Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

Strongly Pure Ideals And Strongly Pure Sub-modules

Nada Khalid Abdullah

University of Tikrit / Department of Mathematics

wathikomar@yahoo.com

Received date: 17 / 9 / 2013 **Accepted date: 17 / 11 / 2014**

ABSTRACT

Let R be a ring with unity , and let M be an unitary R-module . In this work we present strongly pure ideal (sub module) concept as a generalization of pure ideal (sub module) . Also we generalize some properties of strongly pure ideal (sub module) . And we study strongly regular ring (R-module) .

Keywords : *pure ideal , strongly pure ideal , local ring , pure sub-module , strongly pure sub module , flat module , superfluous sub module .*

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

المثاليات النقية بقوة والمقاسات الجزئية النقية بقوة

ندى خالد عبد الله

جامعة تكريت / قسم الرياضيات

wathikomar@yahoo.com

تاريخ استلام البحث: 17 / 9 / 2015 **تاريخ قبول البحث: 17 / 11 / 2014**

الملخص

لتكن R حلقة بمحايد . M مقاس بمحايد معرف على الحلقة R . تناول البحث الحالي مفهوم المثاليات النقية بقوة كتعميم للمثاليات النقية ، حيث عرف الباحث المثالي I بانه مثالي نقي بقوة اذا لكل عنصر I ∈ × يوجد عنصر اولي p∈I بحيث . × = p.× وكذلك دراسة بعض الخصائص وتعميم بعض القضايا من المثاليات النقية الى المثاليات النقية بقوة . بالإضافة الى ذلك عرف الباحث مفهوم المقاس الجزئي النقي بقوة كتعميم للمقاس الجزئي النقي ، حيث عرف المقاس الجزئي N من المقاس M بانه نقي بقوة اذا وجد مثالي اولي P في الحلقة R بحيث يحقق PN = PM ∩ N بالإضافة الى تعميم بعض المميزات للمقاس الجزئي النقي الى المقاس الجزئي النقي بقوة.

الكلمات الدالة : المثالي النقي ، المثالي النقي بقوة ، الحلقة المحلية ، المقاس الجزئي النقي ، المقاس الجزئي النقي بقوة ، المقاس المسطح ، المقاس الجزئي الصغير .

1.INTRODUCTION

Let R be a ring with unity , and let M be a unitary R-module . Let L and K are sub modules of M , the residual of K by L , denoted by [K:L] is the set of all x in R such that xL⊆K .The annihilator of M , denoted by ann M , is [0:M] . For each m∈M , the annihilator of m , denoted by ann(m) , is [0:Rm] . Let M be a multiplication R-module and N is a sub module of M . Then N=IM for some ideal I of R . Note that I⊆[N:M] and hence N=IM⊆[N:M]M⊆N , so that N=[N:M]M . Also K is a multiplication sub module of M if and only if N∩K=[N:K]K ,for every sub module N of M , [24] . Finitely generated faithful multiplication modules are cancellation modules , [29] . We introduce the concept of idempotent sub module as follows .

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

A sub module N of M is an idempotent if and only if N=[N:M]N . If M is a finitely generated faithful multiplication R-module , then N is an idempotent sub module of M if and only if [N:M] is an idempotent ideal of R , [23] . Brown-McCoy(1950) in [9] define regular rings . And Fieldhouse (1969) generalize regular rings to regular modules in [16] .Ware (1971) in [30] and Zelmanowilz (1972) in [31] also study regular modules . E.Anderson and Fuller [5] called the sub module N is a pure of M if IN=N∩IM for every ideal I of R . Although R. Ribenboim [27] define N to be pure in M if rM∩N=rN for each r∈R .

This work include four sections . In section one , we introduce strongly pure ideals concept as generalization of pure ideals . An ideal I of R is said to be strongly pure , if for each x∈I there exists a prime element p∈I such that x=xp . Pure ideal is deferent from strongly pure ideal , thus we give examples which indicate that two classes are different . However , we put some conditions under which the two classes are equivalent as we see in (Prop.2-4) that R is factorial ring , such that each non zero and non unit element of R is irreducible. Moreover , we give some properties of strongly pure ideals as the intersection of two ideals is strongly pure ideal if one of them is strongly pure ideal (Prop.2-7) . In (Prop.2-9) prove that if the direct summand of two ideals is strongly pure , then one of them is strongly pure . If I is strongly pure ideal of R , then I_A is strongly pure ideal of R_A (Prop.2-10). Also , we present in section two strongly regular ring that a ring R is called strongly regular if and only if for each x∈R ,there exists prime element p∈R such that x=xpx.

In section three , we introduce strongly pure sub module concept . A sub module N of an R-module M is called strongly pure , if there exists prime ideal P of R , such that N∩Mp=Np , for each p∈P . Pure su bmodule is deferent from strongly pure sub module , thus we give examples which indicate that two classes are different . However , we put some conditions under which the two classes are equivalent (Prop.4-3) . And we study some properties of strongly pure sub modules . Also , we get the main result of strongly pure sub modules in (Th.4-13) .

Finally in section four , we present strongly regular module concept as generalization of regular module . Also generalize characterization and some properties of regular module to strongly regular module .

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

2.STRONGLY PURE IDEALS

In this section , we introduce a generalization for pure ideal concept namely strongly pure ideal .

Note: From now on , strongly pure ideal means right strongly pure ideal unless otherwise stated . First we recall definitions of pure ideals and prime ideals .

Recall that an ideal I of a ring R is said to be right (left) pure if for each x∈I , there exists y∈I such that x=x.y (x=y.x) , [2] & [17] .

Definition : An ideal I of a ring R is called strongly right (left) pure if for each x∈I , there exists a prime element p∈I such that x=x.p (x=p.x) .

Remarks and Examples :

1- Each strongly pure ideal is pure . But the converse is not true in general , as we see in the following example :

$(\bar{3})=\{\bar{0},\bar{3}\}$ is a strongly pure ideal of a ring Z_6 , since $\bar{0}=\bar{0}.\bar{3}$ and $\bar{3}=\bar{3}.\bar{3}$. But $(\bar{2})=\{\bar{0}, \bar{2}, \bar{4}\}$ is not strongly pure ideal of a ring Z_6 , since there is no prime element p ∈ $(\bar{2})$ such that $\bar{2}=\bar{2}.p$.

2- If I is strongly pure ideal of a ring R , then JI=J∩I , for each ideal J of R .

Recall that an ideal I of a ring R is called idempotent if $I^2=I$, [20] .

3- Each ideal generated by prime idempotent element is strongly pure ideal .

Proof : Let I=(p) be an ideal generated by prime element p , such that $p=p^2$. If x∈I , there exists r∈R such that x=rp , implies $x=rp=rp^2=rpp=xp$. Therefore I is strongly pure ideal of a ring R . In general

4- If I is an ideal of a ring R generated by a prime idempotent elements p_i , where i = 1 , 2 , --- , n . Then I is a strongly pure ideal of R .

Proof : Let I be an ideal generated by prime idempotent elements p_i , where i=1,2,---,n .And let x∈I ,there exists r_i∈R such that $x=\sum_{i=1}^{n} r_i p_i=\sum_{i=1}^{n} r_i p_i^2=\sum_{i=1}^{n} r_ip_ip_i$. Let $p_i=1-\prod_{i=1}^{n}(1-p_i)$,

implies that $xp=\sum_{i=1}^{n} r_ip_i(1-\prod_{i=1}^{n}(1-p_i))=\sum_{i=1}^{n} r_ip_i-\sum_{i=1}^{n} r_ip_i\prod_{i=1}^{n}(1-p_i)$.

$(1-p_i)=x-\prod_{i=1}^{n}(1-p_i)(\sum_{i=1}^{n} r_i p_i-\sum_{i=1}^{n} r_i p_i)=x$ .Therefore I is strongly pure ideal of R .

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

5- Each strongly pure ideal is idempotent .

Proof : Let I be a strongly pure ideal of R , and let x∈I . Then there exists a prime element p∈I , such that x=x p . But x p∈I.I , thus $x\in I^2$. Hence $I\subseteq I^2$, and it is clear that $I^2\subseteq I$, implies $I=I^2$. Therefore I is an idempotent ideal of R . Recall that an element a∈R is called irreducible if it is non zero non unit and whenever a=b c where b,c∈R , then either b or c is a unit element of R , [18]. An integral domain R is a factorial ring if there is a set S of non zero non unit of R such that every non zero element of R can be written uniquely in the form ua_1a_2---a_k ,where u is a unit of R and $a_1,a_2,\text{---},a_k\in S$. Except for the order in which the factors are written , [20] The following lemma taken from [20] .

Lemma : Let R be a factorial ring , and let S be a set whose existence in required in the definition . Then every irreducible element of R is prime , every element of S is prime , and every prime of R is the product of unit of R and an element of S . The following result gives some conditions to get strongly pure ideals from pure ideals .

Proposition : Let R be a factorial ring , and let I be an ideal of R , such that each non zero and non unit element of R is irreducible . Then I is strongly pure ideal if and only if I is pure ideal .

Proof : Let I be a pure ideal of R , and let x∈I , there exists y∈I such that x=xy . Since y∈R is irreducible element of R , so it is prime in I (Lemma2-3) . Therefore I is strongly pure ideal of R .

The converse is clear . In general right strongly pure ideal is not left , and left strongly pure ideal is not right . As the following example .

Example : Let $R=\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}: a,b,d\in Z_2\}$ be the ring of 2×2 matrices over the field Z_2 . Then there are two ideals I and J namely $I=\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\}$, $J=\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\}$. Clearly I is right strongly pure ideal and J is not left strongly pure ideal . Recall that a ring R is said to be reduced if R contains no non zero nilpotent element , [22] . The following proposition gives condition to get right strongly pure ideal from left strongly pure ideal ,

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

Proposition : Let R be a reduced ring , and let I be any ideal of R . Then I is a right strongly pure if and only if I is a left strongly pure .

Proof : Let I be a right strongly pure ideal of R and let x∈I , there exists prime element p∈I such that x=xp . Since R is a reduced ring , then $(x-xp)^2=0$, so x–px=0 , thus x=px . Therefore I is a left strongly pure ideal of R . The converse is similar , hence it is omitted . The following proposition gives some property of strongly pure ideals .

Proposition :

1-Let I and J are two ideals of a ring R . If I is strongly pure ideal of R , then I∩J is a strongly pure ideal of R .

2-Let I and J are two ideals of a ring R such that I⊆J . If I∩J is strongly pure ideal of R , then I is strongly pure ideal of R .

Proof : Clear .

The following corollaries is immediate consequence of previous proposition .

Corollary : Let I and J are two strongly pure ideals of R , then I∩J is also strongly pure ideal of R .

Corollary : Let I and J are two ideals of R , then I is strongly pure ideal if and only if I∩J is strongly pure ideal of R . The following proposition gives another property of strongly pure ideals .

Proposition : Let I and J are two ideals of a ring R , if I⊕J is strongly pure ideal of R . Then either I or J is strongly pure ideal of R .

Proof : Let x∈I and x∈J , implies x+y∈I⊕J . Since I⊕J is strongly pure ideal of R , there exists prime element p∈I⊕J , where p=p+0∈I⊕J , such that x+y=(x+y)p=(x+y)(p+0)=xp+yp∈I⊕J . Since yp∈I∩J , and I∩J={0} , hence yp=0 . Thus x=xp∈I . Therefore I is strongly pure ideal of R . And if p=0+p∈I⊕J , then by the same method we can get J is strongly pure ideal of R.

Next , from (prop. 2–4) and (prop. 2–10) we can get the following result .

Corollary : Let R be a factorial ring , and let I and J are two ideals of R , such that I⊕J is strongly pure ideal of R . Then I and J are strongly pure ideals of R .

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

Proposition : If I is a strongly pure ideal of a ring R , then I_A is strongly pure ideal of a ring R_A , for each maximal ideal A of R .

Proof : Let $a/r \in I_A$, where $a \in I$ & $r \in R-A$. Since I strongly pure ideal of R , there exists prime element $p \in I$ such that a=a p , implies that a/r=a.p/r=a/r.p/1 , where p/1 is prime in I_A .Thus I_A is strongly pure ideal of R_A .

Remark: Let I be strongly pure ideal of R ,such that $I \subseteq J(R)$ .Then I={0} .

Proof : Let $x \in I$, since I is strongly pure ideal of R , there exists a prime element $p \in I$, such that x=xp , implies x(1–p)=0 . And since $I \subseteq J(R)$, then $p \in J(R)$, hence x=0 , so I=(0) . Recall that a ring R is called local ring if R is commutative , with unique maximal ideal , [11] .

Proposition : Let I be an ideal of commutative ring R . Then I is strongly pure ideal of R if and only if $I_A=(0)$ or $I_A=R_A$ for each maximal ideal A of R .

Proof : If I is a strongly pure ideal of R , then I_A is strongly pure ideal of R_A (Prop.2–12) . If $I_A \neq R_A$, implies R_A is local ring , hence $I_A \subseteq J(R_A)$. Therefore $I_A=(0)$ (Remark 2–13) .

3.STRONGLY REGULAR RINGS

Let R be a commutative ring with non zero identity . Before we state some results , we give important definition which will be used here and later . Recall that a ring R is regular if and only if for each $x \in R$, there exists $y \in R$ such tha x = xyx , [26] .

Now, we present strongly regular ring concept as generalization of regular ring

Definition : A ring R is called strongly regular if and only if for each $x \in R$, there exists a prime element $p \in R$ such that x = xpx .

Also: A ring R is called strongly regular if each element of R is strongly regular

Remarks and Examples :

1- Z_4 is strongly regular ring , but Z_6 is not strongly regular ring .

2- Each strongly regular ring is a regular ring . But the converse is not true in general , as we see in the following example .

Z_6 is a regular ring , but it is not strongly regular ring . Recall that right duo ring if all whose right ideals are two sided ideals .A left duo ring is similarly defined .A duo ring is right and left duo ring ,[10] . The following results follow from (Prop.2-4) and from [14 , Prop.2-24] .

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

3- Let R be right duo factorial ring ,such that each ideal of R is irreducible. Then R is strongly regular if and only if each ideal of R is left strongly pure .

4- Let R be a factorial ring , such that each ideal of R is irreducible . Then R is strongly regular ring if and only if each ideal of R is strongly pure .

In this section we gives characterizations and some useful properties of strongly regular ring .

Proposition : Let R be a factorial commutative ring , such that each ideal of R is irreducible . Then R is strongly regular ring if and only if R_A is field , for each maximal ideal A of R .

Proof : Let R_A be a field , for each maximal ideal A of R , and let I be an ideal of R . Then I_A is an ideal of R_A , since R_A is field , then R_A has no proper ideal . Thus either I_A=(0) or $I_A=R_A$ [14] . Implies that I is strongly pure ideal of R (Prop.2-15) . Therefore R is strongly regular ring , (Remark 3-3(5)) .

Conversely ; let J be an ideal of R , there exists an ideal I of R such that $J=I_A$. But R is strongly regular ring , thus I is strongly pure ideal of R . Hence either I_A=(0) or $I_A=R_A$ (Prop.2-15) , its` mean that R_A has no proper ideal . Therefore R_A is field .

We end this section by the following proposition .

Proposition : Let R_1 and R_2 are two rings , such that $R_1 \oplus R_2$ is strongly regular ring . Then either R_1 or R_2 is strongly regular ring .

Proof : Let $r_1 \in R_1$ and $r_2 \in R_2$, implies $r_1+r_2 \in R_1 \oplus R_2$. Put $x=r_1+r_2$, since $R_1 \oplus R_2$ is strongly regular ring , there exists prime element $p=p+0 \in R_1 \oplus R_2$, such that $x=xpx=(r_1+r_2)\, p\, (r_1+r_2)= r_1 p r_1+r_1 p r_2+r_2 p r_1+r_2 p r_2$. But $r_1 p r_2 , r_2 p r_1 \in R_1 \cap R_2$, and $R_1 \cap R_2 =(0)$. Thus $x=r_1 + r_2 = r_1 p r_1+ r_2 p r_2$, implies that $r_1=r_1 p r_1 \in R_1$. Therefore R_1 is strongly regular ring . And if $0+p \in R_1 \oplus R_2$, by the same method we can get $r_2 = r_2 p r_2 \in R_2$, hence R_2 is strongly regular ring .

4.STRONGLY PURE SUB MODULES

Let M be a unitary R-module , and let R be a commutative ring with unity , such that each ideal of R is a proper ideal .Recall that a sub module N of an R-module M is called pure if $N \cap Mr = Nr$ for each $r \in R$, [18].Also ; we define N is pure sub module if $N \cap MI = NI$, for each ideal I of a ring R , [18] Now , we start this section by the following definition.

Definition : A sub module N of an R-module M is called strongly pure , if there exists prime

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

ideal P of a ring R , such that N∩Mp=Np , for each p∈P . Also ; we define N is strongly pure sub module if there exists a prime ideal P of a ring R , such that N∩MP=NP .

Remarks and Examples :

1- $(\bar{3})$ is a strongly pure sub module of Z_6 -module . But $(\bar{2})$ is not strongly pure sub module of Z_6 -module .

2- Each strongly pure sub module is pure . But the converse is not true in general , as we see in the following example $(\bar{3})$ is pure sub module of Z_{12}-module . But it is not strongly pure sub module of Z_{12} -module .Next , we study the relation between pure sub module and strongly pure sub module by the following proposition .

Proposition : Let M be an R-module , and R/I is an integral domain , for each ideal I of R . Then N is strongly pure sub module of M if and only if N is pure sub module of M .

Proof : Let N be a pure sub module of M , thus N∩MI=NI for each ideal I of R . Since R/I is an integral domain , hence I is a prime ideal , [21] . Therefore N is strongly pure sub module of M . The converse is clear . The following remark gives characterization of strongly pure sub module .

Remark : Let M be an R-module , and let R/I be an integral domain , for each ideal I of R . If N is a sub module of M , then the following statements are equivalent :

1- N is strongly pure sub module .

2- N ∩ MI = NI for each ideal I of a ring R .

3- N ∩ MI = NI for each finitely generated ideal I of R . [14 , Prop.(2-8)] .

The following propositions show some properties of strongly pure sub modules .

Proposition : Let N be a sub module of an R-module M , and let L be a sub module of N . If N is pure sub module of M , and L is strongly pure sub module of N , then L is strongly pure sub module of M .

Proof : Since L is strongly pure sub module of N , then there exists a prime ideal P of R , such that L∩NP=LP . And since N is pure sub module of M , then N∩MP=NP , implies that L∩N ∩MP=LP . And since L⊆N , then L∩N=L , implies L∩MP=LP . Therefore L is strongly pure sub module of M . The following corollary follow from the previous proposition.

Corollary : Let N and L are sub modules of an R-module M . If N is a pure sub module of M , and N∩L is strongly pure sub module of N . Then N∩L is strongly pure sub module of M .

Proposition : Let M be an R-module and let R/I be an integral domain for each ideal I of R . Then every direct summand sub module of M is strongly pure sub module of M .

Proof : Suppose N is a direct summand of an R-module M , then $M=N\oplus L$ for some sub module L of M . As $NI\subseteq N\cap MI$ is trivial , where I is an ideal of R . We want to prove the reverse inclusion . Let $x\in N\cap MI$, implies $x\in MI=(N\oplus L)I$, then there exists $a_i\in N$, $b_i\in L$ and $r_i\in I$ such that $x=\sum_{i=1}^{n}(a_i+b_i)r_i=\sum_{i=1}^{n}a_i r_i+\sum_{i=1}^{n}b_i r_i$, so $x-\sum_{i=1}^{n}a_i r_i=\sum_{i=1}^{n}b_i r_i$. But $x\in N$, thus $x-\sum_{i=1}^{n}a_i r_i\in N$. Hence $\sum_{i=1}^{n}b_i r_i\in N\cap L$. Since $N\cap L=(0)$, thus $\sum_{i=1}^{n}b_i r_i=0$, and $x-\sum_{i=1}^{n}a_i r_i=0$. It is mean $x=\sum_{i=1}^{n}a_i r_i$, so $x\in NI$. Thus $N\cap MI\subseteq NI$. Therefore N is pure sub module of M . And since R/I is an integral domain . Thus N is strongly pure sub module of M (Prop. 4-3) . Before , we give the main theorem of pure sub modules , we will need the following well-known lemmas . Recall that an R-module M is said to be flat if given any exact sequence $0\to A\to B$ of right R-modules,the sequence $0\to A\otimes_R M\to B\otimes_R M$ is exact ,[7] .

Lemma :[23] If M is a flat R-module , then for any sub module N of M , the following statements are equivalent .

1- M/N is flat .

2- $N\cap MI=NI$ for each ideal I of R .

3- $N\cap MI=NI$ for each finitely generated ideal I of R .

Recall that a sub module N of an R-module M is called superfluous if and only if $N+L\neq M$, for each proper sub module L of M , [19] .

Lemma : (Nakayamas` lemma) [5] , [19] and [28] .

For a left ideal I of a ring R , the following statements are equivalent :

1- For every finitely generated R-module M , if MI =M , then M = 0 .

2- For every finitely generated R-module M . MI is superfluous in M .

Recall that a right R-module M is said to be a multiplication R-module if for each sub module N of M , there exists an ideal I of a ring R , such that N = MI , [1] ,[8] and [15] . And an R-module M is called faithful if ann M = (0) , [15]

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

Lemma : [23] Let N be a sub module of finitely generated faithful multiplication R-module M If N=[N:M]N , then N is an idempotent sub module of M if and only if [N:M] is an idempotentideal of a ring R .The following theorem gives several characterization of strongly pure sub modules of faithful multiplication R-module with strongly pure annihilator.

Theorem : Let R/I be an integral domain and let M be a multiplication R-module , with strongly pure annihilator . If N is sub module of M . Then statements (1) to (9) are equivalent , and further if M is finitely generated and faithful , then statements (1) to (10) are equivalent .

1- N is a strongly pure sub module of M .

2- N is a multiplication and idempotent sub module of M .

3- N is a multiplication and K=[N:M]K for each sub module K of M .

4- N is a multiplication and [K:N]N=[K:M]N for each sub module K of M .

5- Rx=[N:M]x for each x∈N .

6- R=[N:M]+ann(x) for each x∈N .

7- R= $\sum_{n\in N}$ [Rn:M]+ann(x) for each x∈N .

8- For each x∈N , there exists a∈[N:M] , such that x=x a .

9- For each maximal ideal P of R either $N_P=0_P$ or $N_P=M_P$.

10- I[N:M] = I∩[N:M] for every ideal I of R .

Proof : (1)⇒(2) Let K be a sub module of M , then K=[K:M]M . Since N is a strongly pure sub module of M and R/I is an integral domain for each ideal I of R , thus N is pure sub module of M (Prop.4-3) , we infear that [K:N]N=N∩[K:N]M⊇N∩[K:M]M=N∩K⊇[K:N]N So that [K:N]N=K∩N , and N is a multiplication . Since N is pure sub module of M , we have that [N:M]N=[N:M]M∩N=N , and hence N is idempotent sub module of M

(2)⇒(3)Let K be a sub module of N.Then K=[K:N]N=[K:N][N:M]N=[N:M]K .

(2)⇒(4) Let K be a sub module of M . Then [K:N]N=[K:N][N:M]N⊆[K:M]N ⊆[K:N]N , so that [K:N]N=[K:M]N .

(4) ⇒(2) Take K=N .

(3)⇒(7) Let x∈N , then Rx=[N:M]x . Since N is multiplication , it follows by [3 ,Lemma 1-1(iv)] that Rx=[N:M]x=($\sum_{n\in N}$ [Rn:M])x , and then

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

[29 ,Corollary to Theorem 9] gives that $R=\sum_{n\in N}[Rn:M]+ann(x)$.

(5)⇔(8) Clear .

(5) ⇒(9) Let P be any maximal ideal of R . We discuss two cases .

Case1: If $[N:M]\subseteq P$.Then for each $x\in N$, $(Rx)_P=[N:M]_P(Rx)_P\subseteq P_P(Rx)_P\subseteq(Rx)_P$, so $(Rx)_P = P_P(Rx)_P$. By Nakayamas` Lemma , $(Rx)_P=0_P$. Hence $N_P=0_P$.

Case 2 : If $[N:M]\nsubseteq P$, there exists $p\in P$ such that $1-p\in[N:M]$, and hence $(1-p)M\subseteq N$. It follows that $N_P=M_P$.

(9)⇒(1) If $N_P = 0_P$ or $N_P=M_P$ for every maximal ideal P of R , then for every ideal I of R , $NI=N\cap MI$ is true locally , [21] . Thus N is pure sub module of M , and since R/I is an integral domain for each ideal I of R , thus N is strongly pure sub module of M (Prop.4-3) .

(1)⇒(10) Let M be a finitely generated faithful multiplication R-module and let I be any ideal of R .

Then $NI=N\cap MI$. Hence $[NI:M]=[(N\cap MI):M]=[N:M]\cap[MI:M]=[N:M]\cap I$,[18] . We need to show that $[NI:M]=[N:M]I$. Obviously, $[N:M]I\subseteq[NI:M]$. Conversely , let $x\in[NI:M]$.

Then $xM\subseteq NI=([N:M]I)M$. But M is cancellation . thus $x\in[N:M]I$, and hence $[NI:M]\subseteq([N:M]I)M$.

(10)⇒(2) For this part it is not necessary to assume M is finitely generated or faithful . Assume $[N:M]I=[N:M]\cap I$ for all ideals I of R . Take $I=[N:M]$. Then $[N:M]^2=[N:M]$, and hence [N:M] is an idempotent ideal of R . It follows that N is idempotent sub module of M (Lemma 4-10) .

Now , to prove that N is multiplication , let K be any sub module of M , and let $I=[K:M]$. Then $[(K\cap N):M]=[K:M]\cap[N:M]=[K:M][N:M]\subseteq[K:N][N:M]$, and hence $K\cap N=[(K\cap N):M]M\subseteq[K:N][N:M]M\subseteq[K:N]N\subseteq K\cap N$, so that $K\cap N=[K:N]N$, and N is multiplication . Next , we get the following result .

Corollary : Let R/I be an integral domain , and let M be a finitely generated faithful multiplication R-module . If N is strongly pure sub module of M , then the following statements hold :

1- $N=[N:M]N$.

2- $annN=ann[N:M]$.

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

3- [N:M] is strongly pure ideal of R .

Proof : (1) follows from Theorem 4-11 .

(2) As N is strongly pure sub module of M , we have NP=N∩MP , for some prime ideal P of R Taking P=annN, we get 0=N∩(annN)M , and hence 0=[0:M]=[(N:(annN)M):M]=[N:M]∩[(annN)M:M]=[N:M]∩annN=[N:M]ann.

Hence annN⊆ann[N:M] .

Conversely ; if x∊ann[N:M] , then x[N:M]=0 , and hence xN=x[N:M]N=0 , so that x∊annN and ann[N:M]⊆annN . Therefore annN=ann[N:M] .

(3) Let a∊[N:M] , implies aM⊆N , and by theorem(4-11,(3)) aM = [N:M]aM, and hence Ra=[N:M]a . Therefore [N:M] a strongly pure ideal of R (Th. 4-11). Another consequence of theorem(4-11) is the following .

Corollary : Let R/I be an integral domain . M is multiplication R-module with strongly pure annihilator .

(1) If I is strongly pure ideal of R , and N is a strongly pure sub module of M . Then IN is a strongly pure sub module of M .

(2) If N and L are strongly pure sub modules of M , then N+L is also strongly pure sub module of M .

(3) If N and L are strongly pure sub modules of M , then N⊗L is strongly pure sub module of M⊗M .

Proof : (1) As I is strongly pure ideal of R , and N is strongly pure sub module of M , then by theorem(4-11) , we infer that each ideal I and sub module N are multiplication and idempotent ideale of R (sub module of M) . Hence IN is multiplication [6 ,Corollary of theorem 2] . Moreover IN is idempotent . By theorem 3-11 , we infer IN is strongly pure sub module of M

(2) Let x∊N+L , there exists a∊N and b∊L such that x=a+b .

By theorem(4-11) Ra=[N:M]a and Rb=[L:M]b , and hence Rx=R(a+b)⊆Ra+Rb=[N:M]a+[L:M]b⊆[(N+L):M](a+b)=[(N+L):M]x .

And by theorem(4-11) , N+L is strongly pure sub module of M .

(3) Let a∊N and b∊L . Then Ra=[N:M]a and Rb=[L:M]b and hence R(a⊗b)=Ra⊗Rb=[N:M]a⊗[L:M]b=[N:M][L:M](a⊗b) . It is easy to check that [N:M][L:M]⊆[a⊗b:M⊗M] . Implies that R(a⊗b)⊆[N⊗L:M⊗M](a⊗b) ⊆ R(a⊗b) , so

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

that $R(a\otimes b)\subseteq[N\otimes L:M\otimes M](a\otimes b)$. By theorem(4-11) , we infer that $N\otimes L$ is strongly pure sub module of $M\otimes M$. Finally , we get the following results **.**

Proposition : Let M be an R-module . N and L are sub modules of M such that L is sub module of N . If N is strongly pure sub module of M , then $\frac{N}{L}$ is strongly pure sub module of $\frac{M}{L}$

Proof : Since N is strongly pure sub module of M , then NI=N∩MI , for some finitely generated prime ideal I of R . So $I(\frac{M}{L}) \cap \frac{N}{L} = \frac{IM}{L} \cap \frac{N}{L} = \frac{IM \cap N}{L} = \frac{IN}{L} = I(\frac{N}{L})$. Thus $\frac{N}{L}$ is a strongly pure sub module of $\frac{M}{L}$.

5.STRONGLY REGULAR R-MODULES

In this section we introduce a generalization for regular R-module concept namely strongly regular R-module .Recall that an element $x \in M$ is called regular if there exists an R-module homomorphism $\theta : M \rightarrow R$, such that $\theta(x)x = x$, [4] . If every element of M is regular , we say that M is regular R-module , [4] .

And an R-module M is regular if for each $x \in M$, and each $r \in R$, there exists $t \in R$, such that $x r = x r t r$, [12] .First , we start this section by the following definition .

Definition : Let M be an R-module . An element $x\in M$ is called strongly regular if there exists an R-module homomorphism $\theta:M\rightarrow R$, such that $\theta(x)x=x$ where $\theta(x)$ is strongly regular element in a ring R .

Definition : An R-module M is called strongly regular if every element of M is strongly regular .**Also :** M is strongly regular R-module if for every element $x\in M$, and each $r\in R$, there exists a prime element $p\in R$, such that $xr=xrpr$.

Remark : Every sub module of strongly regular R-module is strongly pure.

Proof : Let N be a sub module of M , and let I be an ideal of R . It is clear that $IN\subseteq N\cap IM$. Conversely , let $x\in N\cap IM$, then $x=\sum_{i=1}^{n} r_i x_i$, where $r_i \in I$ and $x_i \in M$. Since M is a strongly regular R-module , hence x is strongly regular element . Thus there exists an R-module

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

homomorphism θ:M→R , such that x= θ(x)x , so $\theta(x)=\sum_{i=1}^{n} r_i\theta(x_i)$ and $x=\theta(x)x=\sum_{i=1}^{n} r_i\theta(x_i)x$. And since x∈N , hence $x=\sum_{i=1}^{n} r_i\theta(x_i)x \in IN$. Thus N∩IM⊆IN . Therefore N is strongly pure sub module of M .

Next , we want to study the relation between strongly regular ring and strongly regular module , by the following proposition .

Proposition : R is strongly regular ring if and only if R is strongly regular R-module .

Proof : Let R be a strongly regular ring , and let x∈R . Thus there exists prime element p∈R , such that x=xpx . Now defined a function θ : R→R , by θ(x)=xp for each x∈R , then θ(x)x=xpx , so θ(x)x=x . Thus R is strongly regular R-module .

Conversely ; Let R be strongly regular R-module , and let x∈R , there exists an R-module homomorphism θ:R→R , such that x=xθ(x) , where θ(x) is strongly regular element of R . Since θ(x)=θ(1.x)=θ(1)x , so xpx=xθ(1)x . Therefore R is strongly regular ring .

Proposition : If M is strongly regular R-module , and divisible over an integral domain R , then every sub module of M is divisible .

Proof : Let N be a sub module of M , and let 0 ≠ r ∈ R , we show that rN=N . By remark (5-3) N is strongly pure , so ≺r≻N = N ∩ ≺r≻M . We show that rN=N∩rM , if x∈N∩rM , then x=rm , since x is a strongly regular element of M , there exists an R-module homomorphism θ:M→R such that x=θ(x).x , and so x=θ(x).x = r.θ(m).x , as x∈N this implies that x∈rN , so N∩rM⊆rN , hence N∩rM= rN . As rM=M , we see that rN=N , so N is divisible .

Proposition : For every strongly regular R-module M , we have J(R)M=0 .

Proof : Since M is strongly regular R-module , thus every sub module of M is strongly pure (Remark 5-3) . Let J(R)M≠0 , therefore there exists x∈J(R)M , so Rx is strongly pure sub module of M , thus Rx∩J(R)M=JRx . Then Rx=J(R) Rx , and by Nakayams`Lemma Rx=0 , so x=0 , and hence J(R)M=0 .

Kirkuk University Journal /Scientific Studies (KUJSS)
Volume 10, Issue 1, March 2015 , p.p(12-28)
ISSN 1992 - 0849

REFERENCES

[1] A.A. Ahmmid: ***On sub modules in multiplication modules*** , M.Sc. Thesis , college of science , University of Baghdad (1999) .

[2] H. Al-Ezeh: ***The pure spectrum of PF-ring*** , comm. Math. Univer. S.P. Vol. 37 , No. 2 (1988) , 179-183 .

[3] M.M. Ali; D.J. Smith: ***Projective , flat and multiplication modules*** . Newzealand J. Math. 31 (2002) , 115-129 .

[4] N. Amiri, M.Ershad: ***On regular module . Math. Shiraz*** , Vol. 19 , No. 1 (2005) , 43-47

[5] F.W.Anderson; K.R.Fuller: ***Rings and Categories of Modules*** . Springer-Verlag (1974) .

[6] D.D.Anderson: ***Some remarks on multiplication ideals*** . Math. Japanica 4 (1980) , 463-469

[7] M.Auslander , M.Bridger: ***Stable module theory*** , Mem. Amer. Math. Soc. 94 (1969) .

[8] A.Barnard: ***Multiplication modules*** , J. of Algebra , Vol.71(1981) , 174-178.

[9] B.Brown, N.H.McCoy: ***The maximal regular ideal of a ring*** , Proc. Amer. Math. Soc.1(1950) , 165-171.

[10] S.H.Brown: ***Rings over which every simple module is relationally complete caned J. Math***. 25 (1973) pp. 693-701 .

[11] D.M.Burton: ***Introduction to modern Ab***, D.M. : Introduction to modern Abstract Algebra , Addison – wisely publishing company , Inc , (1967) .

[12] T.J.Cheathem; ***J.R.Smith: Regular modules and semi-simple modules*** . Pacific J. Math. 65 (1976) , 315-323 .

[13] A. Derya Keskin: ***study on prime submodules*** . Math. Journal , Vol. 3(1996), 27-32 .

[14] S.M. Easeen: ***F-Regular modules*** , M.SC. Thesis , college of science , University of Baghdad (1992) .

[15] Z.A. El-Bast; ***Smith, P. F. : Multiplication modules*** , Comm. In Algebra , 16(1988), 755-779 .

[16] D.J. Fieldhous: ***Pure theories*** , Math. Ann. 184(1969) , 1-18 .

[17] D.J. Fieldhous: ***Pure simple and indecomposable ring*** . Can. Math. Bull. 13 (1970), 71-78 .

[18] D.J. Fieldhous: ***Regular modules . Proc***. Amer. Math. Soc. 32 (1972) , 49-51.

[19] F. Kasch: ***Modules and rings*** . Academic press , London , (1982) .

[20] M.D. Larsen; ***Mecarlthy , P. J. : Multiplication theory of ideals*** . Academic press , New York , (1971) .

[21] L.J. Mahmood; ***On primary sub modules*** . Ph. D. Thesis , University of Technology , (2005) .

[22] R.D. Mahmood; ***On pure ideals and pure sub modules*** . Ph. D. Thesis , University of Mosule , (2000) .

[23] M. Majid; Ali David ; Smith, J. : ***Pure sub modules of multiplication modules*** . Math. Sultan Qaboos , Vol. 45(2004) , No. 1 , 61-74 .

[24] A.G. Naoum; M.A.K. Hasan: ***The residual of finitely generated multiplication modules*** , ***Arch***. Der Math. 46(1986) , 225-230 .

[25] M.Prest, G.Puninski and P. Rothmaler ; ***Rings described by Various Purites*** , Comm. In Algebra , 27 (5) , (1999) , 2127-2162 .

[26] P. Ribenboim: ***Algebraic Numbers*** . Wiley (1972) .

[27] E.A Shahabaddin, ***F. Farkhondeh***; ***On weakly primary ideals*** . Math. Journal , Vol. 12 , N. 3 , (2005) , 423-429 .

[28] P.F. Smith,; ***Some remarks on multiplication modules*** . Arch. Math. 50(1988), 223-235.

[29] J.Von Neumann; ***On regular rings , proc***. Nat. scad. Sciences U.S.A. vol 22 (1936) , pp. 707-713 .

[30] R.Ware: ***Endomorphism rings of projective modules*** , Trans. Amer. Math. Soc. 155(1971) , 233-256 .

[31] J. Zelmanowitz: ***Regular modules , Trans. Amer***. Math. Soc. 163(1972) , 341-355 .

AUTHOR

Nada Khalid Abdullah: B.s.c: Mathematics from Al Mustansiriya University 1987, **M.s.c:** Tikrit University Education College 1996, Now, Lecture in Tikrit / Pure Science Education College.