β	0.5			β	3		
	μ	0.5			μ	2		
	σ	0.5			σ	0.5		
	Case 2	Value	MLM	LSM	Case 5	Value	MLM	LSM
	α	3	0.0116	0.0254	α	1	0.0067	0.0075
	β	6			β	12		
	μ	3.5			μ	12		
	σ	1			σ	0.5		
	Case 3	Value	MLM	LSM	Case 6	Value	MLM	LSM
	α	100	0.0075	0.0075	α	2	0.0186	0.0303
	β	5			β	100		
	μ	4			μ	2		
	σ	2			σ	6		