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The Hybrid Odd Exponential-Exponential Distribution: Statistical Properties and Applications

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Abstract: In this paper, we study a new statistical distribution called the hybrid odd exponential-exponential distribution (HOE-E distribution for short) we the aim of applying it to real data sets. This new distribution appears as a sub-model of the larger HOEE-Φ family in (Mahdi et al., 2024). We explore many mathematical and statistical properties of this distribution such as the series expansion of the PDF, the quantile function, moments and their generating function, incomplete moments, the shape of the PDF, and the shape of the hazard rate function, order statistics, and parameter estimation using the maximum likelihood estimation. We use this distribution to model a plasma concentration data set. Moreover, this we show that the suggested HOE-E distribution outperformance many other distributions based on various statistical criteria like AIC, CAIC, BIC, HQIC, W, and A.

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التوزيع الهجين الفردي الأسي المزدوج: الخصائص والتطبيقات الاحصائية

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المستخلص
في هذا البحث نقوم بدراسة توزيع احصائي جديد يسمى التوزيع الهجين الفردي الأسي المزدوج (توزيع HOE-E للاختصار) بهدف تطبيقه على مجاميع بيانات حقيقية. هذا التوزيع هو توزيع جزئي من عائلة HOEE-Φفي (Mahdi et al., 2024). نستكشف العديد من الخصائص الرياضية والاحصائية لهذا التوزيع مثل توسيع سلسلة دالة الكثافة الاحتمالية، دالة الكمية، العزوم، العزوم الناقصة، شكل دالة الكثافة الاحتمالية، وشكل دالة معدل الخطر، والإحصاءات المرتبة وتقدير المعلمات باستخدام تقدير الاحتمال الأقصى. نستخدم هذا التوزيع لنمذجة مجموعة بيانات تركيز البلازما. علاوة على ذلك، نُظهر أن التوزيع HOE-E المقترح يتفوق على العديد من التوزيعات الأخرى بناءً على معايير إحصائية مختلفة مثل AIC، CAIC، BIC، HQIC، W، وA.
الكلمات المفتاحية: عائلة HOE-G ، طريقة MLE، Entropies، PDF، CDF.

1. Introduction

Statistical distributions provide an organized framework for analyzing and understanding data in a range of fields, making them useful tools in daily life. Making informed decisions across a range of disciplines, such as sociology, epidemiology, and finance, is made easier when one is aware of the intrinsic variability in phenomena through the study of statistical distributions. A phenomenon that gained attention in recent years is to generate new distributions from old ones. Marshal et al., in (Marshall & Olkin, 1997) defined the MO-G family which is shown to be an excellent alternative to many existing distributions such as the exponential distribution, the gamma distribution, and the Weibull distribution. It is also quite common to use PDFs of certain distributions and use them to define extended versions of them. In particular, Cordeiro et al., in (Cordeiro et al., 2013) defined the exponential-G family which is one of the most common families in recent years. Bourguignon et al., in (Bourguignon et al., 2014), exploited the Weibull distribution to generate the Weibull-G family which was extensively studied and found many applications in real-life situations. By using the truncation method, Abbasi et al., in (Abbasi et al., 2023) introduced the right truncated Xgamma-G family of distributions which extends the PDF of the Xgamma distribution. Alzaatreh et al., in (Alzaatreh et al., 2013), introduced a new method for generating new continuous distributions called the T-X transformer method. This method proved to be a very efficient and easy tool for generating wide range of continuous families by using two different random variables T and X. Many other methods have been presented by many other researchers such as The Marshall-Olkin Topp Leone-G family in (Khaleel et al., 2020), the Gompertz inverse exponential distribution in (Oguntunde et al., 2018), the [0,1] Truncated Inverse Weibull-G family in (Khaleel et al., 2022), and the Transmuted Topp-Leone power function distribution in (Hassan et al., 2021).

Let $r(t)$ be a given PDF of a random variable T that takes values in the interval $[a, b]$ where a and b are two real numbers. The T-X transformer method relies on a function $W(F(X))$ of the CDF of some other random variable X that satisfies the following conditions:

A. $W(F(x)) \in [a, b]$.
B. $W(F(x))$ can be differentiated and is not decreasing monotonically.
C. $W(F(x))$ approaches a as x approaches $-\infty$ and it approaches b as x approaches $+\infty$.

The CDF is then given by the formula:

$$G(x) = \int_{a}^{W(F(x))} r(t)\ dt. \tag{1}$$

A simple differentiation with respect to x gives the PDF:

$$g(x) = \left\{\frac{d}{dx} W(F(x))\right\} r\{W(F(x))\}. \tag{2}$$

In this paper, we use the PDF function of the exponential function in place of $r(t)$ and set $W(F(x))$ to be the function:

$$W(F(x)) = \frac{F(x)}{1 - F(x)} \log \frac{1}{1 - F(x)}, \tag{3}$$

and use them to generate a distribution called the hybrid odd exponential-exponential distribution (HOE-E distribution). This new distribution is shown to be a useful alternative to many other existing models. The rest of the paper is organized as follows: in section 2, we construct the HOE-E distribution using the method outlined before. Section 3 deals with many essential statistical and mathematical properties of this distribution. The method of maximum likelihood estimation is used in section 4 to

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estimate the parameters of the HOE-E distribution. In section 5 we give a simulation study based on the Monte Carlo algorithm while in section 6 we give an application of the newly defined distribution. We conclude the paper in section 7.

2. The Hybrid Odd Exponential-Exponential Distribution (HOE-E): In the previous section and as was discussed in (Mahdi et al., 2024), we gave an outline of the method we will use to generate our targeted model, and now we shall proceed with this goal. By substituting ($t \in (0, \infty)$):

$$r(t) = \lambda e^{-\lambda t}, \tag{4}$$

$$F(x) = 1 - e^{-\theta x}, \tag{5}$$

and equation (3) in equation (1) we get:

$$G(x, \lambda, \theta) = \int_0^{\frac{1-e^{-\theta x}}{e^{-\theta x}} \log \frac{1}{e^{-\theta x}}} \lambda e^{-\lambda t}\, dt. \tag{6}$$

By carrying out the integral we have the CDF of the HOE-E distribution:

$$G(x, \lambda, \theta) = 1 - \exp\left\{-\lambda\theta x \frac{1 - e^{-\theta x}}{e^{-\theta x}}\right\}, \tag{7}$$

and a differentiating with respect to x gives the PDF as:

$$g(x, \lambda, \theta) = \lambda\theta \frac{1 + \theta x - e^{-\theta x}}{e^{-\theta x}} \exp\left\{-\lambda\theta x \frac{1 - e^{-\theta x}}{e^{-\theta x}}\right\}, \tag{8}$$

where x takes it values in the interval $(0, \infty)$. We denote a random variable X with PDF given in equation (8) by X~HOE-E(λ, θ) which is a two-parameters distribution, where θ is a scaling parameter and λ is a shape parameter. We can easily find the survival function $S(x, \lambda, \theta)$ and the hazard rate function $H(x, \lambda, \theta)$ of the distribution and they take the form:

$$S(x, \lambda, \theta) = \exp\left\{-\lambda\theta x \frac{1 - e^{-\theta x}}{e^{-\theta x}}\right\}, \tag{9}$$

and

$$H(x, \lambda, \theta) = \lambda\theta \frac{1 + \theta x - e^{-\theta x}}{e^{-\theta x}}, \tag{10}$$

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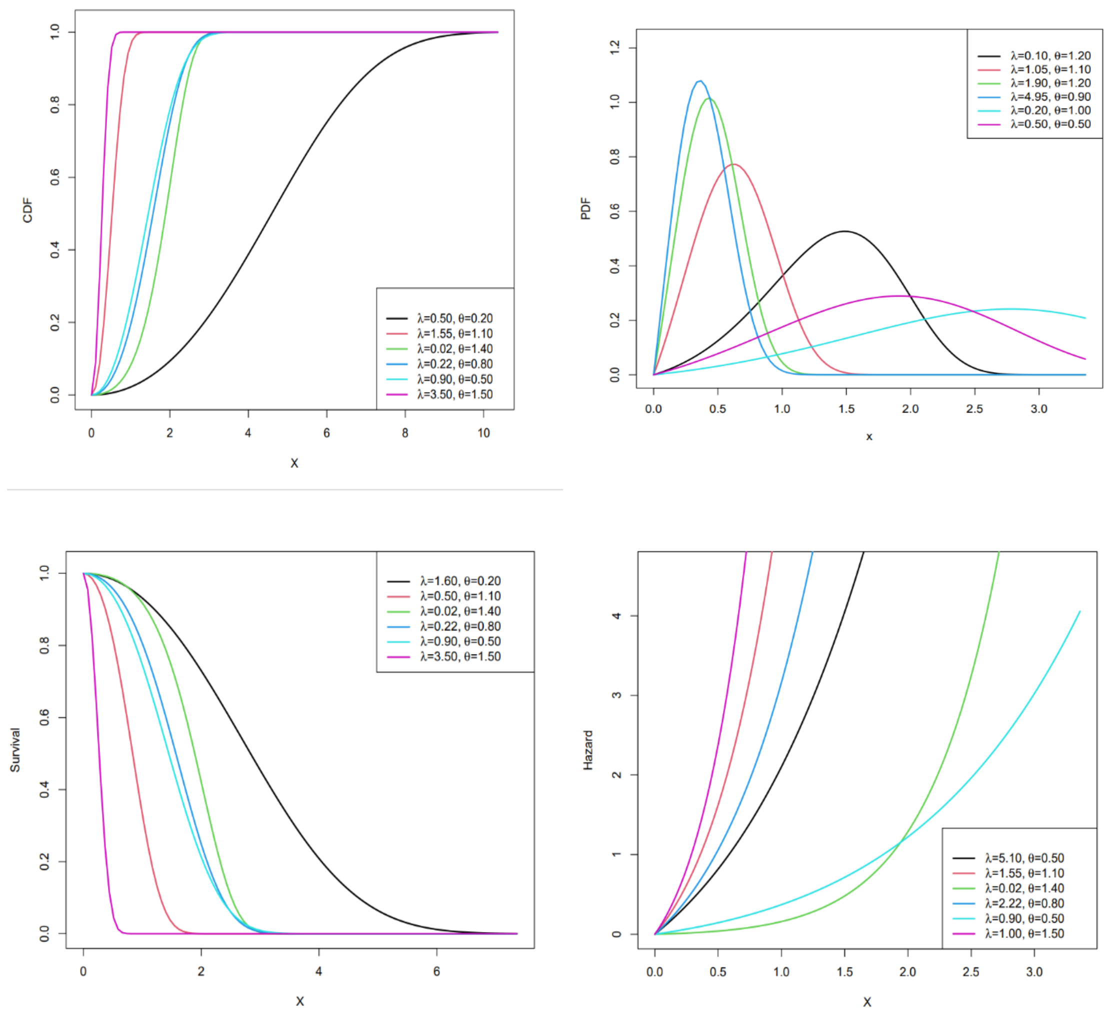

Figure (1): Shapes of the CDF, the PDF, the survival function, and hazard rate function of the HOE-E(λ, θ) distribution. These figures were created by the authors using R program.

respectively. Figure 1 shows a variety of plots of the CDF, PDF, survival, and hazard rate function of the HOE-E(λ, θ) distribution.

3. Statistical and Mathematical Properties of the HOE-E Distribution: In this section, we will derive many statistical and mathematical properties of the HOE-E distribution. The most important derivation of this section is the series expansion of the PDF given in equation (8) which will make many other derivations easy.

3-1. Series expansions of the PDF and CDF: Starting with equation (7), we can put it in the following form using the exponential expansion, (Merovci et al., 2016):

$$G(x,\lambda,\theta) = 1 - \sum_{n=0}^{\infty} \frac{(-1)^n(\lambda\theta)^n}{n!} x^n e^{n\theta x} \left(1 - e^{-\theta x}\right)^n. \qquad (11)$$

Using the generalized Newton expansion, we get:

$$G(x,\lambda,\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(\lambda\theta)^n}{n!} x^n e^{n\theta x} \sum_{m=0}^{\infty} \binom{n}{m} e^{-m\theta x}, \qquad (12)$$

Therefore, the CDF expansion reduces to:

$$G(x,\lambda,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} T_{m,n} x^n e^{-(m-n)\theta x}, \qquad (13)$$

where,

$$T_{m,n} = \frac{(-1)^{n+1}(\lambda\theta)^n}{n!} \binom{n}{m}.$$

Now, if we take the derivative of equation (13) with respect to x, we get the PDF function as:

$$g(x,\lambda,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} T_{m,n}[(n-m)x^n + nx^{n-1}]e^{-(m-n)\theta x}. \qquad (14)$$

As was mentioned earlier, equation (14) will be utilized many times to derive many other properties of interest. In particular, we will prove later that the moments of the HOE-E can be calculated using equation (14) and they are given by a mixing of gamma functions.

3-2. The quantile function: The quantile function $Q(u)$ of the HOE-E distribution can be found by solving for x in the following equation, (Oguntunde et al., 2019):

$$1 - exp\left\{-\lambda\theta x \frac{1 - e^{-\theta x}}{e^{-\theta x}}\right\} = u. \qquad (15)$$

After a few algebraic manipulations, we get:

$$e^{\theta x} \frac{x}{x + \kappa} = 1,$$

where, $\kappa = \frac{1}{\lambda\theta} log \frac{1}{1-u}$. Using Theorem 3 of (Mező & Baricz, 2017), we get the quantile function as:

$$Q(u) = \frac{1}{\theta} W_{-1}(\theta\kappa). \qquad (16)$$

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This equation can be used for theoretical purposes such as finding the mode or can be used in applications to formulate simulation studies.

3-3. Moments and moments generating function: The moments of a random variable X play a pivotal role in statistics theory. The moment μ_p of order p can be found by the formula, (Al-Noor et al., 2022):

$$\mu_p = E(X^p)$$

For our distribution, if we substitute equation (14) in the formula above, we get:

$$\mu_p = \int_0^\infty x^p \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} T_{m,n}[(n-m)x^n + nx^{n-1}]e^{-(m-n)\theta x}$$

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} T_{m,n} \int_0^\infty (n-m)x^{n+p}e^{-(m-n)\theta x} + nx^{n+p-1}e^{-(m-n)\theta x}dx.$$

By using the change of variables, $y = -(m-n)\theta x$, we end up with:

$$\mu_p = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} T_{m,n} \int_0^\infty \frac{1}{(n-m)^{n+p}\theta^{n+p+1}} y^{n+p}e^{-y} + \frac{n}{(n-m)^{n+p}\theta^{n+p}} y^{n+p-1}e^{-y}dy$$

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} T_{m,n}\left[\frac{\Gamma(n+p+1)}{(n-m)^{n+p}\theta^{n+p+1}} + \frac{n\Gamma(n+p)}{(n-m)^{n+p}\theta^{n+p}}\right], \qquad (17)$$

We can find the first and second moments as:

$$\mu_1 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} T_{m,n}\left[\frac{\Gamma(n+2)}{(n-m)^{n+1}\theta^{n+2}} + \frac{n\Gamma(n+1)}{(n-m)^{n+1}\theta^{n+1}}\right], \qquad (18)$$

and

$$\mu_2 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} T_{m,n}\left[\frac{\Gamma(n+3)}{(n-m)^{n+2}\theta^{n+3}} + \frac{n\Gamma(n+2)}{(n-m)^{n+2}\theta^{n+2}}\right], \qquad (19)$$

so the variance is given by

$$Var(X) = \mu_2 - \mu_1^2$$
$$= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} T_{m,n} \left[\frac{\Gamma(n+3)}{(n-m)^{n+2}\theta^{n+3}} + \frac{n\Gamma(n+2)}{(n-m)^{n+2}\theta^{n+2}} \right]$$
$$- \left\{ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} T_{m,n} \left[\frac{\Gamma(n+3)}{(n-m)^{n+2}\theta^{n+3}} + \frac{n\Gamma(n+2)}{(n-m)^{n+2}\theta^{n+2}} \right] \right\}^2 \quad (20)$$

The moment generating function $M_X(t)$ of a random variable X is, by definition, given by:

$$M_X(t) = E(e^{tX})$$

To find this function for our model, we substitute equation (14) in the formula above to get:

$$M_X(t) = \int_0^{\infty} e^{tx} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} T_{m,n} [(n-m)x^n + nx^{n-1}] e^{-(m-n)\theta x}.$$

This last equation simplifies to:

$$M_X(t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} T_{m,n} \int_0^{\infty} (n-m)x^n e^{-[(m-n)-t]\theta x} + nx^{n-1} e^{-[(m-n)-t]\theta x} dx.$$

Changing the variable of integration to $y = [(m-n)-t]\theta x$, we can put it in the form:

$$M_X(t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} T_{m,n} \int_0^{\infty} \frac{(n-m)}{(m-n-t)^{n+1}\theta^{n+1}} y^n e^{-y} + \frac{n}{(m-n-t)^n \theta^n} y^{n-1} e^{-y} dy$$
$$= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} T_{m,n} \left[\frac{(n-m)\Gamma(n+1)}{(m-n-t)^{n+1}\theta^{n+1}} + \frac{n\Gamma(n)}{(m-n-t)^n \theta^n} \right]. \quad (21)$$

3-4 The incomplete moments: For a random variable X the incomplete moment of order p is denoted by $m_p(\omega)$ and computed via the formula, (Khaleel et al., 2017):

$$m_p(\omega) = \int_{-\infty}^{w} x^p f(x)dx,$$

where, $f(x)$ is the PDF function of X. Substituting equation (14) in the formula above and using the change of variable $y = -(m-n)\theta x$, we get:

$$m_p(\omega) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} T_{m,n} \int_0^{(n-m)\theta\omega} \frac{1}{(n-m)^{n+p}\theta^{n+p+1}} y^{n+p} e^{-y} + \frac{n}{(n-m)^{n+p}\theta^{n+p}} y^{n+p-1} e^{-y} dy$$

$$= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} T_{m,n} \left[\frac{\gamma(n+p+1,(n-m)\theta\omega)}{(n-m)^{n+p}\theta^{n+p+1}} + \frac{n\gamma(n+p,(n-m)\theta\omega)}{(n-m)^{n+p}\theta^{n+p}} \right], \quad (22)$$

where, $\gamma(s,t)$ is the lower incomplete gamma function.

3-5 Shapes of PDF and hazard rate function: We can study the shape of the PDF and the hazard rate function of the HOE-E distribution by studying the first derivatives of these functions. The derivative of the PDF function is given by, (Ahmed et al., 2021):

$$\frac{d}{dx} g(x,\lambda,\theta) = \left[\left(e^{\theta x} + \theta x e^{\theta x} - 1\right)\left(\theta e^{\theta x} - \lambda\theta^2 x e^{\theta x} - \lambda\theta e^{\theta x}\right) + \left(2\theta e^{\theta x} + \theta^2 x e^{\theta x}\right)\right]\lambda\theta\, e^{-\lambda\theta x\left(e^{\theta x}-1\right)},$$

Therefore, the critical points of the PDF are the roots of the equation:

$$\left(e^{\theta x} + \theta x e^{\theta x} - 1\right)\left(\theta e^{\theta x} - \lambda\theta^2 x e^{\theta x} - \lambda\theta e^{\theta x}\right) + \left(2\theta e^{\theta x} + \theta^2 x e^{\theta x}\right) = 0. \quad (23)$$

If x_0 is a root of this equation, then it is a local minimum if the function is decreasing for $x < x_0$ and increasing for $x > x_0$, while it is a local maximum if the function is decreasing for $x > x_0$ and increasing for $x < x_0$. The first derivative of the hazard rate function takes the simple form:

$$\frac{d}{dx} H(x,\lambda,\theta) = (2\lambda\theta^2 + \lambda\theta^3 x)e^{\theta x},$$

which is evidently always increasing for $x > 0$ as was suggested in Figure 1. It is worth mentioning that the root of equation (23) can only be approximated by some suitable numerical method.

3-6. Order statistics: A statistical sample's rth order statistic is equal to its rth-smallest value which is one of the most essential techniques in non-parametric statistics and inference, along with rank statistics.

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Let $X_1, X_2, \dots, X_n$ be a random sample taken from some random variable X. The CDF function of the rth random variable is given by, (Al Abbasi et al., 2019):

$$F_r(x) = \sum_{j=r}^{n} \binom{n}{j} [F(x)]^j [1 - F(x)]^{n-j},$$

and the corresponding PDF function is:

$$f_r(x) = \frac{n!}{(r-1)!\,(n-r)!} f(x)[F(x)]^{r-1}[1 - F(x)]^{n-r},$$

where, $F(x)$ and $f(x)$ are the CDF and PDF of the random variable X. In our case, the CDF function of the rth random variable is:

$$F_r(x) = \sum_{j=r}^{n} \binom{n}{j} \left[1 - exp\left\{-\lambda\theta x \frac{1 - e^{-\theta x}}{e^{-\theta x}}\right\}\right]^j exp\left\{-\lambda\theta(n - j) x \frac{1 - e^{-\theta x}}{e^{-\theta x}}\right\}, (24)$$

and, the PDF function becomes:

$$f_r(x) = \frac{n!}{(r-1)!\,(n-r)!} \lambda\theta \frac{1 + \theta x - e^{-\theta x}}{e^{-\theta x}} \left[1 - exp\left\{-\lambda\theta x \frac{1 - e^{-\theta x}}{e^{-\theta x}}\right\}\right]^{r-1} \times exp\left\{-\lambda\theta(n - r + 1) x \frac{1 - e^{-\theta x}}{e^{-\theta x}}\right\}. \quad (25)$$

4. **Parameters estimation:** Here, we will derive the expressions for the maximum likelihood estimators for the distribution under consecration. Let $X_1, X_2, \dots, X_n$ be a random sample of size n, taken from the HOE-E distribution. The log-likelihood function of the HOE-E distribution is, (Hassan et al., 2021):

$$l_n(\lambda, \theta) = n \log \lambda + n \log \theta + \sum_{j=1}^{n} \log\left(e^{\theta x_j} + \theta x_j e^{\theta x_j} - 1\right) - \lambda\theta \sum_{j=1}^{n} x_j\left(e^{\theta x_j} - 1\right).$$

We can find the partial derivatives of this last equation as:

$$\frac{\partial}{\partial\lambda} l_n(\lambda,\theta) = \frac{n}{\lambda} - \theta\sum_{j=1}^{n} x_j\left(e^{\theta x_j} - 1\right), \tag{26}$$

and

$$\frac{\partial}{\partial\theta} l_n(\lambda,\theta) = \frac{n}{\theta} + \sum_{j=1}^{n} \frac{\left(2x_j + \theta x_j^2\right)e^{\theta x_j}}{e^{\theta x_j} + \theta x_j e^{\theta x_j} - 1} - \lambda\sum_{j=1}^{n}\left(\left(\theta x_j^2 + x_j\right)e^{\theta x_j} - x_j\right). \tag{27}$$

By setting these two equations equal to zero and solving the resulting system, we obtain the maximum likelihood estimator for the HOE-E distribution. Note that these two equations are highly non-linear and thus they require a numerical method to find their roots.

5. **Simulation study:** This section uses a Monte Carlo simulation process under the square error loss function to estimate the parameters of the HOE-E distribution using the MLE method and the R programming language. Using HOE-E data, 3,000 random samples are created for Monte-Carlo experiments, where x represents the HOE-E lifespan for a range of real parameter values and sample sizes n as (30,50,100,200). The best estimating techniques reduce the root mean square error (RMSE) and bias of the estimators. The MLE method's simulation results are compiled in Tables 1 and 2. We quantify the accuracy of these guesses using the Mean, the RMSE, and the bias. These tables demonstrate that when we raise the sample size n, we get better results. We also see that bias and RMSE are both exactly zero for a certain set of parameter selections. The parameters Means has no clear behaviors.

6. **Applications:** In this section, we will give an application of the HOE-E distribution on a real-life data set. The criteria used to judge the efficiency of the models are the Akaike information criterion (AIC), the Corrected Akaike information criterion (CAIC), the Bayesian information criteria (BIC), the Hannan-Quinn information criterion (HQIC), the Cramer-von Mises statistic (W) and the Anderson-Darling statistic (A).

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Table (1): MLE estimation for the parameters of the HOE-E distribution using a Monte Carlo method with θ = 2.5.

θ	λ	n		Mean	RMSE	Bias
2.5	0.005	30	θ	0.0084	0.0095	0.0034
			λ	2.3926	0.2909	-0.1073
		50	θ	0.0070	0.0065	0.0020
			λ	2.4172	0.2368	-0.0827
		100	θ	0.0059	0.0030	0.0009
			λ	2.4274	0.1672	-0.0725
		200	θ	0.0053	0.0017	0.0003
			λ	2.4530	0.1156	-0.0469
2.5	0.03	30	θ	0.0583	0.1086	0.0283
			λ	2.4216	0.5247	-0.0783
		50	θ	0.0447	0.0449	0.0147
			λ	2.4538	0.4305	-0.0461
		100	θ	0.0374	0.0233	0.0074
			λ	2.4713	0.3216	-0.0286
		200	θ	0.0336	0.0139	0.0036
			λ	2.4863	0.2290	-0.0136
2.5	0.08	30	θ	0.2850	5.4634	0.2050
			λ	2.4795	0.6802	-0.0204
		50	θ	0.1225	0.1655	0.0425
			λ	2.5091	0.5568	0.0091
		100	θ	0.0992	0.0702	0.0192
			λ	2.4984	0.4055	-0.0015
		200	θ	0.0885	0.0401	0.0085
			λ	2.5046	0.2890	0.0046
2.5	0.1	30	θ	0.4259	8.8161	0.3259
			λ	2.4724	0.7149	-0.0275
		50	θ	0.1573	0.2408	0.0573
			λ	2.5186	0.5965	0.0186
		100	θ	0.1255	0.0927	0.0255
			λ	2.4965	0.4225	-0.0034
		200	θ	0.1113	0.0516	0.0113
			λ	2.5036	0.3026	0.0036

Source: This table was created by the authors using R program.

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Table (2): MLE estimation for the parameters of the HOE-E distribution using a Monte Carlo method with θ = 1.1.

θ	λ	n		Mean	RMSE	Bias
1.1	0.005	30	θ	0.0070	0.0057	0.0020
			λ	1.1023	0.0777	0.0023
		50	θ	0.0055	0.0016	0.0005
			λ	1.1143	0.0443	0.0143
		100	θ	0.0057	0.0016	0.0007
			λ	1.0982	0.0280	-0.0017
		200	θ	0.0050	0	0
			λ	1.1000	0	0
1.1	0.03	30	θ	0.0551	0.1136	0.0251
			λ	1.0842	0.2304	-0.0157
		50	θ	0.0412	0.0371	0.0112
			λ	1.0985	0.1910	-0.0014
		100	θ	0.0356	0.0228	0.0056
			λ	1.1018	0.1453	0.0018
		200	θ	0.0330	0.0139	0.0030
			λ	1.0985	0.1030	-0.0014
1.1	0.08	30	θ	0.3133	6.0228	0.2333
			λ	1.0959	0.2874	-0.0040
		50	θ	0.1144	0.1241	0.0344
			λ	1.1142	0.2453	0.0142
		100	θ	0.0958	0.0698	0.0158
			λ	1.1106	0.1795	0.0106
		200	θ	0.0871	0.0404	0.0071
			λ	1.1079	0.1316	0.0079
1.1	0.1	30	θ	0.4713	9.8935	0.3713
			λ	1.0975	0.3098	-0.0024
		50	θ	0.1490	0.1729	0.0490
			λ	1.1080	0.2538	0.0080
		100	θ	0.1205	0.0929	0.0205
			λ	1.1160	0.1953	0.0160
		200	θ	0.1097	0.0523	0.0097
			λ	1.1075	0.1383	0.0075

Source: This table was created by the authors using R program.

The data set used here is the plasma concentration data from pediatric studies used previously in (Kindblom et al., 2009). The data set reads: 1.50, 0.94, 0.78, 0.48, 0.37, 0.19, 0.12, 0.11, 0.08, 0.07, 0.05, 2.03, 1.63, 0.71, 0.70, 0.64, 0.36, 0.32, 0.20, 0.25, 0.12, 0.08, 2.72, 1.49, 1.16, 0.80, 0.80, 0.39, 0.22, 0.12, 0.11, 0.08, 0.08, 1.85, 1.39, 1.02, 0.89, 0.59, 0.40, 0.16, 0.11, 0.10, 0.07, 0.07, 2.05, 1.04, 0.81, 0.39, 0.30, 0.23, 0.13, 0.11, 0.08, 0.10, 0.06, 2.31, 1.44, 1.03, 0.84, 0.64, 0.42, 0.24, 0.17, 0.13, 0.10, 0.09. We compare the performance of the HOE-E with other distributions Like: the [0, 1] truncated exponentiated exponential exponential distribution

([0, 1] TEEE) (Ismael, 2021,), the Beta exponential distribution (BeE) (Nadarajah & Kotz, 2006). Kumaraswamy exponential distribution (KuE) (Cordeiro & de Castro, 2011), Exponentiated Generalized exponential distribution (EGE) (Okagbue et. al., 2017). Weibull exponential distribution (WeE) (Bourguignon et. al., 2014), Gompertz exponential distribution (GoE) (Bashir & QURESHİ, 2022), Marshal Olkin exponential distribution (MOE) (Marshall & Olkin, 1997), and the exponential distribution (E).

The results are summarized in Table 3 below. The first row in this table confirms that the HOE-E distribution has the smallest values for all measures used in the study, hence our model is considered the best choice in this situation. Figure 2 also demonstrates various analysis aspects of this study.

Table (3): The performance of the HOE-E distribution compared to the other models for the plasma concentration data set.

Distributions	Estimates	AIC	CAIC	BIC	HQIC	W	A
HOE-E	λ =0.1004 θ =17.0351	59.44	59.63	63.82	61.17	0.19	1.26
[0, 1] TEEE	λ =1.8411 θ =1.3102 γ =1.2759	65.96	66.34	72.52	68.55	0.23	1.49
BeE	λ =0.9783 θ =2.2557 γ =0.7357	68.73	69.12	75.30	71.33	0.25	1.62
KuE	λ =0.9557 θ =5.8691 γ =0.2664	68.59	68.98	75.16	71.19	0.25	1.61
EGE	λ =1.3555 θ =0.9836 γ =1.2331	68.74	69.13	75.31	71.34	0.25	1.62
WeE	λ =0.9546 θ =0.0531 γ =0.0918	68.51	68.89	75.08	71.10	0.25	1.60
GoE	λ =1.7152 θ =0.1144 γ =1.0608	68.46	68.85	75.03	71.06	0.25	1.60
MOE	λ =0.5919 θ =1.2949	65.16	65.35	69.54	66.89	0.23	1.50
E	λ =1.6897	64.75	64.82	66.94	65.62	0.25	1.62

Source: This table was created by the authors using R program.

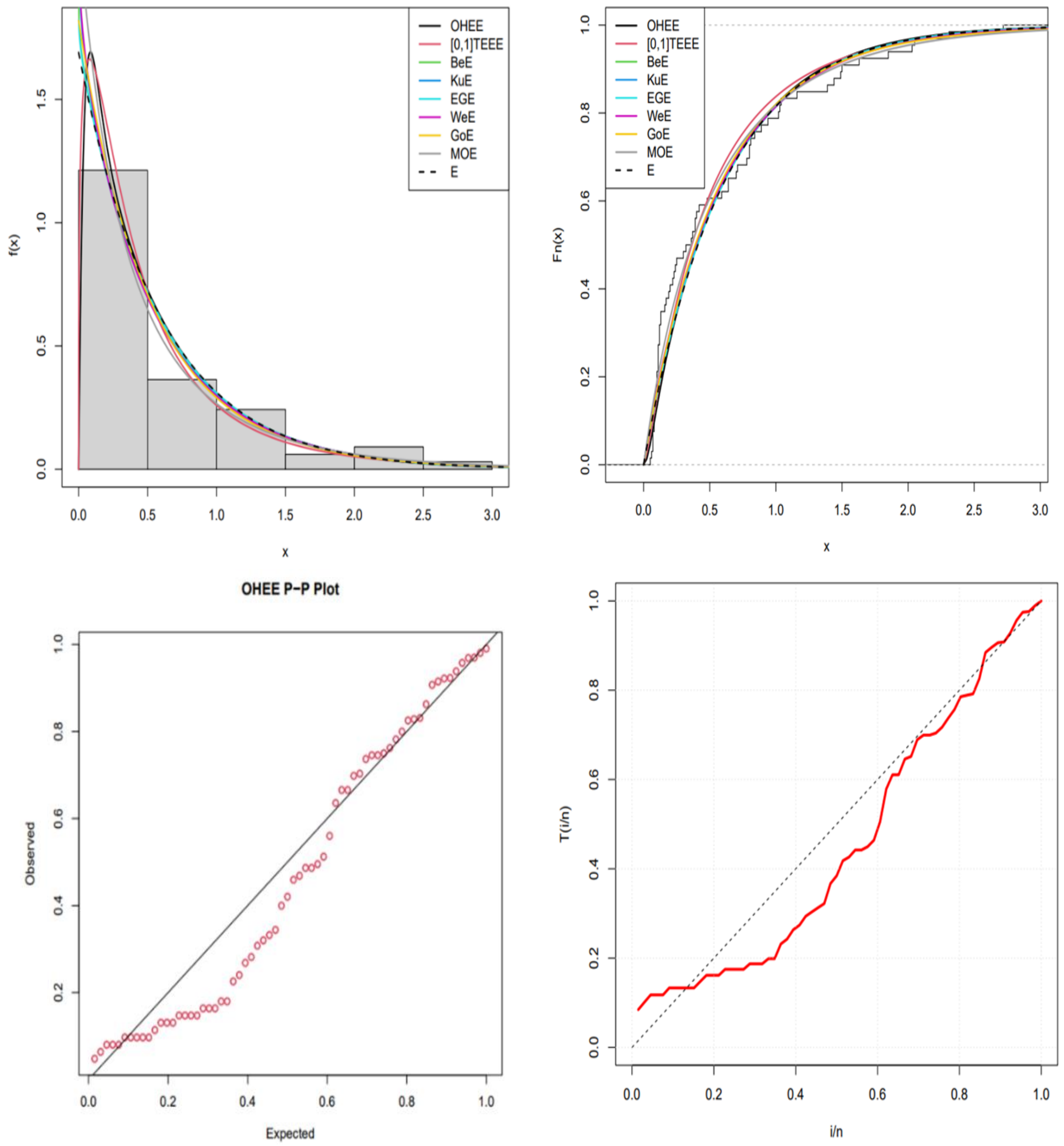

Figure (1)**:** The empirical histogram, the empirical CDF, the P-P plot, and the expected of the HOE-E distribution for the plasma concentration data set.

Source: These figures were created by the authors using R program.

7. Conclusions: The study examines a new distribution called the hybrid odd exponential-exponential distribution or HOE-E for short. It studies various statistical and mathematical properties of this new distribution such as series expansion, quantile function, moments and their generating function, incomplete moments, and order statistics. It uses the maximum likelihood method to estimate the parameters of the new model and includes a simulation study of this estimators as well. An application of the HOE-E distribution on a data set from the health sector reveals its superiority over many other distributions.

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