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The Gompertz Inverted Nadarajah-Haghighi (GoINH) Distribution Properties with Application to real data

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Abstract: Over the past years, many academics in mathematical statistics have introduced new continuous distributions to respond to real-world data growth, and this adaptability or flexibility has become essential to this development. In this study, a new distribution called Gompertz Inverted Nadarajah-Haghighi (GoINH) is proposed, according to the Gompertz family which is a new four-parameter probability distribution model. In addition, we will present a number of properties of the proposed distribution such as moment generating function, survival function, risk function, characteristic function, quantile function, pdf expansion, and ordered statistics which represent a few of the many mathematical and statistical features of the recently proposed model. The model parameters were also estimated using the maximum likelihood method. In the end, a practical application was carried out on real data, represented by failure times for 40 heavy-duty locomotive engines, and comparison with a number of new or known distributions to determine the efficiency of the distribution using a number of informatics criteria is using the R programming language.

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خصائص توزيع معكوس جومبرتز ندراجا حجيجي مع التطبيق على بيانات حقيقية

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المستخلص

على مدى السنوات الماضية، قدم العديد من الأكاديميين في مجال الإحصاء الرياضي توزيعات مستمرة جديدة للاستجابة لنمو البيانات في العالم الحقيقي، وأصبحت هذه القدرة على التكيف أو المرونة ضرورية لهذا التطور. في هذه الدراسة تم اقتراح توزيع جديد يسمى Gompertz Inverted Nadarajah-Hagighi (GoINH)، وفقًا لعائلة Gompertz وهو نموذج توزيع احتمالي جديد مكون من أربع معلمات. بالإضافة إلى ذلك، سنقدم عددًا من خصائص التوزيع المقترح مثل الدالة المولدة للعزوم، ودالة البقاء، ودالة المخاطرة، والدالة المميزة، والدالة الكمية، وتوسيع pdf، والإحصائيات المرتبة التي تمثل عددًا قليلاً من الميزات الرياضية والإحصائية العديدة للتوزيع المقترح. تم تقدير معلمات النموذج أيضًا باستخدام طريقة الامكان الاعظم. وفي النهاية تم اجراء تطبيق عملي على بيانات حقيقية تمثلت ب أوقات فشل ل 40 محرك للمكان الثقيلة والمقارنة بعدد من التوزيعات الجديدة او المعرفة لمعرفة مدى كفاءة التوزيع باستخدام عدد من معايير المعلوماتية باستخدام لغة البرمجة R.

الكلمات المفتاحية: معكوس توزيع Nadarajah-Haghighi، العزوم، الدالة الكمية، الدالة المولدة للعزوم، وMLE.

1. Introduction

Statistical distributions were presented in order to be considered the basic tool in analysis for many areas of life, such as engineering, science, geography, finance, and insurance. New attempts to extend well-known continuous distributions have led to improved data modeling services in various fields.

Inverted distributions are often very useful for exploring additional properties of phenomena that normal distributions cannot. An Inverted Nadarajah-Haghighi distribution model that shows decreasing and unimodal (right-skewed) density was first presented in 2018 by [Tahir et al., (2018)]. It is right to provide expansions of this distribution, including, for example: Consider the problems estimating hazard functions, dependability, and model parameters for an inverted Nadarajah-Haghighi dist. When The example may be obtained from a progressive kind II censorship system by [Elshahhat and Rastogi. (2021)], the inverse Nadarajah-Haghighi power with three parameters is proposed by [Ahsan-ul-Haq et al., (2022)], and a new class of distribution called the inverted Nadarajah-Haghighi power series is

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introduced by [Ahsan-ul-Haq et al., (2024)]. This dis has pdf, and CDF given respectively by:

$$g(x)_{INH} = \alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1}\left(e^{1-(1+\lambda x^{-1})^{\alpha}}\right) \quad (1)$$

$$G(x)_{INH} = e^{1-(1+\lambda x^{-1})^{\alpha}} \quad (2)$$

When α, λ are the tow parameter [Tahir et al., (2018)]

In this study, our interest focuses on the Gompertz family distributions presented by [Alizadeh et al., (2017)], according to which we will generate the new distribution. Many studies have been presented, including: They proposed the Gompertz-Frechet (GoFr) distribution was studied and presented [Bodhisuwan and Aryuyuen. (2021)]. Four real datasets are used to show the Discrete Gompertz (DGz-G) family by [Eliwa et al., (2020)]. Eghwerido established the transformed Gompertz-G (ShiGo-G) family. Which is a type of generator for classical statistical distributions. It is used to create new continuous distributions [Eghwerido and Agu. (2021)].The family Gompertz-G's CDF and pdf are given respectively:

$$F(x)_{Go} = 1 - e^{\frac{\theta}{\gamma}\{1-[1-G(x)]^{-\gamma}\}} \quad (3)$$

$$f(x)_{Go} = \theta g(x)[1-G(x)]^{-\gamma-1} e^{\frac{\theta}{\gamma}\left[1-(1-G(x))^{-\gamma}\right]}, x \geq 0, \gamma, \theta > 0 \quad (4)$$

[Alizadeh et al., (2017)]

The study aims to find a new statistical distribution with four parameters within the Gompertz family called Gompertz Inverted Nadarajah-Haghighi, along with finding and drawing the main functions of the new distribution and many of its statistical and mathematical properties. In addition to estimating its parameters using the maximum possibility method, as well as a practical application on real data to demonstrate the suitability of the new distribution. The study comprised five components. The initial section presented the suggested statistical distribution, followed by a subsequent section outlining various statistical characteristics of the distribution. The third section provided an estimation of the distribution parameters, and the fourth section demonstrated a practical application using real data. After completing these sections, the task was concluded. The study's most significant findings are reported in the final section.

2. **Gompertz inverted Nadarajah-Haghighi distribution:** When substituting equation (1) into equation (4), we obtain the pdf of the new distribution, which is called GoINH for short. When substituting equation (2) into

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equation (3), we obtain CDF of the distribution GoINH, whose equations are given as follows:

$$F(x)_{GoINH} = 1 - e^{\frac{\theta}{\gamma}\left\{1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right\}} \quad (5)$$

$$f(x)_{GoINH} = \theta\alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1}\left(e^{1-(1+\lambda x^{-1})^{\alpha}}\right)\left[1 - e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma-1} * e^{\frac{\theta}{\gamma}\left\{1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right\}} \quad (6)$$

Where equation (5) represents the CDF, while equation (6) represents the pdf of the GoINH distribution, while $\theta, \alpha, \lambda, \gamma$ are represents the parameters of the proposed distribution. Figures (1) and (2), represent the plotting of GoINH's pdf and CDF functions using different parameter values to depict a file which was obtained from programming in R language.

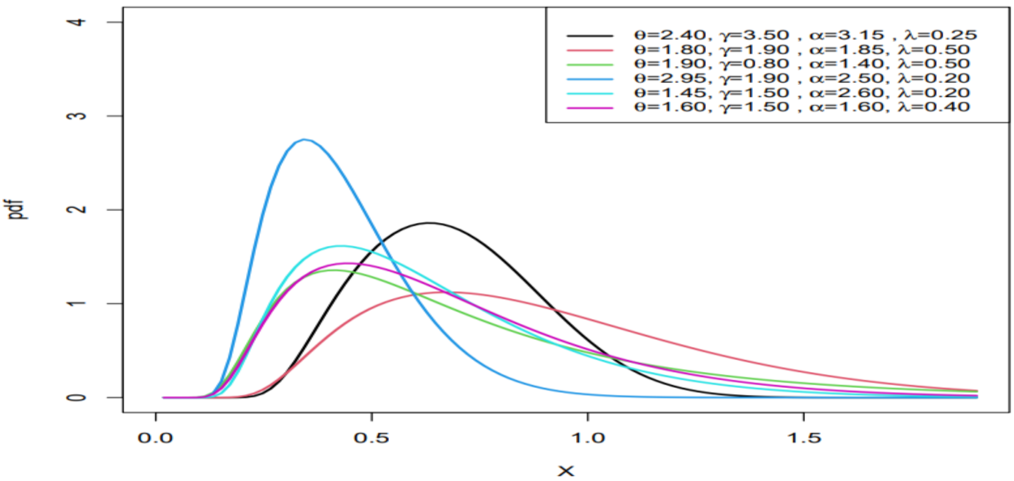

Fig. 1: PDF of GoINH dist.

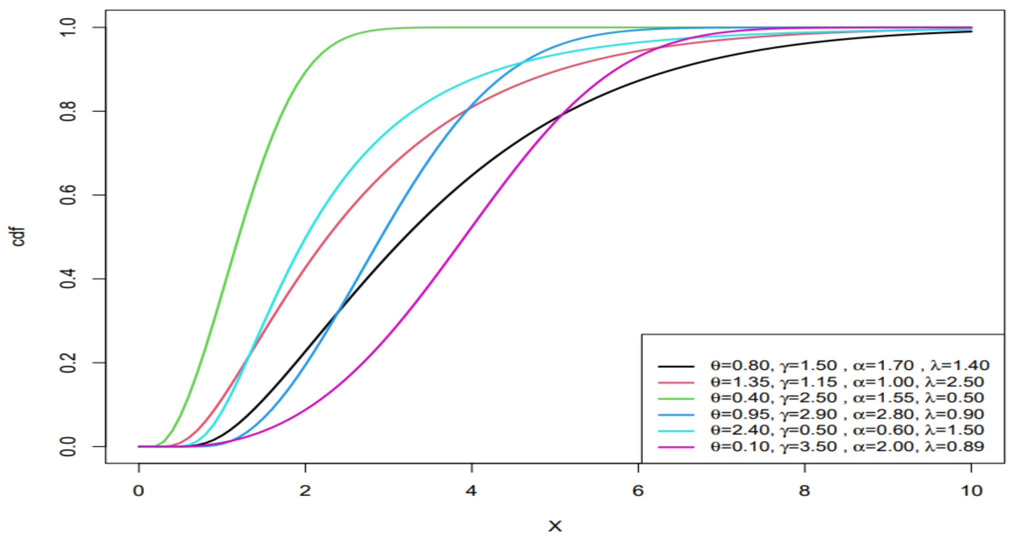

Fig. 2: CDF of GoINH dist.

3. Some statistical properties of the GoINH distribution: Following is an extraction of a few of the fundamental statistical characteristics of the GoINH:

First, the relation is used to generate the dependability function, function is the probability that a system will not fail after a period of time known by form: $S(x)_{GoINH} = 1 - F(x)$

Thus, the GoINH distribution's dependability function is:

$$S(x)_{GoINH} = e^{\frac{\theta}{\gamma}\left\{1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right\}} \quad (7)$$

Where Figure (3) expresses the survival function for the GoINH distribution which was obtained from programming in R language.

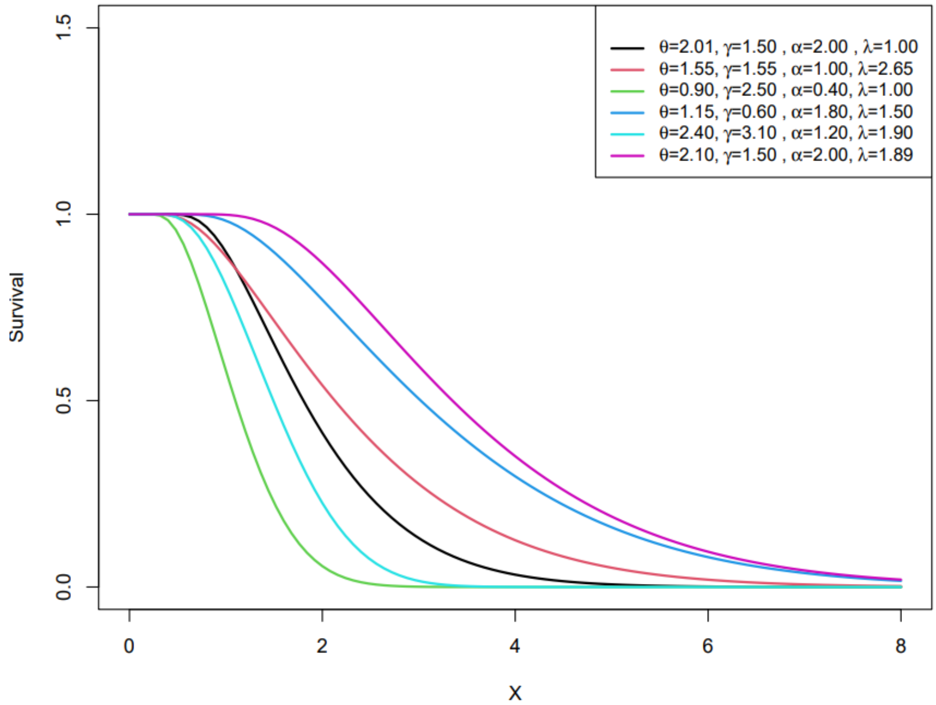

Fig. 3: S(x) of GoINH dist.

To get the Hazard function, we divide equation (6) by equation (7):

$$h(x)_{GoINH} = \theta\alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1}\left(e^{1-(1+\lambda x^{-1})^{\alpha}}\right)\left[1 - e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma-1} \quad (8)$$

Which represents the GoINH distribution failure rate. Where Figure (4) expresses the Hazard function for the GoINH distribution which was obtained from programming in R language.

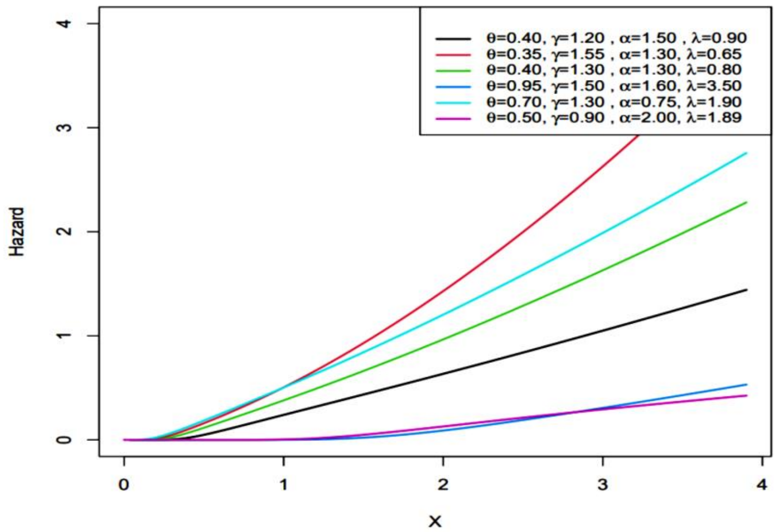

Fig. 4: $h(x)$ of GoINH dist.

By dividing equation (5) by equation (6), we obtain the reliability function in the form:

$$r(x,\theta,\gamma,\alpha,\lambda)_{GoNH} = \frac{F(x)_{GoINH}}{h(x,\alpha,\theta,\gamma,\lambda)_{GoINH}}$$

$$r(x)_{GoINH} = \frac{1-\ e^{\frac{\theta}{\gamma}\left\{1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right\}}}{\theta\alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1}\left(e^{1-(1+\lambda x^{-1})^{\alpha}}\right)\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma-1}} \quad (9)$$

The role represented by the risk inverse function is very important, as it has importance in analyzing data that is subject to censorship, and it also has importance that qualifies it for analyzing reliability in stochastic modeling.

3-1. Quantile function: From the relationship, one may get the quantile function $Q(u)$ by [Mahdi et al., (2021)], :[Oguntunde, et al., (2018)], and [Khalaf et al., (2023)]:

$$Q(u) = F^{-1}(u)$$

Thus, the GoNH distribution's quantile function may be obtained as follows:

$$x=\lambda\left[\left(1-\left(\log\left(1-\left(1-\left(\frac{\gamma}{\theta}(\log(1-u))\right)\right)^{\frac{1}{-\gamma}}\right)\right)\right)^{\frac{1}{\alpha}}-1\right]^{-1} \quad (10)$$

It is easy to obtain the median GoINH distribution by substituting the value of 0.5 for u in equation (10) to become:

$$\text{median} = \lambda\left[\left(1-\left(\log\left(1-\left(1-\left(\frac{\gamma}{\theta}(\log(0.5))\right)\right)^{\frac{1}{-\gamma}}\right)\right)\right)^{\frac{1}{\alpha}} - 1\right]^{-1} \qquad (11)$$

3-2. Expansions: In this part, we will expand on the pdf file of the new distribution, which is used to find quite a few properties of the distribution. By taking the pdf file in equation (6) as follows [Khalaf A. A. (2022)], and [Khaleel et al., (2022)]:

By using the exponential expansion of the equation:

$$e^{\frac{\theta}{\gamma}\left\{1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right\}} = \sum_{i=0}^{\infty}\frac{\left(\frac{\theta}{\gamma}\right)^{i}\left(1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right)^{i}}{i!}$$

We substitute the last expression into the equation and get the new simplified function:

$$f(x)_{GoINH} = \sum_{i=0}^{\infty}\frac{\left(\frac{\theta}{\gamma}\right)^{i}}{i!}\left(1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right)^{i}\theta\alpha\lambda x^{-2}(1 + \lambda x^{-1})^{\alpha-1}\left(e^{1-(1+\lambda x^{-1})^{\alpha}}\right)\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma-1}$$

Using the binomial series expansion on the last term in the numerator of the equation

$$\left(1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right)^{i} = \sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{i}{j}\left(\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right)^{j}}{i!}$$

Such that we get:

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$$f(x)_{GoINH} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\left(\frac{\theta}{\gamma}\right)^{i}\binom{i}{j}(-1)^{j})}{i!}[1 - e^{1-(1+\lambda x^{-1})^{\alpha}}]^{-\gamma(j+1)-1}\theta\alpha\lambda x^{-2}(1 + \lambda x^{-1})^{\alpha-1}(e^{1-(1+\lambda x^{-1})^{\alpha}}$$

Using the binomial series expansion on the last term in the numerator of the equation

$$[1 - e^{1-(1+\lambda x^{-1})^{\alpha}}]^{-(\gamma(j+1)+1)} = \sum_{k=0}^{\infty}\frac{(-1)^{k}\binom{\gamma(j+1)+1}{k}(e^{1-(1+\lambda x^{-1})^{\alpha}})^{k}}{k!}$$

We substitute the last expression into the equation and get the new simplified function:

$$f(x)_{GoINH} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{-\gamma(j+1)-1}{k}\theta\alpha\lambda\frac{\left(\frac{\theta}{\gamma}\right)^{i}\binom{i}{j}(-1)^{j+k}}{i!\,k!}x^{-2}(1 + \lambda x^{-1})^{\alpha-1}\left(e^{1-(1+\lambda x^{-1})^{\alpha}}\right)^{k+1}$$

Let $\vartheta_{i,j,k} = \sum_{i,j,k=0}^{\infty}\frac{\left(\frac{\theta}{\gamma}\right)^{i}\binom{i}{j}(-1)^{j+k}}{i!k!}(-1)^{j+k}\binom{-\gamma(j+1)-1}{k}$, we get:

$$f(x)_{GoINH} = \vartheta_{i,j,k}\theta\alpha\lambda x^{-2}(1 + \lambda x^{-1})^{\alpha-1}\left(e^{1-(1+\lambda x^{-1})^{\alpha}}\right)^{k+1} \qquad (12)$$

3-3. Moments: The moment function [Hammed and Khaleel. (2023)], and [Al-Noor, et al., (2022)] is given by:

$$\mu_n = e(x^n)_{GoINH} = \int_0^{\infty} x^n f(x)_{IGoNH}dx \qquad (13)$$

After substituting the pdf function into equation (12) in equation (13), we get the following equation:

$$\mu_n = \vartheta_{i,j,k}\theta\alpha\lambda\int_0^{\infty} x^{n-2}(1 + \lambda x^{-1})^{\alpha-1}e^{-(k+1)(1-(1+\lambda x^{-1})^{\alpha})}dx \qquad (14)$$

Let $t = (1 + \lambda x^{-1})^{\alpha}$,then, and By using the binomial theory, then we get:

$$\mu_n = \sum_{i,j,k,l=0}^{\infty} \frac{\left(\frac{\theta}{\gamma}\right)^i \binom{i}{j} (-1)^{j+k-n} \binom{-\gamma(j+1)-1}{k} \theta\lambda^{1+n}}{i!\,k!} \binom{n+l-1}{l} \int_0^{\infty} t^{\frac{l}{\alpha}} e^{-(k+}$$

The n^{th} moment of the IGoNH distribution may be found by applying the gamma function $(\Gamma(.,.))$, and let $\phi_{i,j,k,l} = \sum_{i,j,k,l=0}^{\infty} \frac{\left(\frac{\theta}{\gamma}\right)^i \binom{i}{j} (-1)^{j+k-n} \binom{-\gamma(j+1)-1}{k}}{i!k!} \binom{n+l-1}{l}$, we get:

$$\mu_n = \phi_{i,j,k,l} \frac{\theta\lambda^{1+n} \Gamma\left(\frac{l}{\alpha}+1\right)}{\alpha(j+1)^{\frac{l}{\alpha}+1}} \tag{15}$$

3-4 Moment Generating Function: Let X denote a R.v, the moment generating function (mgf) by [Khalaf and Khaleel. (2020)], [Khaleel et al., (2018)], and [Yusur, and Khaleel, (2023, December)] is given by:

$$M_x(t)_{GoINH} = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)_{GoINH} dx \tag{16}$$

Used series expansion for e^{tx},

$$M_x(t)_{GoINH} = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(x^n) = \sum_{n=0}^{\infty} \frac{t^n}{n!} [\mu_n]$$

$$M_x(t)_{GoINH} = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(x^n)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[\phi_{i,j,k,l} \frac{\theta\lambda^{1+n} \Gamma\left(\frac{l}{\alpha}+1\right)}{\alpha(j+1)^{\frac{l}{\alpha}+1}}\right] \tag{17}$$

3-5. Characteristic function: Can discover the GoINH distribution characteristic function by doing the following [Khaleel, and Hammed, (2023)], and [Oguntunde, et al., (2018)]:

$$Q_x(t)_{GoINH} = E\left(e^{itx}\right) = \int_0^{\infty} e^{itx} f(x)_{GoINH} dx$$

Applying the expansion of the exponential function and simplifying the equation based on the expansion equation, we obtain the characteristic function as follows:

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$$Q_x(t)_{GoINH} = \sum_{i,j,k,l=0}^{\infty} \sum_{z=0}^{\infty} \frac{(it)^z}{z!} \left[\phi_{i,j,k,l} \frac{\theta\lambda^{1+n}\Gamma\left(\frac{l}{\alpha}+1\right)}{\alpha(j+1)^{\frac{l}{\alpha}+1}} \right] \quad (18)$$

3-6. Order statistic: Regarding a size-n R.s. from a dist. function F(x)_GoINH, the pdf of the j − th order statistic and its corresponding pdf GoINH as follows: [Abdal et al., (2020)], :[Khaleel et al., (2017)], and:[Khaleel et al., (2023)]

$$f_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} [F(x)_{GoINH}]^{j+r-1} f(x)_{GoINH}$$

$$f_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} \left[1 - e^{\frac{\theta}{\gamma}\left\{1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right\}}\right]^{j+r-1} \times \theta\alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1}\left(e^{1-(1+\lambda x^{-1})^{\alpha}}\right)\left[1 - e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma-1} e^{\frac{\theta}{\gamma}\left\{1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right\}} \quad (19)$$

The pdf of least order statistics may thus be derived by changing Equation to require j = 1.

$$f_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} \left[1 - e^{\frac{\theta}{\gamma}\left\{1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right\}}\right]^{r} \times \theta\alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1}\left(e^{1-(1+\lambda x^{-1})^{\alpha}}\right)\left[1 - e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma-1} e^{\frac{\theta}{\gamma}\left\{1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right\}}$$

However, by substituting j = n in Equation as follows, the equivalent pdf of maximum order statistics may be obtained:

$$f_{j:n}(x) = \sum_{r=0}^{n-j} k(-1)^r \binom{n-j}{r} \left[1 - e^{\frac{\theta}{\gamma}\left\{1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right\}}\right]^{n+r-1} \times \theta\alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1}\left(e^{1-(1+\lambda x^{-1})^{\alpha}}\right)\left[1 - e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma-1} e^{\frac{\theta}{\gamma}\left\{1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right\}}$$

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3-7. Skewness and Kurtosis: Skewness is defined as a measure that shows the shape of the data (whether the data are symmetrical or not). What is meant by asymmetry is the extent to which the probability distribution curve is skewed towards the right or left. As for flatness, it is a measure of the flatness or pointedness of the peak of the distribution. The peak of the distribution of the random variable is compared with the peak of the normal distribution.

$$SK = \frac{\mu_3}{\mu_2^{\left(\frac{3}{2}\right)}}$$

$$SK = \frac{Q_{(3/4)} + Q_{(1/4)} - 2Q_{(1/2)}}{Q_{(3/4)} - Q_{(1/4)}} \quad (20)$$

$$KU = \frac{\mu_4}{\mu_2^2} - 3$$

$$KU = \frac{Q_{\left(\frac{7}{8}\right)} - Q_{\left(\frac{5}{8}\right)} - Q_{\left(\frac{3}{8}\right)} - Q_{\left(\frac{1}{8}\right)}}{Q_{\left(\frac{6}{8}\right)} - Q_{\left(\frac{1}{8}\right)}} \quad (21)$$

3-8. Rényi Entropy: We can obtain the Rényi Entropy function using the equation [Habib et al., (2023)],[Mohammad et al., (2020)]:

$$I_R(\sigma)_{GoINH} = \frac{1}{1-\sigma} \log \int_0^{\infty} \left(f(x)\right)^{\sigma} dx \quad (22)$$

By substituting the pdf function of the new distribution into equation (22), we obtain the entropy function for GoINH in the form:

$$I_R(\sigma)_{GoINH} = \frac{1}{1-\sigma} \log \int_0^{\infty} \left(\phi_{i,j,k,l} \frac{\theta\lambda^{1+n}\Gamma\left(\frac{l}{\alpha}+1\right)}{\alpha(j+1)^{\frac{l}{\alpha}+1}} \right)^{\sigma} dx \quad (23)$$

The equation represents the Rennie entropy, and if the value of $\sigma \to 1$, the entropy turns into another type called (Shanon Entropy).

4. Estimation: Parameters for the GoINH distribution. It is estimated using the maximum likelihood method, by deriving the logarithm of the likelihood function for a random sample $x_1, x_2, \ldots, x_n$ distributed in line with the CDF file of the GoINH dist.:[Ibrahim, et al., (2017)], and:[Khaleel, (2017)].

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$$L(\Theta, X) = \prod_{i=1}^{n} \theta\alpha\lambda x^{-2}(1+\lambda x^{-1})^{\alpha-1}(e^{1-(1+\lambda x^{-1})^{\alpha}})\left[1 - e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma-1} * e^{\frac{\theta}{\gamma}\left\{1-\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\right\}} \quad (24)$$

The log-likelihood function L is obtained as:

$$L = n\log(\theta) + n\log(\alpha) + n\log(\lambda) - 2n\sum_{i=1}^{n}\log x_i + (\alpha-1)\sum_{i=1}^{n}\log\left(1+\frac{\lambda}{x_i}\right) + 1 - (1+\lambda x_i^{-1})^{\alpha} - (\gamma+1)\sum_{i=1}^{n}\log\left(1-e^{1-(1+\lambda x_i^{-1})^{\alpha}}\right) + 1 - \left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma} \quad (25)$$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} \quad (26)$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1+\frac{\lambda}{x_i}\right) + \frac{\alpha\lambda}{x^2}(1+\lambda x_i^{-1})^{\alpha-1} - (\gamma+1)\sum_{i=1}^{n}\frac{\frac{\alpha\lambda}{x^2}(1+\lambda x_i^{-1})^{\alpha-1}e^{1-(1+\lambda x_i^{-1})^{\alpha}}}{1-e^{1-(1+\lambda x_i^{-1})^{\alpha}}} + \gamma e^{1-(1+\lambda x_i^{-1})^{\alpha}}\left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma-1}(1+\lambda x^{-1})^{\alpha}\log(1+\lambda x^{-1})^{\alpha} \quad (26)$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n}\frac{\lambda}{x_i\left(1+\frac{\lambda}{x_i}\right)} - \alpha(1+\lambda x_i^{-1})^{\alpha-1}(1+x_i^{-1})(\gamma+1)\sum_{i=1}^{n}\frac{e^{1-(1+\lambda x_i^{-1})^{\alpha}}\alpha(1+\lambda x_i^{-1})^{\alpha-1}}{x_i\left(1-e^{1-(1+\lambda x_i^{-1})^{\alpha}}\right)} - e^{1-(1+\lambda x_i^{-1})^{\alpha}}\alpha(1+\lambda x_i^{-1})^{\alpha-1} \quad (27)$$

$$\frac{\partial L}{\partial \gamma} = -\sum_{i=1}^{n}\log\left(1-e^{1-(1+\lambda x_i^{-1})^{\alpha}}\right) + \left[1-e^{1-(1+\lambda x^{-1})^{\alpha}}\right]^{-\gamma}\log\left(1-e^{1-(1+\lambda x_i^{-1})^{\alpha}}\right) \quad (28)$$

The solution to the system of non-linear equations $\frac{\partial L}{\partial \theta}\frac{\partial L}{\partial \alpha}, \frac{\partial L}{\partial \lambda},$ and $\frac{\partial L}{\partial \gamma}$ the ML estimates of the parameters θ, α, λ and γ are provided as findings. An analytical solution for the problem is not feasible and so numerical approaches, such as those provided by software like R, MAPLE, and SAS, must be used.

5. **Model Application:** The data set contains 40 observations and it is studied in many papers such as (Alobaidi et al., (2021)), and (Khalaf et al., (2022)). The data are (1.6 2.0 2.6 3.0 3.5 3.9 4.5 4.6 4.8 5.0 5.1 5.3 5.4 5.6 5.8 6.0 6.0 6.1 6.3 6.5 6.5 6.7 7.0 7.1 7.3 7.3 7.3 7.7 7.7 7.8 7.9 8.0 8.1 8.3 8.4 8.4 8.5 8.7 8.8 9.0).

Our first step is to derive Maximum Likelihood Estimates (MLEs) for several competing models given their unknown parameters. The Anderson-Darling (A), Cramér-von Mises (2018), (W), Hannan and Quinn Information Criteria (HQIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criterion (BIC), and Akaike Information Criterion (AIC) are among the goodness-of-fit statistics that we compare next. The model that performs better is found when these criteria are minimized. We compare the GoNH distribution to several counterparts in order to assess its effectiveness, including the Beta Nadarajah-Haghighi (BeINH) (New), the Weibull Nadarajah-Haghighi (WeINH) (New), the [0,1]Truncated Exponentiated Exponential Nadarajah-Haghighi ([0,1] TEEINH) (New), and Kumaraswamy Nadarajah-Haghighi (KuINH) (New), Exponentiated Generalized Nadarajah-Haghighi (EGINH) (New), log Gamma Nadarajah-Haghighi (LGamINH) (New), and Nadarajah Haghighi (Nadarajah, S., & Haghighi, F).

Table (1): Estimates of models for data 1

Dist.	Estimates	NLL	AIC	CAIC	BIC	HQIC
GoINH	$\hat{\theta}$ =0.0379 $\hat{\gamma}$ =4.2943 $\hat{\alpha}$ =5.3629 $\hat{\lambda}$ =0.4524	81.93	171.868	173.0108	178.6235	174.3106
[0,1]TE EINH	$\hat{\theta}$ =252.01 $\hat{\gamma}$ =0.4030 $\hat{\alpha}$ =0.8094 $\hat{\lambda}$ =73.205	84.98	177.9754	179.1183	184.7309	180.418
BeINH	$\hat{\theta}$ =35.316 $\hat{\gamma}$ = 21.978 $\hat{\alpha}$ =0.2062 $\hat{\lambda}$ =33.736	90.34	188.6843	189.8272	195.4398	191.1269

Dist.	Estimates	NLL	AIC	CAIC	BIC	HQIC
KuINH	$\hat{\theta} = 19.89$ $\hat{\gamma} = 55.134$ $\hat{\alpha} = 0.224$ $\hat{\lambda} = 8.5856$	86.51	181.034	182.1768	187.7895	183.4765
EGINH	$\hat{\theta} = 126.1$ $\hat{\gamma} = 0.4268$ $\hat{\alpha} = 0.852$ $\hat{\lambda} = 54.840$	86.23	180.4646	181.6074	187.2201	182.9072
WeINH	$\hat{\theta} = 16.82$ $\hat{\gamma} = 2.742$ $\hat{\alpha} = 0.065$ $\hat{\lambda} = 11.320$	82.26	172.5346	173.6774	179.2901	174.9771
LGamINH	$\hat{\theta} = 70.06$ $\hat{\gamma} = 48.582$ $\hat{\alpha} = 0.125$ $\hat{\lambda} = 33.364$	89.54	187.0932	188.2361	193.8488	189.5358
INH	$\hat{\theta} = 26.58$ $\hat{\gamma} = 0.0045$	102.1	208.3033	208.6276	211.681	209.5246

The previous two tables show that the values of the AIC, BIC, CAIC, and HQIC criteria for the GONH distribution are less standard values, and this indicates how efficient the distribution is compared to other distributions. To confirm the validity of this, Figures 5 and 6 show the extent to which the GoINH distribution matches the data used which was obtained from programming the pdf and cdf functions, the new distribution with some other distributions, and comparison with each other in the R language.

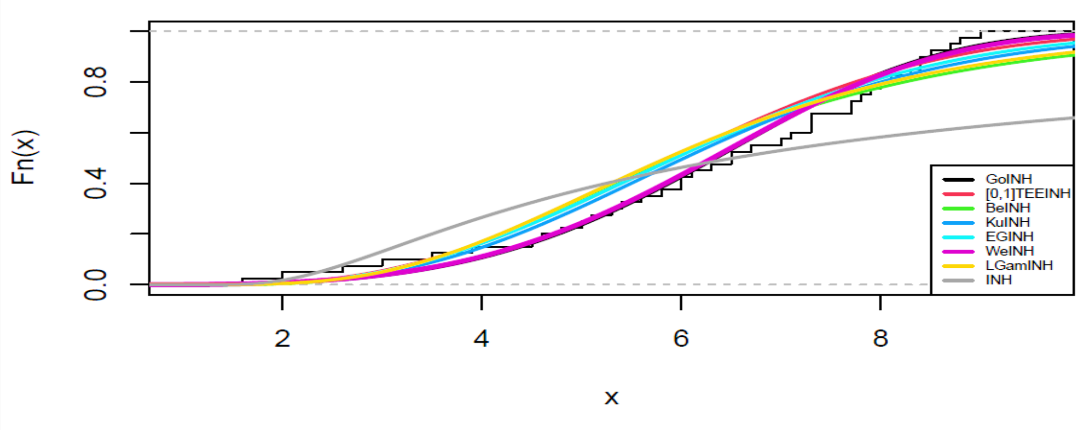

Fig. 5: fitted densities of eight distribution

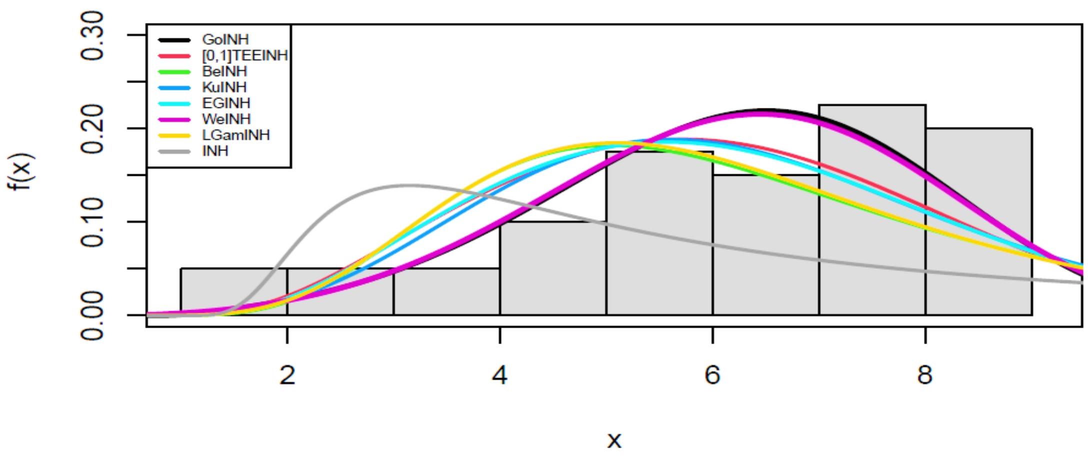

Fig. 6: fitted densities

6. **Conclusions**: We propose a new distributions class, designated as (GoINH). The recently suggested model has several mathematical and statistical properties, including explicit equations for its moments, survival function, moment-generating function, risk function, characteristic function, quantile function, expansion of pdf, and ordered statistics. The Maximum Likelihood method was utilized to estimate the model's parameters. Extensive simulation studies were carried out to provide a comprehensive quantile evaluation of the small sample of MLEs. On actual data sets, the new distribution's usage fullness is explained. As future studies, I suggest estimating the parameters of the new distribution using other estimation methods, since we obtained estimates from the maximum potential method during this study, while performing simulations to identify bias in the estimated parameter values and find out which methods are best for estimating the new distribution.

Conflicts of Interest: The authors declare no conflict of interest.

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