Journal of University of Babylon for Pure and Applied Sciences, Vol. (28), No. (3): 2020

Online ISSN: 2312-8135, Print ISSN: 1992-0652

On Quotient Semigroup

Saud Mohammed Hassan [1] **Dunia Mohamad Kreem Al-Ftlawy** [2] **Zainab Fahad Mhawes** [3]

1. Iraq, Karbala,Secondary School of Jummana Bant Abi-Talib. suadmohmmadhasanalsafee@gmail.com
2. Iraq, Karbala,Secondary School of Al-Motafogat. suadmohmmadhasanalsafee@gmail.com.
3. Iraq, Al-Qadisiyah university, Education College, Mathematics Department. suadmohmmadhasanalsafee@gmail.com

Article Information

Submission date: 23 / 2 / 2020

Acceptance date: 20 /10/ 2020

Publication date: 31/ 12/ 2020

Abstract

In this paper, we introduce a new type of semigroup, namely (**Quotient semigroup**) in differential equations with the functional analytic. This semigroup constructs the solution of the partial differential equations as the form:

$$\frac{\rho(t)}{h(x)}\frac{\partial u(t,x)}{\partial t}=\frac{\rho'(t)}{h'(x)}\frac{\partial u(t,x)}{\partial x},\quad where\quad \rho(t)\ be\ a\ function\ such\ that\ \rho(0)=1$$

Key Words: semigroup, strongly continuous, operators, generator operator,C_0-semigroup.

I. Introduction

Many Scientists ([1],[2],[3]) introduce several generations of analysts working in the area of operator semigroups. In particular, the progress has been made in the asymptotic theory of strongly continuous semigroups. One of the major results in this direction a strongly continuous semigroup on a Banach space with the norm of the resolvent of its generator A is uniformly bounded in the right half-plane.

We consider the following equations:

$$\frac{\rho(t)}{h(x)}\frac{\partial u(t,x)}{\partial t}=\frac{\rho'(t)}{h'(x)}\frac{\partial u(t,x)}{\partial x},\qquad where\quad \rho(t)\ be\ a\ function\ such\ that\ \rho(0)=1$$

We introduce a new type of semigroup namely (Quotient semigroup) and its define by: $U_D(t)\emptyset(x)=\emptyset\left[h^{-1}\left(\frac{h(x)}{\rho(t)}\right)\right]$ such that h^{-1} exists.

And also we introduce strongly continuous generalized Quotient semigroup defined by: $T_D(t)\phi(x)=exp\left[\int_{h(x)\circledast_D t}^{x}\rho(\xi)\,dh(\xi)\right]\phi[h(x)\circledast_D t]$

Journal of University of Babylon for Pure and Applied Sciences, Vol. (28), No. (3): 2020
Online ISSN: 2312-8135, Print ISSN: 1992-0652

II. Strongly continuous semigroup and its generator operator.

2.1 Definition [4]:

Let E be Banach space then a function f(t) is called continuous at a point t_0 if $\|f(t)-f(t_0)\|_E \to 0\,, at\ t \to t_0$, continuous on the interval [a,b], if it is continuous at each point of this segment.

2.2 Definition [5]:

The function f(t) is called differentiable in point t_0, if there is an element $f' \in E$ such that:

$$\left\| \frac{f(t_0+h)-f(t_0)}{h} - f' \right\|_E \to 0\,, at\ h \to 0$$

The element f' is called the derivative of the function f (t) at point t_0 and denoted by:

$$f' = f'(t_0)).$$

2.3 Definition [6]:

We will say that the operator function A(t) is continuous in norm at point $t_0 \in$ [a,b] if

$$\lim_{t\to t_0} \|A(t) - A(\,t_0\,)\|_E = 0.$$

2.4 Definition [7]:

The operator-function A(t) is strongly continuous in a point $t_0 \in$[a,b] if at any fixed $x \in E_1$

$$\lim_{t\to t_0} \|A(t)x - A(\,t_0\,)x\|_{E_2} = 0.$$

2.5 Theorem [8]:

An operator–function A(t) is strongly continuous at $t_0 \in$[a,b] on all E_1 if its norms are bounded, i.e.

$$\|A(t)\| \le M\ ,\ \ M > 0$$

2.6 Definition [8]:

We say that an operator A is closed if for every $x_n \in D$ (A), then $\|x_n - x_0\| \to 0$ and $Ax_0 = y_0$.

2.7 Definition [7]:

A family of bounded operators T (t) (t >0), define on the Banach space E, is called strongly continuous semigroup of operators if T(t) strongly continuous and satisfies the condition T (t)T (s) = T(t+s) (t>0, s > 0).

2.8 Definition [6]:

It is said that T (t) is a semigroup of class C_0 if it is strongly continuous and the following condition holds

$\lim_{t\to 0}\|T(t)x-x\|_E=0.$ for any x $\in$E.

2.9 Theorem [9]:

The linear operator A is a generating operator (generator) of a semigroup T (t) of class C_0 iff its closed with a dense in E.

2.10 Definition[6]:

A family of bounded operators T (t) (t >0), define on the Banach space E, is called strongly continuous multiplicative semigroup of operators if T(t) strongly continuous and satisfies the conditions:

1)T(0)=I

2) T (t)oT (s) = T(t*s) (t, s > 0).

III. Quotient semigroup.

3.1 Definition:

Let t $\in$ (t_1, t_2)$\subseteq$ $\mathcal{R}$, x $\in$ (a,b)$\subseteq$ $\mathcal{R}$, $\rho(t)$ *and* $h(t)$ are real functions with domains D (ρ) = (t_1, t_2), D (h) = (a, b), continuously differentiable and strictly monotone. In addition, $h(x).\rho(t)\in D(h^{-1})\cap D(\rho^{-1})$, where h^{-1} *and* ρ^{-1}- inverse functions. Consider the differential equation:

$$\frac{\rho(t)}{h(x)}\frac{\partial u(t,x)}{\partial t}=\frac{\rho'(t)}{h'(x)}\frac{\partial u(t,x)}{\partial x},\quad \rho(0)=1,\ \ h(x)\neq 0,\forall\, x$$

It is easy to see that the general solution of this equation is:

$$u(t,x)=\emptyset\left[\frac{h(x)}{\rho(t)}\right]\ \dots\dots\dots\dots\dots\dots\dots(2)$$

Where φ is an arbitrary differentiable function.

we can assign the one-parameter equation (1) to a one-parameter family of operators:

$$U_D(t)\emptyset(x)=\emptyset\left[h^{-1}\left(\frac{h(x)}{\rho(t)}\right)\right]\dots\dots\dots\dots.(3)$$

under the assumption that φ belongs to the space of continuous and bounded functions C(a,b) with the norm:

$$\|\emptyset\| = \sup_{x\epsilon(a,b)} |\emptyset(x)|$$

3.2 Definition:

We define a binary operation ⊙ by:

$$s \circledast_D t = h^{-1}(h(s).h(t)) \quad \dots\dots\dots\dots\dots.(4)$$

3.3 Lemma:

The operational family $U_D(t)$ defined by (3) is a semigroup of linear and bounded in C(a, b) of operators with the binary operation in (4).

Proof:

We note that:

1-$U_D(0) = \emptyset\left[h^{-1}\left(\frac{h(x)}{\rho(0)}\right)\right] = \emptyset[h^{-1}(h(x))] = \emptyset[x]$

2-$U_D(t)U_D(s)\emptyset(x) = U_D(t)\left[\emptyset\left[h^{-1}\left(\frac{h(x)}{\rho(s)}\right)\right]\right] = \emptyset\left[h^{-1}\left(\frac{h(x)}{\rho(t)\rho(s)}\right)\right]$

$$= \emptyset\left[h^{-1}\left(\frac{h(x)}{\rho(\rho^{-1}[\rho(t)\rho(s)])}\right)\right] = \emptyset\left[h^{-1}\left(\frac{h(x)}{\rho(t \circledast_D s)}\right)\right] = U_Q(t \circledast_D s)\emptyset(x)$$

Therefore $U_D(t)$ is semigroup operator.

3.4 Remark:

The semigroup $U_D(t)$ is called a **Quotient semigroup**.

3.5 Remark:

The function h(t) which given in the semigroup $U_D(t)$ is invariant relative to the functions h(x) on h(x)+c , where c is real constant.

3.6 Proposition:

Let D be a semigroup generated by the equation (1) then there exists a point $t_0\epsilon(t_1,t_2)$, such that $U_D(t_0)\varphi(x) = \varphi(x)$.

Online ISSN: 2312-8135, Print ISSN: 1992-0652

Proof:

It follows from continuity and monotony of the function $h_c(t) = \frac{c}{h(t)}$, which the appropriate selection of constant c, it becomes zero at unique point $t_0 \epsilon (t_1, t_2)$. In this case, It follows that the Cauchy problem for equation (1) with the initial condition u (t_0, x) = φ (x) has a unique solution and it can be represented as:

$$u(x,t) = U_D(t_0)\varphi(x) \ldots\ldots\ldots\ldots(5)$$

3.7 Definition:

If $\rho(t) = h(t)$ then the semigroup D-semigroup can be written by the form:

$$U_D(t)\varphi(x) = \varphi(t \circledast^h x) \ldots\ldots\ldots\ldots\ldots(6)$$

Where $t \circledast^h x = h^{-1}\left(\frac{h(x)}{h(t)}\right)$

And its called h-symmetric and the equation (1) is called symmetric generating equation.

2.8 Lemma:

The family of semigroups produced by symmetric differential equation, contains only one symmetric semigroup.

Proof:

Let $U_D(t)\emptyset(x) = \emptyset\left[h^{-1}\left(\frac{h(x)}{h(t)}\right)\right]$, for c≠1 and $h_c(t) = \frac{c}{h(t)}$, thus we have:

$$U_{h_c}(t)\emptyset(x) = \emptyset\left[h_c^{-1}\left(\frac{h_c(x)}{h_c(t)}\right)\right] = \emptyset\left[h^{-1}\left(\frac{h(x)}{h(t)}.c\right)\right]$$

Therefore U_{h_c} is not symmetric for c≠1 .

3.9 Definition:

The semigroup $U_D(t)$ is called a strongly continuous at a point $t_0 \epsilon (t_1, t_2)$, if for all φ∈E the inequality holds:

$$\lim_{t \to t_0} \|U_D(t)\varphi - \varphi\|_E = 0 \quad \ldots\ldots\ldots\ldots\ldots(7)$$

3.10 Definition:

The semigroup $U_D(t)$, $\rho(t) = \frac{1}{t+1}$, $t \neq -1$ is a class

$$U_h^{(0)}(t) = U_D(t) = \emptyset[h^{-1}(h(x).(t+1))]$$

We note that $U_D^{(0)}(t)$ with the binary operation $t \circledast^h x = h^{-1}(h(x).(t+1))$ is called Arithmetic semigroup.

3.11 Definition:

Let f(t) be a vector function , define on $t\epsilon(t_1,t_2)$ with valued in E. $\mu(t)$ be a strictly monotonic function ,define on $D(\mu) \subseteq R$, $R(\mu) = (t_1,t_2)$, then a function $g(t0 = f(\mu(t))$ is called μ-deformation of function f(t).

3.12 Remark:

Every D-semigroup is ρ-deformation of semigroup $U_D^{(0)}(t)$.

3.13 Definition :

The function φ ∈C(a,b) is called uniformly continuous if its μ^{-1}-deformation

$\psi = \phi(\mu(x))$ is bounded and uniformly continuous function. we note that :

$$\|\phi\|_{C(a,b)} = \sup_{x\in(a,b)} |\phi(x)| = \sup_{s\in(\mu^{-1}(a),\mu^{-1}(b))} |\phi(\mu(s))| = \|\psi\|_{C\mu}$$

3.14 Proposition:

Every $U_D^{(0)}(t)$ strongly continuous semigroup in the space of h^{-1}-uniformaly continuous functions.

Proof:

We note that:

$$\left\|U_D^{(0)}(t)\phi(x) - \phi(x)\right\| = \sup_{x\in(a,b)} |\phi[h^{-1}(h(x).(t+1))] - \phi[h^{-1}(h(x))]|$$

$$= \sup_{\tau\in(h^{-1}(a),h^{-1}(b))} |\psi(\tau+t) - \psi(t)| = \|\psi(\tau+t) - \psi(t)\| \to 0, t \to 0$$

3.15 Theorem:

A generator operator of the semigroup $U_D^{(0)}(t)$ given by the differential expression :

$$A_h^{(0)}\phi(x) = \frac{h(x)}{h'(x)}\frac{\partial\phi}{\partial x} \quad ,h(x) = 0 \quad ;0 < x < 1, \lim_{x\to b} h(x) = \infty$$

And a domain $D\left(A_h^{(0)}\right) = \{\phi: \phi \in C_{h^{-1}}, L\phi \in C_{h^{-1}}\}$.

Journal of University of Babylon for Pure and Applied Sciences, Vol. (28), No. (3): 2020
Online ISSN: 2312-8135, Print ISSN: 1992-0652

Proof:

We have :

$$R(\lambda, A) = (\lambda I - A)^{-1} \quad , \quad Re(\lambda) > \omega, where\ R(\lambda, A)\ stand\ for\ reslventa\ of\ \ \lambda$$

Thus:

$$R\left(\lambda, A_h^{(0)}\right) = \int_0^\infty e^{-\lambda t} U_D^{(0)}(t)\phi(x)dt$$

$$\left\| R\left(\lambda, A_h^{(0)}\right) \right\| < \frac{1}{Re(\lambda)} \quad , \qquad Re(\lambda) > 0$$

$$\forall \ n \in N \quad , \qquad J_n = nR(n, A_h^{(0)})$$

Thus we have:

$$A_h^{(0)} J_n = n(J_n - I)$$

$$y_n(x) = (J_n\phi)(x) = n\int_0^\infty e^{-nt} U_D^{(0)}(t)\phi(x)dt = n\int_1^\infty e^{-nt} U_D^{(0)}(t)\phi(x)dt$$

$$= n\int_0^\infty e^{-nt}\phi[h^{-1}(h(x).(t+1))]dt = \int_x^b e^{-n[\frac{h(\tau)}{h(x)}-1]} h'(\tau)\phi(\tau)d\tau$$

$$\frac{h(x)}{h'(x)} y'_n(x) = n(J_n - I)\phi(x) = A_h^{(0)} y_n(x)$$

3.17 Definition:

Let $(a, b) \subseteq R$ be an interval and let h(x) be a differentiable function such that $\lim_{x\to b} h(x) = \infty$, we define a space $L_{p,\omega,h}$ by :

$$L_{p,\omega,h} = \left\{ \phi\colon\ \|\phi\|_{p,\omega,h,g} = \left[\int_a^b |\exp[\omega h(x)]\, g(x)\phi(x)|^p\, dh(x)\right]^{\frac{1}{p}} , p \geq 1 , \omega > 0 , g(x) > 0, g'(x) > 0 \right\}$$

On the space $L_{p,\omega,h}$ we define the following family of operators:

$$T_D(t)\phi(x) = exp\left[\int_{h(x)\circledast_D t}^{x} \rho(\xi)\, dh(\xi)\right] \phi[h(x) \circledast_D t]$$

Journal of University of Babylon for Pure and Applied Sciences, Vol. (28), No. (3): 2020

Online ISSN: 2312-8135, Print ISSN: 1992-0652

3.18 Theorem:

The family of operators $T_D(t)$ is strongly continuous generalized canonical semigroup defines on the space $L_{p,\omega,h}$ and the following estimation holds:

$$\|T_D(t)\| \le |t|^{-p}$$

Proof:

$$\|T_D(t)\phi\|^p_{p,\omega,h} = \int_a^b \exp\left[\omega h(x) + p \int_{h(x)\circledast_D t}^{x} \rho(\xi)\, dh(\xi)\right] g(x)|\phi(h(x) \circledast_D t)|^p dh(x) \le$$

$$\le \int_a^b \exp[\omega h(x)]\, g(x)|\phi(h(x) \circledast_D t)|^p dh(x)$$

$$\le \frac{1}{|t|} \int_{h(a)\circledast_D t}^{b} \exp[\omega h(\tau)]\, g(h^{-1}(h(\tau).(t+1)))|\phi(\tau)|^p dh(\tau)$$

$$\le \frac{1}{|t|} \int_a^b \exp[\omega h(\tau)]\, g(\tau)|\phi(\tau)|^p dh(\tau)$$

$$\|T_D(t)\phi\|^p_{p,\omega,h} \le \frac{1}{|t|} \|\phi\|^p_{p,\omega,h}$$

$$\|T_D(t)\phi\| \le |t|^{-p} \|\phi\|$$

Clearly $T_D(0)\phi(x) = \phi(x)$

$$T_D(t)T_D(s)\phi(x) = T_D(t) exp\left[\int_{h(x)\circledast_Q s}^{x} \rho(\xi)\, dh(\xi)\right] \phi[h(x) \circledast_D s]$$

$$= exp\left[\int_{h(x)\circledast_D t}^{x} \rho(\xi)\, dh(\xi)\right] exp\left[\int_{h(x)\circledast_D (t+s)}^{h(x)\circledast_D t} \rho(\xi)\, dh(\xi)\right] \phi[h(x) \circledast_D (t+s)]$$

$$= exp\left[\int_{h(x)\circledast_D (t.s)}^{x} \rho(\xi)\, dh(\xi)\right] \phi[h(x) \circledast_D (t.s)] = T_D(t+s)\phi(x)$$

Generating operator of semigroup $T_D(t)$.

We can constriction the grating of the semigroup $T_D(t)$ by:

$$A_D\phi(x) = \lim_{t\to 1}\frac{1}{t}[T_D(t) - I]\phi(x)$$

And this means,

$$A_D\phi(x) = \frac{d}{dt}T_D(t)\phi(x)|_{t=0} = h(x)\frac{d\phi(x)}{dh(x)} - \rho(x)\phi(x)$$

Conflict of Interests.

There are non-conflicts of interest .

References

1. Kostin, V.A., *On the uniform correct solvability of boundary value problems for abstract equations with a Keldysh–Feller operator.* Differentsial'nye Uravneniya, 1995. **31**(8): p. 1419-1425.
2. Kostin, V., A. Kostin, and D. Kostin. *C 0-operator laplace integral and boundary value problems for operator degenerate equations*. in *Doklady Mathematics*. 2011. Springer.
3. Keyantuo, V. and P. Vieten, *On analytic semigroups and cosine functions in Banach spaces.* Stud. Math, 1998. **129**(2): p. 137-156.
4. Hille, E. and R.S. Phillips, *Functional analysis and semi-groups*. Vol. 31. 1996: American Mathematical Soc.
5. Kreui, S., *Linear differential equations in Banach space*. Vol. 29. 2011: American Mathematical Soc.
6. Daleckii, J.L. and M.G.e. Kre_n, *Stability of solutions of differential equations in Banach space*. 2002: American Mathematical Soc.
7. Evgrafov, M.A., *Analytic functions*. 2019: Courier Dover Publications.
8. Костин, В.А., М. Муковнин, and М. Гим, *О КОЭРЦИТИВНОСТИ СИСТЕМ Co-ОПЕРАТОРНЫХ МНОГОЧЛЕНОВ.* Вестник Воронежского государственного университета. Серия: Физика. Математика, 2014(4): p. 150-159.
9. Cioranescu, I. and V. Keyantuo. *On operator cosine functions in UMD spaces*. in *Semigroup Forum*. 2001. Springer.

الخلاصة

في هذا البحث، قدمنا نوع جديد من شبه الزمرة يسمى (شبه زمرة القسمة) في المعادلات التفاضلية مع استخدام التحليل الدالي. شبه الزمرة هذه تكون حل للمعادلات التفاضلية الجزئية على الشكل الاتي:

$$\frac{\rho(t)}{h(x)}\frac{\partial u(t,x)}{\partial t} = \frac{\rho'(t)}{h'(x)}\frac{\partial u(t,x)}{\partial x}, \quad where \quad \rho(t) \; be \; a \; function \; such \; that \; \rho(0) = 1, h(0) = 1$$

الكلمات الدالة: شبه الزمرة، الاستمرارية القوية، المؤثرات، المؤثر المولد، شبه الزمرة– C_0 .