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The Numerical Investigations of Non-Polynomial Spline for Solving Fractional Differential Equations

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Abstract

We present a crossing approach based on the new construction of non-polynomial spline function to investigate the numerical solution of the fractional differential equations. We find the accuracy of the spline method and to presenting the completion of non-polynomial spline two examples for problems are used. To clarify, we present the numerical computations that can be used to solve difficult problems while the results are found and got to be in good error estimation with comparing exact solutions.

Keywords: Spline approximation; fractional derivative; convergence analysis; error bound.

MATHEMATICS SUBJECT CLASSIFICATION: 41A15; 26A33; 65Bxx; 65L70.

1. Introduction

In the modern era, a non-polynomial quadratic spline method is demonstrated solving fourth boundary value problems. This method is not reducing the order of the problem even if it is confirmed literally for the solution of the problem. Convergence analysis of the fourth order method is discussed on the topic by Arshad Khan, Shahna [10], [25]. A. H. Bhrawy et al. [11] reveals a quadrature tan method for fractional differential equations with variable coefficients to use a quadrature shifted Legendre tau (Q-SLT) method. Several FDEs with variable coefficients are analyzed and confirmed to the efficiency of the proposed method by numerical results. A. Lotfi et al. [12] present a numerical direct method for solving a general class of fractional optimal control problems (FOCPs). M. A. Ramadan et al. [15] present two new second and fourth-order methods based on a septic non-polynomial spline function for the numerical solution of sixth-order two-point boundary value problems. A. Pervaiz et al. [16] construct a numerical technique for recovering sixth order boundary value problems (BVPs) by showing non-polynomial spline method. Pankaj Kumar Srivastava et al. [17] use non-polynomial quantic spline functions to grow a numerical algorithm for analyzing the approximation to the solution of a system of second order boundary value problems when it is discussed with heat transfer. Non-polynomial quadratic spline method compares our algorithm and discussed Shahid S. and et al. [18] reveal how to find the numerical solution of linear fifth-order boundary value problems. A novel approach to numerical solution of a class of fourth-order time fractional partial differential equations (PDEs) is demonstrated by Muhammad A. et al. [1] who doing several tests for the problems that they have got and compare with the computational results or consequences in existing paper. To compare with other species on this topic is to demonstrate more accurate scheme that revealed. Faraidun K. and Pshtiwan O. [2] present numerical solution of fractional differential equations by using fractional spline functions to introduce the technique that is efficient and simple to get. Mehrdad and et al [3] works on creating the operational matrix of fractional derivatives of sequencing α in the Caputo sense by using the linear B-spline functions. Faraidun K. and Amina H. [4] solve differential equations of fractional order by an algorithm cubic spline with lacunary fractional derivatives. Discovering and getting a better error bound, we use and work on small field. Zahra and Elkholy [5] proposed the use of cubic splines in the numerical solution of fractional differential equations. Fractional calculus was going to be greatly important material to demonstrating and analyzing in several conditions such as physics, chemistry and engineering states. To get more information, jointing and connecting cubic polynomial spline function are worked on method with shooting method which is how to reveal and demonstrate approximate solution for a class of fractional boundary value problems (FBVPs). A non-polynomial collocation approximation of solution to fractional differential equations is employed by Neville J. and et al. [6]. To clarify more, fractional differential equations Volterra integral equations are equal in classic making and producing a scheme, optimal order of convergence will be the result to get without trying to be inconvenient on the answer or solution. Ali Akgul and Esra [7] reveal a novel method for solutions of fourth-order boundary value problems by using the reproducing Kernel Hilbert space method. The tests used to compare with other tests in the other papers to get approximate and more accurate solutions. Mohammad and Rezvan [8] discuss one-dimensional fractional sub-diffusion equations on an unbounded domain. The numerical tests describe and analyze the power and order of exacting of the methods that represented. Mingzhu Li et al. [9] reveals non-polynomial spline method for the time-fractional nonlinear Schrodinger equation, the method that proposed is used numerical experiments to accurate with the exact. In this paper, a capable numerical solution based on non-polynomial function as fractional term spline basis has been derived using the fractional boundary of the spline function. In section 2, we assume that the preliminary definitions used to drive the non-polynomial spline by apply to solve fractional differential equations; explaining and discussing of quantic non-polynomial spline scheme which is revealed the non-change relations between the values of approximating spline and it is derivatives are nodal and section 3; we derive the formulation of non-polynomial spline approximation of fractional order; we present the formulation of temporal discretization. In section 4, the theoretical analysis and convergence of the method, for Caputo will be carried out; we work on calculation of truncation error and matrix form of the scheme that proposed. Finally, in the last section, numerical evidence will be reported to demonstrate of the accuracy of the method with maple programming; we compare and discuss the efficiency of the method on numerical results.

2. Preliminaries and Basic Definitions

Definition 2.1 [19] Suppose that $\alpha > 0,\ x > a, \alpha, a, x \in \mathbb{R}$. Then the Caputo fractional derivative of order $\alpha > 0$ is defined by the following fractional operator.

$$D_a^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(s)}{(x-s)^{\alpha+1-n}} ds & \text{for } n-1 < \alpha < n\,, n \in \mathbb{N} \\ \frac{d^n}{dx^n} f(x) & \text{for } \alpha = n, \quad n \in \mathbb{N} \end{cases}$$

Definition 2.2 [13], [14] Suppose that $\alpha > 0,\ x > a, \alpha, a, x \in \mathbb{R}$. Then we have

$$D_a^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(s)}{(x-s)^{\alpha+1-n}} ds & \text{for } n-1 < \alpha < n\,, n \in \mathbb{N} \\ \frac{d^n}{dx^n} f(x) & \text{for } \alpha = n, \quad n \in \mathbb{N} \end{cases}$$

This is named the Riemann-Liouville fractional derivative of order α.

Definition 2.3 [19], [20] Suppose that $D_a^{k\alpha} f(x) \in C[a,b]$ for $k = 0,1,\dots,n$ where $0 < \alpha \le 1$, then we have the Taylor Series expansion about $x = \tau$

$$f(x) = \sum_{i=0}^{n} \frac{(x-\tau)^{i\alpha}}{\Gamma(i\alpha+1)} D_a^{i\alpha} f(\tau) + \frac{(D_a^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-\tau)^{(n+1)\alpha} \quad \text{With } a \le \xi \le x, \text{for all } x \in (a,b],$$

Where $D_a^{k\alpha} = D_a^{\alpha}.D_a^{\alpha} \dots D_a^{\alpha}$ (k times).

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3. Description of method Non-polynomial Spline Function

Let us consider a mesh points of uniform partition that $(x_i, y_i), i = 0,1, \dots, n$ are $n + 1$ distinct points, such that

$$a = x_0 < x_1 < \dots < x_n = b,$$

Where $h = \frac{b-a}{n+1}$, $x_i = a + ih$, for $i = 0,1, \dots, n + 1$. For each segment $[x_i, x_{i+1}]$ for $k = 0,1, \dots, n - 1$, we will formulate a non-polynomial spline function with boundary conditions in equation (1), which are develop as in [10], as $Q(x)$in $C^4[a, b]$,

$$Q_i(x) = a_i \sin\tau(x - x_i) + b_k e^{\tau(x-x_i)} + c_i(x - x_i) + d_i \quad (1)$$

Where a_i, b_i, c_i and d_i are constants, and also $i = 0,1, \dots, n$, let $y(x)$ be the exact solution, and let Q_i be an approximation to y_k get by the part of $Q_i(x)$ passing through the nodes(x_{i+1}, Q_{i+1}). The non-polynomial spline of fractional order that satisfy the boundary conditions as follows:

$Q(x) = Q_i(x)$, $x \in [x_i, x_{i+1}]$ for $i = 0,1, \dots, n - 1$

$Q(x) \in C^4$ [a, b].

To evaluate the coefficients a_i, b_i, c_i and d_i , we first define the following conditions:

$$Q_i(x_i) = y_i, \quad Q_i(x_{i+1}) = y_{i+1},$$

$$Q_i^{(2)}(x_i) = \frac{1}{2}(D_i + D_{i+1}), \quad Q_i^{(1)}(x_i) = y_{i+1}^{(1)} \quad (2)$$

Where $i = 0,1, \dots, n$

Using the conditions of equation (2), the coefficients in equation (1) can be calculated as

$$a_i = \frac{y_{i+1} - y_i - hy_{i+1}^{(1)}}{\sin\theta - \theta\cos\theta} + \frac{h^2(D_i + D_{i+1})[1 - e^\theta + \theta e^\theta]}{2\theta^2(\sin\theta - \theta\cos\theta)},$$

$$b_i = \frac{h^2(D_i + D_{i+1})}{2\theta^2},$$

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$$c_i = \frac{y_{i+1} - y_i}{h} + \frac{h(D_i + D_{i+1})[1 - e^{\theta}]}{2\theta^2} - \frac{[y_{i+1} - y_i - hy_{i+1}^{(1)}]\sin\theta}{h[\sin\theta - \theta\cos\theta]} - \frac{h(D_i + D_{i+1})[1 - e^{\theta} + \theta e^{\theta}]\sin\theta}{2\theta^2(\sin\theta - \theta\cos\theta)},$$

$$d_i = y_i - \frac{h^2(D_i + D_{i+1})}{2\theta^2}.$$

Where $\theta = h\tau$, and $i = 0,1,2, ..., n.$

The following consistency relation is derived by applying the one half fractional derivative continuities at knots with Caputo derivative, i.e.

$Q_{i-1}^{(m)}(x_i) = Q_i^{(m)}(x_i)$, $m = \frac{1}{2}$, we obtain

$$y_{i-1} - 2y_i + y_{i+1} - h(y_{i+1}^{(1)} - y_i^{(1)}) = \alpha(D_{i+1} - D_{i-1}) \tag{3}$$

Let $\alpha = -\dfrac{h^{\frac{3}{2}}(\sin\theta - \theta\cos\theta)}{2\theta^{\frac{3}{2}}\sin(\frac{\theta\pi}{4h})} - \dfrac{h^2[1 - e^{\theta} + \theta e^{\theta}]}{2\theta^2}$

Where $i = 0,1, \dots, n$

4. Non-Polynomial spline solutions

The spline solution of fractional differential equations can be determine, from the equation (3), which are Caputo derivatives can be written in the following:

$$(-1 - \frac{a}{h^2})y_i + (1 + \frac{2a}{h^2})y_{i+1} - \frac{a}{h^2}y_{i+2} = (\frac{-h^2}{2} - a)y_i^{(2)} + (\frac{1}{6}h^3 + ah)y_i^{(3)} + (\frac{-1}{24}h^4 - \frac{7}{12}ah^2)y_i^{(4)} + hy_{i+1}^{(1)} \tag{4}$$

Where $i = 0, 1, 2, ..., n+1$.

$$(-h^2-\alpha)y_i+(h^2+2\alpha)y_{i+1}-\alpha y_{i+2}=h^2(\frac{-h^2}{2}-\alpha)y_i^{(2)}+ h^3(\frac{1}{6}h^2+\alpha)y_i^{(3)}+h^4(\frac{-1}{24}h^2-\frac{7}{12}\alpha)y_i^{(4)}+h^3y_{i+1}^{(1)}$$

$$Y=\left[h^{\alpha-\beta}D_{x_{i-1}}^{\alpha}p(x)Q(x)+g(x)\right]_{x=x_i} \qquad (5)$$

From the system (4) gives (n) linear equations in the (n) unknowns $y_i,\ i=1,2,...,n,$ therefore two more equations are required. The two end condition can be obtained using non-polynomial spline with expanding Taylor series method as follows:

$$AY(x)=B\hat{Y}(x)+C\tilde{Y}(x)+D\breve{Y}(x)+E\bar{Y}(x) \qquad (6)$$

Substitute equation (5) using equation (6), we have

$$AY_i=g_i-\mu h^{\alpha+2}\hat{Y}_i-\eta h^{\alpha}D^{(\alpha)}Y(x)\Big|_{x=x_i}+T\ ,\ i=1,2,...,n, \qquad (7)$$

Where T is the truncation error, A, B, C, D and E are $(n\times n)$ square matrices such that:

Suppose $Y(x)=[y_1,y_2,...,y_n]^T$, $\bar{Y}(x)=[y_1^{(1)},y_2^{(1)},...,y_n^{(1)}]^T$, $\hat{Y}(x)=[y_1^{(2)},y_2^{(2)},...,y_n^{(2)}]^T$, $\tilde{Y}(x)=[y_1^{(3)},y_2^{(3)},...,y_n^{(3)}]$, $\breve{Y}(x)=[y_1^{(4)},y_2^{(4)},...,y_n^{(4)}]$ are column vectors with dimension (n), the system of equations presented by equation (4) can be expressed in matrix from as:

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$$A=\begin{pmatrix} -1-\frac{\alpha}{h^2} & 1+\frac{2\alpha}{h^2} & -\frac{\alpha}{h^2} & 0 & 0 & \cdots & 0 \\ 0 & -1-\frac{\alpha}{h^2} & 1+\frac{2\alpha}{h^2} & -\frac{\alpha}{h^2} & 0 & \cdots & 0 \\ 0 & 0 & -1-\frac{\alpha}{h^2} & 1+\frac{2\alpha}{h^2} & -\frac{\alpha}{h^2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ 0 & 0 & 0 & \cdots & & & \\ 0 & 0 & 0 & \cdots & -1-\frac{\alpha}{h^2} & 1+\frac{2\alpha}{h^2} & -\frac{\alpha}{h^2} \\ 0 & 0 & 0 & \cdots & 0 & -1-\frac{\alpha}{h^2} & 1+\frac{2\alpha}{h^2} \end{pmatrix},$$

$$B=\begin{pmatrix} \frac{-h^2}{2}-\alpha & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{-h^2}{2}-\alpha & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{-h^2}{2}-\alpha & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ 0 & 0 & 0 & \cdots & \frac{-h^2}{2}-\alpha & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{-h^2}{2}-\alpha & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-h^2}{2}-\alpha \end{pmatrix},$$

$$C=\begin{pmatrix} \frac{1}{6}h^3+\alpha h & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{6}h^3+\alpha h & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{6}h^3+\alpha h & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ 0 & 0 & 0 & \cdots & \frac{1}{6}h^3+\alpha h & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{6}h^3+\alpha h & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{6}h^3+\alpha h \end{pmatrix},$$

$$D=\begin{pmatrix} \frac{-1}{24}h^4-\frac{7}{12}\alpha h^2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{-1}{24}h^4-\frac{7}{12}\alpha h^2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{-1}{24}h^4-\frac{7}{12}\alpha h^2 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ 0 & 0 & 0 & \cdots & \frac{-1}{24}h^4-\frac{7}{12}\alpha h^2 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{-1}{24}h^4-\frac{7}{12}\alpha h^2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-1}{24}h^4-\frac{7}{12}\alpha h^2 \end{pmatrix},$$

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$$E=\begin{pmatrix} h & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & h & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & h & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \\ 0 & 0 & 0 & \cdots & h & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & h & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & h \end{pmatrix}.$$

Using the condition of spline function in equation (2) can be solving the values of vector $Y(x)=[y_1, y_2,..., y_n]^T$ then we can write the system given by (4) and (7) as follows:

$$AY_i + \mu h^{\alpha+2}\hat{Y}_i = G(x)\big|_{x=x_i} + T\,,\ i=1,2,...,n, \tag{8}$$

Where $A,\ \mu$ are matrices or order $(n\times n)$, and

$$G(x)\big|_{x=x_i} = g_i - \eta h^{\alpha} D^{(\alpha)}Y(x_i)\,,\ i=1,2,...,n,$$

Form equation (8), can be written the error equation, also see [22], [25]

$$AE = GE - \mu h^{\alpha+2}\, E + T.$$

$$AE - GE + \mu h^{\alpha+2}\, E = T \rightarrow E = (A - G + \mu h^{\alpha+2})^{-1}T,$$

Which is implies that

$E=(A-G+\mu h^{\alpha+2})^{-1}T$, in order to get a bound on error $\|E\|$ (the maximum error).

5. Convergence Analysis of the Method

In this section, we want to found the unique solution of the resulting of linear system of equations which are obtained from non-polynomial spline scheme in equation (1), and also conduct a convergence analysis as follows.

Theorem 1 The final system of non-polynomial spline in equation (5) has a unique solution.

Proof Based on the coefficient of matrices in equation (5) it is seen that strictly diagonal dominant, hence they are invertible, such as of the diagonalizable matrix A has the determinate not zero. We will take in some proper elementary convert to A in order to confirm determinate of A is not zero. Also using technique of theorem 1 in [18] can be shown that A is invertible, the solution of non-polynomial spline method exist and is unique.

Theorem 2 Consider the function $y(x)$ of class $C^{\infty}[a,b]$ and $Q(x)$ be the non-polynomial spline function with the conditions in equations (1) and (2), we have

$$||e^{(\alpha k)}(x)||_{\infty} = o(h^{6-\alpha k}), \quad k = 0,1,2,3 \text{ and } 4. \tag{9}$$

where $||y(x)||_{\infty} = \max_{a\le x\le b} |y(x)|$, $e_i = s_i - y_i$ and $0 < \alpha \le 1$, $i = 0,1,2,\ldots,n-1$. .

Proof Let $Q(x)$ is a non-polynomial spline function, hence $Q^{(4)}(x)$is a piecewise continuous function on$[a,b]$, for $j = 1,2,\ldots,n$. As [23], suppose that $Q^{(4)}(x)$ denoted the restriction of $Q(x)$ over $[x_{j-1}, x_j]$, then

$$Q^{(4)}(x) = Q^{(4)}(x_{j-1})\frac{x_j - x}{h} + Q^{(4)}(x_j)\frac{x - x_{j-1}}{h}$$

Now define another linear function $g(x)$on $[x_{j-1}, x_j]$as follows

$$g(x) = y^{(4)}(x_{j-1})\frac{x_j - x}{h} + y^{(4)}(x_j)\frac{x - x_{j-1}}{h}$$

Clearly $g(x)$is a linear interpolation of $y^{(4)}(x)$, $\alpha = 1$,

$$||Q^{(4)}(x) - g(x)||_{\infty} = \max_{x_{j-1}\le x\le x_j}\left|\left(Q^{(4)}(x_{j-1}) - y^{(4)}(x_{j-1})\right)\frac{x_j - x}{h} + \left(Q^{(4)}(x_j) - y^{(4)}(x_j)\right)\frac{x - x_j}{h}\right|$$

$$= o(h^2)$$

$$||Q^{(4)}(x) - y^{(4)}(x)||_{\infty} = ||Q^{(4)}(x) - g(x) + g(x) - y^{(4)}(x)||_{\infty}$$

$$\le ||Q^{(4)}(x) - g(x)||_{\infty} + ||g(x) - y^{(4)}(x)||_{\infty}$$

$$\le o(h^2) + o(h^2) = o(h^2),$$

And approximate y_i by non-polynomial spline $Q(x)$ where

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$$Q^{(3)}(x)-y^{(3)}(x)=\frac{y_{i+1}^{(2)}-y_i^{(2)}}{h}-y^{(3)}(x)+o(h^3),$$

$$\begin{aligned}\left\|Q^{(3)}(x)-y^{(3)}(x)\right\| &\le \left\|y^{(3)}(\phi_i)-y^{(3)}(x)\right\|+o(h^3)\\ &\le \left\|y^{(4)}(\lambda_i)(\phi_i-x)\right\|+o(h^3)\\ &=o(h^3),\end{aligned}$$

The others also can be proven similarly; therefore the convergence of the method has been obtained, also see theorem 1 in [25].

Theorem 3 let $y(x)$ be a function of class $C^{\infty}[a,b]$ and $Q(x)$ is non-polynomial spline function from equation (1) and (2), and $y^{(\alpha)}\in C^4[0,1]$ then we have

$$||D^{(\alpha)}y(x)-D^{(\alpha)}Q(x)||_\infty \le \frac{h^r[x(h-x)]^{m-r}}{r!(2m-2r)!}\parallel y^{(\alpha+2)}(\varepsilon)\parallel, \qquad (10)$$

Holds $r=0,1,\dots,m, 0\le x\le h$ and $\varepsilon\in(a,b)$.

Proof Since we have the interpolation of non-polynomial spline function with all derivatives same as Hermite interpolation polynomial and using Taylor series method, also see [21], [24],

Let $g=y^{(\frac{1}{2})}, p_3=Q^{(\frac{1}{2})}, m=2, \alpha=\frac{1}{2}$ and $r=0,$ we get

$$\left\|y^{(\frac{1}{2})}(x)-Q^{(\frac{1}{2})}(x)\right\|\le\frac{h^4}{24}\left\|D^2D^{(\frac{1}{2})}y(\varepsilon)\right\|$$

$$\to\left\|y^{(\frac{1}{2})}(x)-Q^{(\frac{1}{2})}(x)\right\|\le\frac{h^4}{24}\left\|D^{(\frac{5}{2})}y(\varepsilon)\right\|$$

$\to\left\|E^{(\frac{1}{2})}(x)\right\|\le\frac{h^4}{24}\left\|D^{(\frac{5}{2})}y(\varepsilon)\right\|$, Where $\varepsilon\in(a,b)$, and $E^{(\frac{1}{2})}(x)=y^{(\frac{1}{2})}(x)-Q^{(\frac{1}{2})}(x)$.

Similarly $g=y^{(\frac{3}{2})}, p_5=Q^{(\frac{3}{2})}, m=3, \alpha=\frac{3}{2}$ and $r=0,$ we get

$$\left\| y^{(\frac{3}{2})}(x) - Q^{(\frac{3}{2})}(x) \right\| \le \frac{h^6}{720} \left\| D^2 D^{(\frac{3}{2})} y(\varepsilon) \right\| \to \left\| y^{(\frac{3}{2})}(x) - Q^{(\frac{3}{2})}(x) \right\| \le \frac{h^6}{720} \left\| D^{(\frac{7}{2})} y(\varepsilon) \right\|$$

$\left\| E^{(\frac{3}{2})}(x) \right\| \le \frac{h^6}{720} \left\| D^{(\frac{7}{2})} y(\varepsilon) \right\|$, Where $\varepsilon \in (a, b)$, and $E^{(\frac{3}{2})}(x) = y^{(\frac{3}{2})}(x) - Qf^{(\frac{3}{2})}(x)$.

Thus we have proved the theorem.

6. Numerical Illustrations

In this present paper, we have performed our method for solving some of the fractional differential equations with various values of *h*. The error estimates in solutions of the methods are tabulated in tables. The obtain error bound results are compared with the exact solution.

Example 1 Consider the fractional differential equation [11]

$$D^{(2)} y(x) + \sin(x) D^{(\frac{1}{2})} y(x) + xy(x) = f(x)$$

The initial conditions $y(0) = y'(0) = 0$,

$$f(x) = x^9 - x^8 + 56x^6 - 42x^5 + \sin(x)\left(\frac{32768}{6435} x^{\frac{15}{2}} - \frac{2048}{429} x^{\frac{13}{2}}\right)$$

The exact solution of this problem is $y(x) = x^8 - x^7$

Example 2 Consider the fractional boundary value problems [5]

$$y^{(2)}(x) + \theta D^{0.3} y(x) + \beta y(x) = -12x^2 + x^3[20 + \theta(\frac{120}{\Gamma(5.7)} x^{1.7} - \frac{24}{\Gamma(4.7)} x^{0.7}) + \beta(x^2 - x)]$$

$y(0) = y(1) = 0$, Where $\theta = 0.5$, $\beta = 1$.

And the exact solution is $y(x) = x^4(x-1)$

Table 1: L_1, L_2, L_∞ errors of example 1 where $t = 0.5$

h	$L_\infty - error$	$L_1 - error$	$L_2 - error$
0.01	2.3833×10^{-7}	3.4680×10^{-8}	1.0940×10^{-2}
0.02	2.6532×10^{-5}	3.8732×10^{-6}	3.5724×10^{-2}
0.03	3.8483×10^{-4}	5.7197×10^{-5}	7.0242×10^{-2}
0.04	2.3655×10^{-3}	3.5850×10^{-4}	1.1881×10^{-1}
0.001	2.6607×10^{-14}	3.8512×10^{-15}	1.9927×10^{-4}
0.002	3.3667×10^{-12}	4.8751×10^{-13}	6.6853×10^{-4}
0.003	5.6832×10^{-11}	8.2361×10^{-12}	1.3557×10^{-3}
0.004	4.2083×10^{-10}	6.1006×10^{-11}	2.2370×10^{-3}

Table 2: L_1, L_2, L_∞ errors of example 1 where $t = 4.7$

n	$L_\infty - error$	$L_\infty(Bhrawy)$	$L_1 - error$	$L_1(Bhrawy)$	$L_2 - error$	$L_2(Bhrawy)$
8	1.63	8.61×10^{-12}	3.07×10^{-1}	7.11×10^{-9}	6.02×10^{-1}	4.99×10^{-10}
12	5.12×10^{-2}	3.17×10^{-15}	7.95×10^{-3}	2.37×10^{-12}	2.48×10^{-1}	9.81×10^{-14}
16	1.83×10^{-3}	2.35×10^{-17}	5.22×10^{-4}	1.52×10^{-14}	1.37×10^{-1}	4.47×10^{-16}

The maximum absolute error in [11] with 4 steps is analyzed in a case which has result 4.01×10^{-4} while the maximum absolute error, using our method, which is presented in Table 1 with different steps. Numerical results of this problem show that our method converges exponentially and is more accurate than the method [11] in table 4.3 from their paper.

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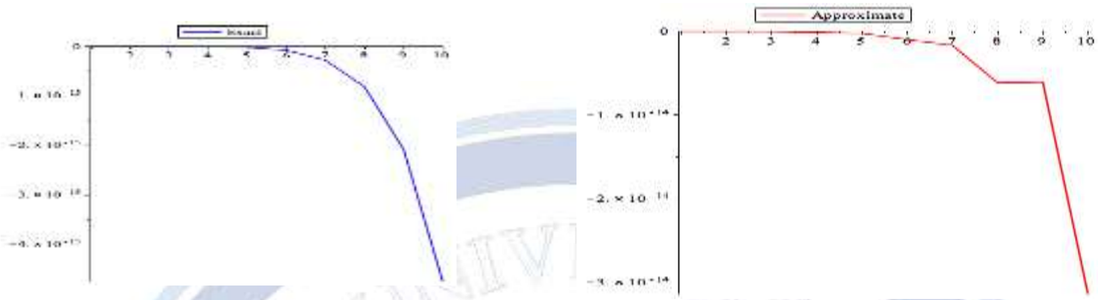

Figure 1: Exact and approximate solution for example1when $Tau = 0.5\ and\ h = 0.001$.

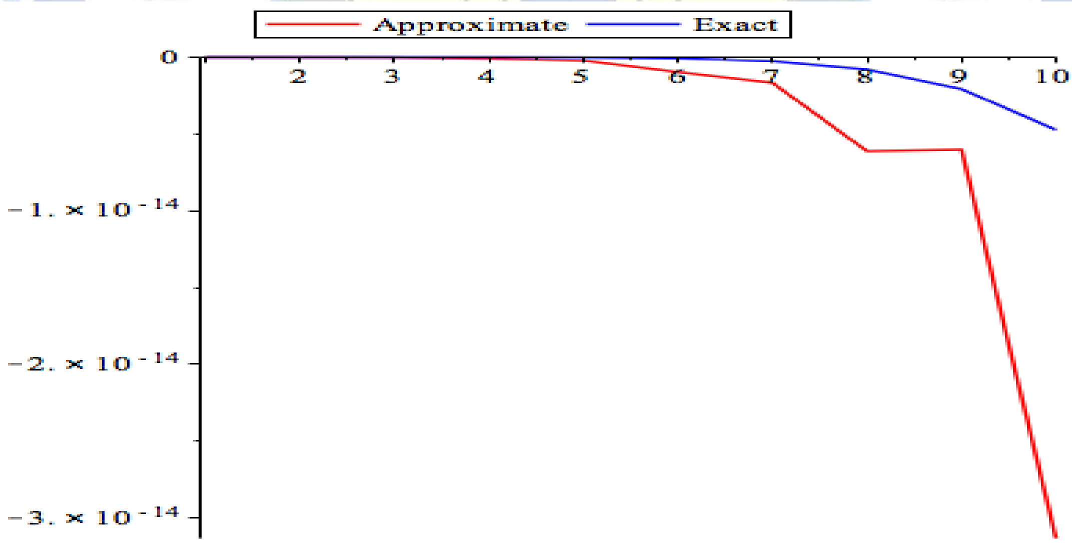

Figure 2: Exact and Approximate solution for example 1 when Tau= 0.5 and h=0.001

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Table 3: L_1, L_2, L_∞ errors of example 2 where $t = 3$

h	$L_\infty - error$	$L_1 - error$	$L_2 - error$
0.01	1.3684×10^{-4}	2.6830×10^{-5}	6.3869×10^{-2}
0.02	1.9586×10^{-3}	3.7914×10^{-4}	1.2376×10^{-1}
0.03	8.6494×10^{-3}	1.6624×10^{-3}	1.7893×10^{-1}
0.04	2.2999×10^{-2}	4.4311×10^{-3}	2.2823×10^{-1}
0.001	1.4888×10^{-8}	2.9619×10^{-9}	6.5493×10^{-3}
0.002	2.3615×10^{-7}	4.6898×10^{-8}	1.3063×10^{-2}
0.003	1.1849×10^{-6}	2.3492×10^{-7}	1.9543×10^{-2}
0.004	3.7112×10^{-6}	7.3454×10^{-7}	2.5968×10^{-2}

Table 4: The maximum absolute errors L_∞ of example 2 where $t = 3$

$if\ n = 8$ $h = 0.125$	$L_\infty - error$	$L_\infty(Zahra)$
0.125	3.7710×10^{-5}	1.73×10^{-4}
0.250	9.4802×10^{-3}	5.35×10^{-4}
0.375	2.1497×10^{-2}	7.98×10^{-4}
0.500	4.1214×10^{-4}	6.74×10^{-4}
0.625	5.3314×10^{-2}	9.50×10^{-5}
0.750	6.1940×10^{-2}	1.78×10^{-3}
0.875	6.2796×10^{-2}	3.42×10^{-3}
1	2.9910×10^{-1}	9.44×10^{-4}

The maximum absolute error in [5] with 8 steps is analyzed in a case which has result 3.42×10^{-3} while the maximum absolute error, using our method, which is presented in Table 3 with different steps. Numerical results of this problem show that our method converges exponentially and is more accurate than the method [5] in table 4 from their paper.

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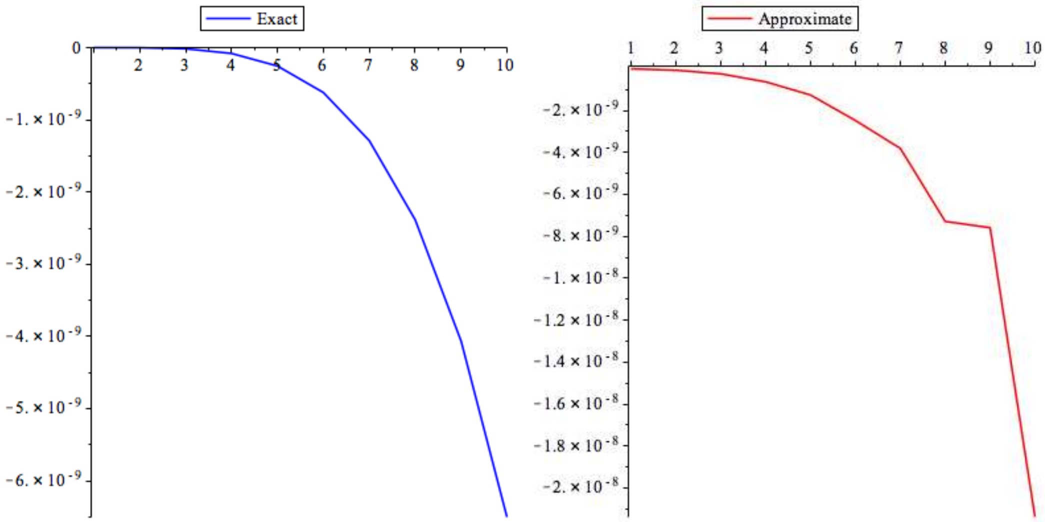

Figure 3: Exact and approximate solution for example2 when $\tau = 3$ *and* $h = 0.001$.

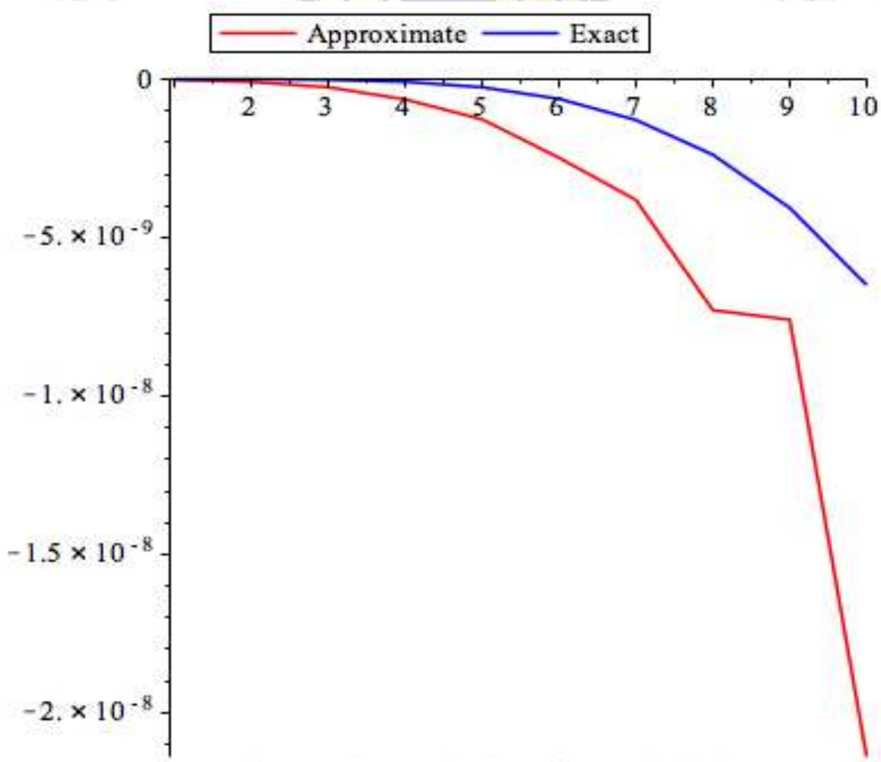

Figure 4: Exact and Approximate solution of example 2 when Tau=3 and h=0.001

7. Conclusion

In this paper, an attempt that is different form the ones in [10, 25] is made to construct a new kind of non-polynomial function with fractional continuity condition. To show the applicability of the proposed method, several model examples have been computed for the different fractional differential equations and various mesh sizes. Numerical results of our problems that we mentioned in our paper in tables 1, 4 and figures demonstrated apparently revealed that our method converges exponentially and more accurate than the method of [5, 11]. The order of convergence has been estimated for the method is extensively analyzed; finally the absolute errors are found and indicated our results by figures and tables to reveal the validity and creativity of the new technique.

Conflict of Interests.

There are non-conflicts of interest .

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