Journal of University of Babylon for Pure and Applied Sciences, Vol. (28), No. (3): 2020

Online ISSN: 2312-8135, Print ISSN: 1992-0652

Jacobi Weighted Moduli of Smoothness for Approximation by Neural Networks Application

Eman S. Bhaya [1] **Najlaa A. Hadi**[2]

1. Mathematics Department, College of Education for Pure Sciences, University of Babylon, Babylon, Iraq, emanbhya@itnet.uobabylon.edu.iq
2. Computer Sciences Department, College of Sciences for Girls, University of Babylon, Babylon, Iraq. najlaaadnan@yahoo.com

Article Information

Submission date: 8 / 9 / 2018

Acceptance date: 11 /11/ 2018

Publication date: 31/ 12/ 2020

Abstract

Moduli of smoothness are intended for mathematicians working in approximation theory, numerical analysis and real analysis. Measuring the smoothness of a function by differentiability is two grade for many purposes in approximation theory. More subtle measurement are provided by moduli of smoothness.

Many versions of moduli of smoothness and K-functionals introduced by many authors. In this work we choose two of these moduli and prove that they are equivalent themselves once and with a version of K-functional twice, under certain conditions.

As an application of our work we introduce a version of Jackson theorem for the approximation by neural networks.

Keywords: Jacobi modulus of smoothnessn. Neural networks. Best approximation. Modulus of smoothness.

1. Introduction, Definitions, Preliminaries, and Main Results

For f: $[-1,1] = I \rightarrow IR$, $L_p(I)$ is the space of all functions satisfying $\|f\|_{L_p(I)} < \infty$, we use the norm $\|f\|_{L_p(I)} = (\int_I |f|^p)^{1/p}$, $0 < p < \infty$ and

$L_{\omega,p}(I) := \{f : \|\omega f\|_{L_p(I)} < \infty\}$.

Definition 1.1 [1]

For $r \in N_o$, $r \geq 1$ and $0 < p \leq \infty$

$$C_p^r(\omega) = \{f : f^{r-1} \in AC_{loc}(-1,1) \text{ and } Q^r f^r \in L_{\omega,p}\},$$

where $AC_{loc}(-1,1)$, the space of all absolutely continuous functions on (-1,1).

Definition 1.2 [1]

For a weight function ω, and I ⊆ [-1,1]

$$L_{\omega,p}(I) := \{ f : \|\omega f\|_{L_p} < \infty\}.$$

Definition 1.3 [1]

For k,r ∈ N and $f \in C_p^r(\omega_{\alpha,\beta})$, $0 < p \leq \infty$

$$\omega_{k,r}^{Q}(f^r,\delta)_{\alpha,\beta,p} := \underbrace{sup}_{0\leq h\leq\delta} \left\| W^{\frac{r}{2}+\alpha,\frac{r}{2}+\beta}(.)\Delta_{hQ}^{k}(.)(f^r,.)\right\|_p .$$

where, $Q^r = \sqrt{1-x^2}$ and $r \in N$.

Natation 1.4

For $\propto$, $\beta \in J_p$

$$\omega_{\propto,\beta}(x) := (1-x)^{\propto}(1+x)^{\beta},$$

$$J_p = \begin{cases} \left(\frac{-1}{p}, \infty\right) & if\ 1 < P < \infty \\ [0,\ \infty) & if\ p = \infty \\ (-p,\ \infty) & if\ 0 < P < 1 \end{cases} .$$

Definition 1.5 [1]

For k, ∈ N and $h \geq 0$

$$\Delta_{hQ}^{k}(f,x;J) = \begin{cases} \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} f\left(x - \frac{khQ}{2} + ihQ\right) & if\ \left[x - \frac{khQ}{2}, x + \frac{khQ}{2}\right] \subseteq J \\ 0 & otherwise \end{cases} .$$

We denote the k^{th} forward and the k^{th} backward differences by $\vec{\Delta}_h^k(f,x) := \Delta_h^k\left(f, x + \frac{kh}{2}\right)$ and $\overleftarrow{\Delta}_h^k(f,x) := \Delta_h^k\left(f, x - \frac{kh}{2}\right)$ respectively.

Definition 1.6 [1]

Adopting the weighted Dt moduli which were defined in [1,p. 218 and (8.2.10)] for a weight ω on D:= [-1,1] , we set for $f \in L_{\omega,p}$,

$$\omega_Q^k(f,\delta)_{\omega,p} := \underbrace{\sup}_{0<h\le\delta} \left\|\omega(.)\Delta_{hQ}^k(f,.)\right\|_{L_p[-1+\delta^*,1-\delta^*]}$$

$$+ \underbrace{\sup}_{0<h\le\delta^*} \left\|\omega(.)\vec{\Delta}_h^k(f,.)\right\|_{L_p[-1,-1+12\delta^*]}$$

$$+ \underbrace{\sup}_{0<h\le\delta^*} \left\|\omega(.)\overleftarrow{\Delta}_h^k(f,.)\right\|_{L_p[1-12\delta^*,1]},$$

where $\delta^* := 2k^2\delta^2$, and $\omega(x) = \sqrt{1-x^2}\,, x \in J \subseteq [-1,1]$ and real δ.

Definition 1.7 [1]

The weighted K-functional was defined in [1,p.55(6.1.1)] as

$$K_{k,Q}(f,\delta^k)_{\omega,p} := \underbrace{\inf}_{g\in C_p^k(\omega)} \{\|\omega(f-g)\|_p + \delta^k\|\omega Q^k g^k\|_p\}.$$

In the articles [2],[3],[4],[5],[6],[7],[8] the authors introduced two versions of moduli of smoothness and K-functional under certain conditions we prove that these moduli are equivalent and also they are equivalent to a version of K-functional, as we see in the following results:

As a property of the modulus $\omega_{k,r}^Q$ the following raesult will be proved

Lemma 1.8

Suppose k is a natural number, r is a natural number or zero, satisfy $\frac{r}{2} + \alpha \ge 0, \frac{r}{2} + \beta \ge 0$ for 0<p<1, then whenever

$g \in C_p^{r+k}(\omega_{\alpha,\beta})$ that

$$\omega_{k,r}^Q(g^r,\delta)_{\alpha,\beta,p} < c(p)\delta^k\left\|\omega_{\alpha,\beta}Q^{k+r}g^{k+r}\right\|_p.$$

Lemma 1.9

Let $k \in N, r \in N_0$ and $\frac{r}{2} + \propto \ge 0, \frac{r}{2} + \beta \ge 0$ and p<1 if

$f \in C_p^r(\omega_{\alpha,\beta})$ then

$$\omega_{k,r}^Q(f^r,\delta)_{\alpha,\beta,p} \le c(p)k_{k,r}^Q(f^r,\delta^k)_{\alpha,\beta,p},\ \delta>0.$$

Our theorem for equivalence is

Journal of University of Babylon for Pure and Applied Sciences, Vol. (28), No. (3): 2020
Online ISSN: 2312-8135, Print ISSN: 1992-0652

Theorem 1.10

Let k be a natural number, $r \in N_0$ and $r/2+\propto \geq 0$, $r/2+\beta \geq 0$,p< 1 and $f \in C_p^r$ $\omega_{\alpha,\beta}$ then there exists natural N depending on k,r,p, ∝ and β such that for all 0< $\delta \leq 2/k$ and n ∈ N Satisfying

max {N , c_1/δ } ≤ n ≤ c_2/δ , then

$k_{k,r}^{Q}(f^r,\delta^k)_{\propto,\beta,p} \leq c(p)A_{k,r}^{Q}(f^r,n^{-k})_{\propto,\beta,p} \leq c(p)\omega_{k,r}^{*Q}(f^r,\delta)_{\propto,\beta,p}$

$\leq c(p)\omega_{k,r}^{Q}(f^r,\delta)_{\propto,\beta,p} \leq c(p)k_{k,r}^{Q}(f^r,\delta^k)_{\propto,\beta,p}$.

As a corollary of our main theorem above is

Corollary 1.11

For naturals k and r ,satisfies $\frac{r}{2}+\propto \geq 0$, $\frac{r}{2}+\beta \geq 0$,

$f \in C_p^r(\omega_{\alpha,\beta})$, thus for any $0<\delta \leq \frac{2}{k}$ and p < 1 , that

$$A_{k,r}^{Q}(f,n^{-k})_{\propto,\beta,p} \sim \omega_{k,r}^{*Q}(f^r,\delta)_{\propto,\beta,p} \sim \omega_{k,r}^{Q}(f^r,\delta)_{\propto,\beta,p}.$$

Now let us introduce an important theorem that give a property for the modulus $\omega_{k,r}^{Q}$,which is not esealy proved using the definition of $\omega_{k,r}^{Q}$.

Theorem 1.12

For the natural r and k that satisfies $\frac{r}{2}+\propto \geq 0$, $\frac{r}{2}+\beta \geq 0$, for $f \in C_{\beta}^{r}$, $\lambda \geq 1$, $\delta > 0$, $0<p<1$, we have :

$$\omega_{k,r}^{Q}\left(f^{(r)},\lambda\,\delta\right)_{\propto,\beta,p} \leq c(p)\omega_{k,r}^{Q}\left(f^{(r)},\delta\right)_{\propto,\beta,p}$$

For the equivalence of the moduli $\omega_{k,r}^{Q}$ and ω_{Q}^{k} the following result is

Theorem 1.13

If k and r are natural numbers satisfying $\frac{r}{2}+\propto \geq 0$and $\frac{r}{2}+\beta \geq 0$, $f \in C_{\beta}^{r}$. then

$$\omega_{k,r}^{k}\left(f^{(r)},\ \delta\right)_{\propto,\beta,p} \sim \omega_{Q}^{k}\left(f^{(r)},\ \delta\right)\omega_{\propto,\beta}\, Q_p^r.$$

Artificial forward neural networks are non-linear expressions representing functions.

Journal of University of Babylon for Pure and Applied Sciences, Vol. (28), No. (3): 2020
Online ISSN: 2312-8135, Print ISSN: 1992-0652

In the articles [1],[2],[3],[6],[7],[8],[9],[10],[11],[12],],[13],[14] the authors introduced Jackson version theorems used the first degree usual modulus of smoothness.

In [15] the author improved this degree of approximation using the k[th] order usual modulus of smoothness.

In our work we improve the results in [1],[2],[3],[6],[7],[8],[9],[10],[11],[12],],[13],[14] and in [15] to direct neural network theorem using weighted Jacobi modulus of smoothness, and prove:

Theorem 1.14

$\|f-p\|_{L_p[0,1]^d} \leq c(p,r,d)\omega_r(f,\frac{1}{n})_{L_p[0,1]^d}$

where r,d are naturals.

2 Auxiliary Results

This section consists of the lemmas that we need in our proofs of the main results.

Lemma 2.1

If r is a natural number, $p < 1$ and $\frac{r}{2}+\propto, \frac{r}{2}+\beta \in J_p$, then

$$C_p^{r+1}(\omega_{\propto,B}) \subseteq C_p^r(\omega_{\alpha,\beta}).$$

Proof/

$|\omega_{\propto,\beta}(x)Q^r(x)g^r(x)| < \pi 2^{\beta-\propto-1}|\omega_{\propto,\beta}Q^{r+1}(x)g^{r+1}(x)|,$

$|\omega_{\propto,\beta}(x)Q^r(x)g^r(x)|^p < \pi^p 2^{(\beta-\propto-1)^p}|\omega_{\propto,\beta}Q^{r+1}(x)g^{r+1}(x)|^p.$

Taking the integral over the interval I, then:

$$\int_I |\,\omega_{\propto,\beta}(x)Q^r(x)g^r(x)|^p\,dx$$

$$< \int_I \pi^p 2^{(\beta-\propto-1)^p}|\omega_{\propto,\beta}Q^{r+1}(x)g^{r+1}(x)|^p\,dx$$

Using the hypothesis $f \in C_p^{r+1}(\omega_{\propto,\beta})$, then

$$\int_I |\,\omega_{\propto,\beta}Q^{r+1}(x)g^{r+1}(x)|^p\,dx < \infty$$

Thus

$$\left\|\omega_{\propto,\beta}(x)Q^r g^r\right\|_p < \infty.$$

Lemma 2.2 [1]

Let k∈ N , r ∈ N_o , r/2 + ∝≥ o , r /2 + β≥ o, and 0 < p ≤∞ if $f \in C_p^r(\omega_{\propto,\beta})$, then

$$\omega_{k,r}^Q(f^r,\delta)_{\propto,\beta,p} \le c(p)\left\|\omega_{\propto,\beta}Q^r f^r\right\|_p. \qquad \delta > 0$$

where c is a constant depends only on k and p.

Lemma 2.3 [16]

If $0 < q < p$ then

$$\left(\sum|f|^p\right)^{1/p} < \left(\sum|f|^q\right)^{1/q}.$$

Lemma 2.4

Suppose $k \in N$, $r \in N_0$ or zero satisfy $\frac{r}{2}+\propto \ge 0$, $\frac{r}{2}+\beta \ge 0$ for $0 < p < 1$, then whenever

g ∈ $C_p^{r+k}(\omega_{\propto,\beta})$ that

$$\omega_{k,r}^Q(g^r,\delta)_{\propto,\beta,p} \le c(p)\delta^k\left\|\omega_{\propto,\beta}Q^{k+r}g^{k+r}\right\|_p.$$

In this work $c(p)$ I a constant depending on p only and may differ from step to another step.

Proof/

For the proof, we shall use a method from [17]

$$\omega_{k,r}^Q(g^r,\delta)_{\propto,\beta,p} := \underbrace{sup}_{0<h\le\delta}\left\|W_{kh}^{\frac{r}{2}+\propto,\frac{r}{2}+\beta}\ \Delta_{hQ}^{k}(g^r,\ .)\right\|_{L_{p(D_{hk})}} \quad \omega_{k,r}^Q(g^r,\delta)_{\propto,\beta,p} :=$$

$$\underbrace{sup}_{0<h\le\beta}\left\|W_{kh}^{\frac{r}{2}+\propto,\frac{r}{2}+\beta}\int_{\frac{-hQ}{2}}^{\frac{hQ}{2}}\ldots\ldots\int_{\frac{-hQ}{2}}^{\frac{hQ}{2}} g^{r+k}(x+u_1\ldots\ldots u_k)du_1du_2\ldots\ldots du_k\right\|_{L_p(D_{hk})},$$

Since $\omega_{kh}^{\frac{r}{2}+\propto,\frac{r}{2}+\beta}(x) \le \omega_{\propto,\beta}(y)Q^r(y)$.For any $y \in (x-\frac{khQ}{2}, x+\frac{khQ}{2})$, so we get

Journal of University of Babylon for Pure and Applied Sciences, Vol. (28), No. (3): 2020

Online ISSN: 2312-8135, Print ISSN: 1992-0652

$\omega_{k,r}^{Q}(g^r,\delta)_{\propto,\beta,p}$

$$\leq \underbrace{sup}_{0<h\leq\delta} \left\| \int_{\frac{-hQ}{2}}^{\frac{hQ}{2}} \dots \dots \int_{\frac{-hQ}{2}}^{\frac{hQ}{2}} Q^{-k} (\omega_{\alpha,\beta} Q^{k+r} g^{k+r})(\mathrm{x} + \mathrm{u}_1 \dots \dots$$

$$+ \mathrm{u}_k)\, \mathrm{du}_1\, \mathrm{du}_2 \, \dots .. \mathrm{du}_k \right\|_{Lp(D_{hk})}$$

$$\omega_{k,r}^{Q}(g^r,\delta)_{\propto,\beta,p} \leq \underbrace{\sup}_{0<h\leq\delta} \left\| \int_{\frac{-hQ}{2}}^{\frac{hQ}{2}} \dots \int_{\frac{-hQ}{2}}^{\frac{hQ}{2}} . \|Q^{-k}\|_q \left\|\omega_{\propto,\beta}\, Q^{k+r}\, g^{k+r}\right\|_p du_2 \dots . du_k \right\|_{\ell} \quad \ell \geq 1$$

$$\leq \; c(p) \, \|(hQ)^{k-1}\|_{\mathrm{p}} \; \|Q^{-k}\| \left\|\omega_{\propto,\beta}\, Q^{k+r}\, g^{k+r}\right\|_{\ell} \quad \ell \geq 1$$

$$\leq \; c(p)\, \delta^{\,\mathrm{k}} \left\|\omega_{\propto,\beta}\, Q^{k+r}\, g^{k+r}\right\|_{\ell} \quad \ell \geq 1$$

$$\leq \; c(p)\, \delta^{\,\mathrm{k}} (\int_{-I}^{1} |\, \omega_{\propto,\beta}\, Q^{k+r}\, g^{k+r}(\mathrm{x_i})|^{\ell}\; |\Delta x_i|^{1/\ell} \,,$$

where $x_1 . x_2 \dots x_n$ is a partition for [-1,1]with $|[x_i, x_{i+1}]| = \Delta\, x_i$,

such that $|\, \Delta\, x_i| \leq z$, z is a positive constant depending on p only.

Using Lemma 2.3, it will

$$\omega_{k,r}^{Q}(g^r,\delta)_{\propto,\beta,p} \leq c(p)\delta^{\mathrm{k}} (\left[\sum_{i=1}^{\Lambda} \left|\omega_{\alpha,\beta}\; Q^{r+k} g^{r+k}(x_i)\right|^{p} \Delta x_i)^{\frac{1}{p}} (z)^{\frac{1}{\ell}+\frac{1}{p}} \right]$$

$$\leq c(p)\delta^{\,\mathrm{k}} \left\|\omega_{\propto,\beta}\; Q^{r+k}\; g^{r+k}\right\|_{\mathrm{p}}.$$

Lemma 2.5

Let $k \in N$, $r \in N_o$ and $r/2+ \propto \geq o$, $\delta/2 + \beta \geq o$ and $p < 1$.

If $f \in C_p^r(\omega_{\propto,\beta})$, then

$$\omega_{k,r}^{Q}(f^r,\delta)_{\propto,\beta,p} \leq c k_{k,r}^{Q}(f^r,\delta^k)_{\propto,\beta,p} \quad \delta > 0.$$

Proof/

Let $g \in C_p^{r+k}(\omega_{\propto}, \beta)$. By Lemma 2.1, then $g \in C_p^{r}\, \omega_{\propto,\beta}$

$$\omega_{k,r}^{Q}(f^r,\delta)_{\propto,\beta,p} \leq \omega_{k,r}^{Q}(f^r,\delta)_{\propto,\beta,p} - \;\; \omega_{k,r}^{Q}(g^r,\delta)_{\propto,\beta,p} + \omega_{k,r}^{Q}(g^r,\delta)_{\propto,\beta,p}$$

$$\omega_{k,r}^{Q}(g^r,\delta)_{\propto,\beta,p} \leq \omega_{k,r}^{Q}((f^r - g^r),\delta)_{\propto,\beta,p} + \omega_{k,r}^{Q}(g^r,\delta)_{\propto,\beta,p}.$$

By Lemma 2.2 we get

$$\omega^{Q}_{k,r}(f^r,\delta)_{\propto,\beta,p} \leq c\|\omega_{\propto,\beta,p}(f^r-g^r)\|_p+\omega^{Q}_{k,r}(g^r,\delta)_{\propto,\beta,p}$$

By Lemma 2.4 we get

$$\omega^{Q}_{k,r}(f^r,\delta)_{\propto,\beta,p} \leq C\ \|\omega_{\propto,\beta,p}(f^r-g^r)\|_p+c\delta^k\|\omega_{\propto,\beta,p}Q^{k+r},g^{k+r}\|_p$$

This completes the proof.

Lemma 2.6 [1]

For $k \in N$, $r \in N_o$ and $h \geq o$

$$\Delta^k_h(f,x) = \int_{-h/2}^{h/2} \dots \int_{-h/2}^{h/2} f^k(x+u_1+\cdots+u_k)du_1 \dots du_k .$$

Lemma 2.7

$$K^{Q}_{k,r}(f^r,\delta^k)_{\propto,\beta,p} \leq \mu^k k^{Q}_{k,r}(f^r,(\delta/\mu)^k{}_{\alpha,\beta,p})$$

$$\leq c(p)\mu^k A^{Q}_{k,r}(f^r,n^{-k})_{\alpha,\beta,p}$$

$$\leq c(p)\omega^{*Q}_{k,r}(f^r,Q/n)_{\alpha,\beta,p}$$

$$\leq c(p)\omega^{*Q}_{k,r}(f^r,\delta)_{\alpha,\beta,p}$$

$$\leq c(p)\omega^{Q}_{k,r}(f^r,\delta)_{\alpha,\beta,p}.$$

Lemma 2.8

$$\omega^{*Q}_{k,r}(f^r,\delta)_{\propto,\beta,p} \leq c(p)\omega^{Q}_{k,r}(f^r,\delta)_{\propto,\beta,p},$$

when $\delta > 0$, $0 < p \leq \infty$.

3 The Equivalence Results

In this section, we shall introduce and prove our theorems for the equivalence of the moduli of smoothness themelves and the moduli of smoothness with K-functional. As an application we introduce a neural network approximation theorem.

Journal of University of Babylon for Pure and Applied Sciences, Vol. (28), No. (3): 2020
Online ISSN: 2312-8135, Print ISSN: 1992-0652

Theorem 3.1

Let k be a natural number, $r \in N_o$ and $r/2+\propto \geq 0, r/2+\beta \geq 0, p<1$ and $f \in C_p^r(\omega_{\propto,\beta})$ then there exists natural N depending on $k, r, p, \propto$ and β such that for all 0< $\delta \leq 2/k$ and $n \in$ N Satisfying

max {N , c_1/δ } ≤ n ≤ c_2/δ , then

$k_{k,r}^Q(f^r,\delta^k)_{\propto,\beta,p} \leq c(p)A_{k,r}^Q(f^r,n^{-k})_{\propto,\beta,p} \leq c(p)\omega_{k,r}^{*Q}(f^r,\delta)_{\propto,\beta,p}$

$\leq c(p)\omega_{k,r}^Q(f^r,\delta)_{\propto,\beta,p} \leq c(p)k_{k,r}^Q(f^r,\delta^k)_{\propto,\beta,p}.$

Proof/ By Lemma 2.6

$$\begin{aligned} K_{k,r}^Q(f^r,\delta^k)_{\propto,\beta,p} &\leq c(p)\mu^k K_{k,r}^Q(f^r,(\delta/\mu)^k)_{\propto,\beta,p} \\ &\leq c(p)\mu^k A_{k,r}^Q(f^r,n^{-k})_{\propto,\beta,p} \\ &\leq c(p)\omega_{k,r}^{*Q}(f^r,Q/n)_{\propto,\beta,p} \\ &\leq c(p)\omega_{k,r}^{*Q}(f^r,\delta)_{\propto,\beta,p}. \end{aligned}$$

Now by Lemma 2.8, then

$\omega_{k,r}^{*Q}(f^r,\delta)_{\propto,\beta,p} \leq \omega_{k,r}^Q(f^r,\delta)_{\propto,\beta,p},$

$K_{k,r}^Q(f^r,\delta^k)_{\propto,\beta,p} \leq c(p)\omega_{k,r}^Q(f^r,\delta)_{\propto,\beta,p}.$

By lemma 2.5 we get

$K_{k,r}^Q(f^r,\delta^k)_{\propto,\beta,p} \leq c(p)A_{k,r}^Q(f^r,n^{-k})_{\propto,\beta,p}$

$\leq c(p)\omega_{k,r}^{*Q}(f^r,\delta)_{\propto,\beta,p} \leq c(p)\omega_{k,r}^Q(f^r,\delta)_{\propto,\beta,p} \leq c(p)\omega_{k,r}^Q(f^r,\delta^k)_{\propto,\beta,p}.$

Corollary 3.2

For naturals k and r , satisfies $\frac{r}{2}+\propto \geq 0$, $\frac{r}{2}+\beta \geq 0$ and $f \in C_p^r(\omega_{\propto,\beta})$, then for any $0<\delta \leq \frac{2}{k}$ and $p<1$, we have

$$A_{k,r}^Q(f,n^{-k})_{\propto,\beta,p} \sim \omega_{k,r}^{*Q}(f^r,\delta)_{\propto,\beta,p} \sim \omega_{k,r}^Q(f^r,\delta)_{\propto,\beta,p}$$

Then, let us introduce an important theorem that give a property for the modulus $\omega_{k,r}^Q$, which is not esealy proved using the definition of $\omega_{k,r}^Q$.

Journal of University of Babylon for Pure and Applied Sciences, Vol. (28), No. (3): 2020
Online ISSN: 2312-8135, Print ISSN: 1992-0652

Theorem 3.3

For the natural $r, k \in N$, that satisfy $\frac{r}{2} + \propto \geq 0$, $\frac{r}{2} + \beta \geq 0$, for $f \in C_{\beta}^{r}$, $\lambda \geq 1$, $\delta > 0$, $0 < P < 1$, then :

$$\omega_{k,r}^{Q}\left(f^{(r)}, \lambda\, \delta\right)_{\propto,\beta,p} \leq c(p)\omega_{k,r}^{Q}\left(f^{(r)}, \delta\right)_{\propto,\beta,p}$$

Proof/ Using Theorem 3.1, it is

$$\omega_{k,r}^{Q}\left(f^{(r)}, \lambda\, \delta\right)_{\propto,\beta,p} \leq c(p)K_{k,r}^{Q}\left(f^{(r)}, (\lambda\, t)^{k}\right)_{\propto,\beta,p}$$

$$\leq c(p)(\left\|\omega_{\alpha,\beta}Q^{r}(f^{r}-g^{r})\right\|_{p} + (\lambda\, \delta)^{k}\left\|\omega_{\alpha,\beta}Q^{k+r}\, P_{n}{}^{k+r}\right\|)$$

$$\leq \lambda^{k}\ c(p)\ (\left\|\omega_{\alpha,\beta}Q^{r}(f^{r}-g^{r})\right\|_{p} + \delta^{k}\left\|\omega_{\alpha,\beta}Q^{k+r}\, P_{n}{}^{k+r}\right\|_{p})$$

$$= \lambda^{k}\ K_{k,r}^{Q}(f, \delta^{k})_{\alpha,\beta,p}$$

Now by using Theorem 3.1 again to have

$$\omega_{k,r}^{Q}\ \left(f^{(r)},\ \lambda\, \delta\right)_{\propto,\beta,p} \leq c(p)\ \lambda^{k}\ \omega_{k,r}\left(f^{(r)},\ \delta\right)_{\propto,\beta,p}.$$

Theorem 3.4

If $k, r \in N$ satisfying $\frac{r}{2} + \propto \geq 0$ and $\frac{r}{2} + \beta \geq 0$, $f \in C_{\beta}^{r}$. Then

$$\omega_{k,r}^{k}\ \left(f^{(r)},\ \delta\right)_{\propto,\beta,p} \sim \omega_{Q}^{k}\ \left(f^{(r)},\ \delta\right)\omega_{\propto,\beta}\ \mathrm{Q}_{\mathrm{p}}^{\mathrm{r}}.$$

Proof/ The proof is directly using Corollary 3.2, it means:

$$\omega_{k.r}^{Q}\ \left(f^{(r)},\ \delta\right)_{\propto,\beta,p} \leq K_{k.r}^{Q}\ \left(f^{(r)},\ \delta^{k}\right)_{\propto,\beta} \leq c(p)\omega_{k,r}^{*Q}(f^{r}, \delta)_{\propto,\beta,p} \leq$$

$$c(p)\omega_{k,r}^{Q}(\ f^{r}, \delta^{r})_{\propto,\beta,p} \leq c(p)\omega_{k,r}^{*Q}(\ f^{r}, \delta^{r})_{\propto,\beta,p}.$$

4. The Neural Networks Application

Artificial forward neural networks are nonlinear expressions representing multivariate numerical function. In connection with such paradigms there arise mainly three problems: a density problem, a complexity problem, and an algorithmic problem. The density problem deals with the following question: which function can be approximated in a suitable sense.

"The mathematical expression of neural networks is:

$$N(x) = \sum_{i=1}^{m} c_{i}\, \sigma_{i}\ (\sum_{j=1}^{d} \omega_{ij}x_{j} + \theta_{i}),\ \ x \in R^{d}\, , d \geq 1,$$

where for any $1 \leq i \leq m$, $\theta_i \in R$, is the threshold, $\omega_i = (\omega_{i1}, \omega_{i2}, \dots \dots, \omega_{is})^T \in R^d$ is the connection weight of neuron I in the hidden layer with the input neurons, and $c_i \in R$ is its connection weight with the output neuron , and $\sigma_i(0)$ is the sigmoidal activation function."

In [9],[10],[11],[13],[14],[18], the authors introduced direct theorems using the first degree usual modulus of smoothness. Then in [15] the authors improved the above results to kth degree modulus of smoothness. In this section we improve the result in [15] to the weighted Jacobi modulus of smoothness. In [15], the author prove the following result

Lemma 4.1 [15]

For any f in $L_{p[0,1]^d}$, there exists a neural networ $p(x) = \sum_{a\in(0.....n)^d} a_\propto \pi_{i=1}^d X_i^{ai}$ satisfy

$$\|f - p\|_{L_p[0,1]^d} \leq c(p,r,d)\omega_r\left(f, \frac{1}{n}\right)_{L_p[0,1]^d},$$

where r, and d are naturals.

Using the same lines used for Theorem2.1 in [15] to get the following Corollary4.2

$$\|f - p\|_{L_p[0,1]^d} \leq c(p,r,d)\omega_r^k(f, \tfrac{1}{n})_{L_p[0,1]^d}$$

$$\leq c\,(p\,,r,d)A_{k,r}^Q(f, \tfrac{1}{n})_{L_p[0,1]^d}.$$

Now let us introduce our man theorem here

Theorem 4.3

$$\|f - p\|_{L_p[0,1]^d} \leq c(p,r,d)\omega_r^k(f, \frac{1}{n})_{L_p[0,1]^d}$$

Where $r, d \in N$

Proof/ Using Corollary 3.2**,** for $f \in [0,1]^d$, we can find a neural network

$$p(x) = \sum_{a\in(0.....n)^d} \; a_\propto \; \pi_{i=1}^d X_i^{ai}$$

$$\|f - p\|_{L_p[0,1]^d} \leq c(p,r,d)\omega_r^k(f, \frac{1}{n})_{L_p[0,1]^d}$$

For $r = 0$ in Theorem 3.4. Using Theorem 3.4, to obtain

$$\|f-p\|_{L_p[0,1]^d} \le c(p,r,d)\omega_{k,r}^{Q}\left(f,\frac{1}{n}\right).$$

Conclusions

In recent years some authors defined some versions of moduli of smoothness and K-functionals. We conclude that under certain conditions we can show that these moduli and K-functional are equivalent. So we can use them to determine the degree of neural networks approximation of functions in L_p space for $p < 1$.

Conflict of Interests.

There are non-conflicts of interest .

References

[1] K.A. Kopotun, D. Leviatan and I.A. shevchuk, " On Moduli of Smoothness with Jacobi Weights, " to appear, (December 30, 2017).

[2] V. K. Dzyadyk and I.A.Shevchuk, "Theory of Uniform Approximation of Functions by Polynomials", Walter de Gruyter, Berlin, (2008).

[3] Z. Ditzian and V. Totikm Modululi of Smoothness, "Springer Series in Computational Mathematics", vol.9, Springer-Verlag, New York, (1987).

[4] D.A. Grave, "Demonstration Dun Theoreme Detche-Ycheff Generalize, " Crelle Journ, 140, (1911).

[5] K.A. Kopotun, D. Leviatan and I.A. Shevchuk, "On Weighted Approximation With Jacobi Weights Preprint, ".

[6] K.A. Kopotun, D. Leviatan and I.A. Shevchuk, "Are the Degrees of the Best (co) Convei and Unconstrained Polynomial Approximation the same?, " II, Ukrain. Mat. Zh. 62, no.3, 369-386 (Russian, with Russian summary); English transl., Ukrainian Math. J. 62 (2010), no. 3, 420-440, (2010).

[7] K.A. Kopotun, D. Leviatan, A. Prymak and I.A. Shevchuk, "Uniform and Pointwise Shape Preserving Approximation by Algebraic Polynomials, " Surv, Approx, Theory 6, 24-74, (2011).

[8] K.A. Kopotun, "Weighted Moduli of Smoothness of K-mamatone Functions and Application, " J. Approx. Theory, 192, 102-131, (2015).

[9] G. Cybenko, "Approximation Superpositions of Asigmoidal Function", Math. Contr. Signals Syst., vol. 2, pp. 303-314, (1989).

[10] K.I. Funahashi, "On the Approximate Realization of Continuous Mappings by Neural Networks, " Neural Networks, vol. 2, pp. 183-192, (1989).

[11] K.M. Hornik Stinchcombe and H. White, "Multilayer-Feed Forward Networks are Universal Approximators," Neural Networks, vol, 2, pp. 359-366, 1989, (2011).

[12] K.A. Kopotun, D. Leviatan, and I.A. Shevchuk, "New Moduli of Smothness: Weighted Dt Moduli Revisited and Applied, " Constr. Approx. 42, 129-159, (2015).

[13] G. Ritter, "Efficient Estimation of Neural Weights by Polynomial Approximation," IEEE Transactions on information Theory, vol. 45, No. 5, July, pp. 1541-1550, (1999).

[14] J. Wang and X. Zongben , "New Study on Neural Networks: the Essential Order of Approximation, " science in chin, Neural Networks 23 618_624a, (2010).

[15] E. Bhaya and Z. Mahmoud, "Nearly Expountial Approximation for Neural Networks", Journal of Babylon University/ Pure and Applied Sciences, vol. 26, No. 1, PP.103-106, (2018).

[16] A.E.Taylor and D.C. Lay, Introduction to Functional Analysis, California, Copyright 1958. 1980 by John wiley and sons. Inc.

[17] K.A. Kopotun, "Uniform Polynomial Approximation with A^+ Weights having Finitely Many Zeros, " J. Math, Anal. Appl. 435, no. 1, 677-700, (2016).

[18] M.A. Kareem, "On the Approximation in Suitable Space," M.SC. thesis, University of Babylon, (2011).

Journal of University of Babylon for Pure and Applied Sciences, Vol. (28), No. (3): 2020

Online ISSN: 2312-8135, Print ISSN: 1992-0652

الخلاصة

اعدت مقاييس النعومة للرياضيين الذين يشتغلون في نظرية التقريب والتحليل العددي والتحليل الحقيقي. ان قياس نعومة دالة باستمرارية اشتقاقها اكثر من مرة هي طريقة مجة جدا. ان الطريقة الافضل والاكثر ملائمة لقياس نعومة دالة هو استخدام مقياس النعومة.

تم تعريف العديد من مقاييس النعومة وانواع من الدالي−K , من قبل الكثير من المشتغلين في نظرية التقريب. في هذا البحث تم اختيار اثنين من مقاييس النعومة وأحد انواع الدالي−K التي تم تعريفها مسبقا. بعدها قمنا ببرهان ان هذين المقياسين متكافئين تحت شروط معينة بالاضافة الى انهما يكافئان الدالي−K تحت نفس الشروط.

وكتطبيق للعمل اعلاه قمنا بتقديم احد انواع مبرهنة جاكسون للتقريب باستخدام الشبكات العصبية.

الكلمات الدالة: مقياس النعومة جاكوبيا. الشبكة تاعصبية. افضل تقريب. مقياس النعومة.