P- ISSN 1991-8941 E-ISSN 2706-6703 Journal of University of Anbar for Pure Science (JUAPS) Open Access
2017,11 (3) :102-109

2-Maximal Submodules and Related Concepts.

Haibat K. Mohammad ali Akram S. Mohammed Mohammad D. Sallman

College of Computer Science and Mathematics-Tikrit University

ARTICLE INFO

Received: 7 / 5 /2017
Accepted: 3 / 12 /2017
Available online: 22/6/2018
DOI: 10.37652/juaps.2017.148087
Keywords:

ABSTRACT

"Throughout this paper R represents commutative ring with identity and M is unitary leftR-module", the purpose of this paper is to study new concept (up to our knowledge) , named 2-maximal submodule which is a generalization of maximal submodule , "where a submodule N of an R-module M is called 2-maximal" submodul of M if and only if $\frac{M}{N}$ is2-regular R-module. Many characterizations and properties of 2-maximal submodules are given. Moreover we studied the behavior of 2-maximal submodule in some classes of module. Finally we give the sufficient condition 2-maximal submodules to be semi-maximal weak-maximal submodules are introduced.

1. Introduction

"Let R be commutative ring with identity and M is unitary R-module". "A proper submodule N of an R-module M is called maximal if and only if there is no proper submodule of M different from N containing N". Equivalently "N is maximal in M if and only if $\frac{M}{N}$ is simple R-module". kalaf in[1] generalized the concept of maximal submodule to semi-maximal submodule , where he called "a proper submodule N of an R-module M is semi-maximal submodule if and only If $\frac{M}{N}$ is semi-simple R-module". Another generalization of maximal submodule is introduce by shwkaea in [2] called weak-maximal submodule , where "a submodule N of an R-module M is called weak-maximal submodule of M if and only if $\frac{M}{N}$ is regular R-module". Ghaleb in [3] introduce the concept 2-reguler R-module," where an R-module M is called 2-reguler if every submodule of M is 2-pure ", where a submodule N of an R-module M is "called 2-pure submodule of M if for each ideal I of R" ,$I^2M \cap N = I^2N$. Every regular R-module is 2-regular but the convers is not true, this lead us to introduce another generalization of maximal submodule called 2-maximal submodule, also a generalization of both semi-maximal and weak-maximal submodule. "where a submodule N of an R-module M is called 2-maximal if and only If $\frac{M}{N}$ is 2-regular R-module".

The main purpose behind writing this paper is to give comprehensive investigation of the properties, characterizations and examples of 2-maximal submodule , and "we look for any connection between these concept, and other classes of modules".

2. 2-Maximal Submodules

" In this section we introduce the definition of2-maximal submodule and try to give module theoretic characterizations and properties of 2-maximal submodule".

Definition (2.1) " A submodule N of an R-module M is called 2-maximal submodule of M if and only if $\frac{M}{N}$ is 2-regular R-module".

* Corresponding author at: College of Computer Science and Mathematics-Tikrit University
.E-mail address:

P- ISSN 1991-8941 E-ISSN 2706-6703 Journal of University of Anbar for Pure Science (JUAPS) Open Access
2017,11 (3) :102-109

Remarks and Examples (2.2) (1) Every maximal submodule ofan R-module M is2-maximal, while the convers is not true in general, for example 6Z is 2-maximal submodule of Z-module Z , But 6Z is not maximal submodule of Z-module Z , Since $\frac{Z}{6Z} \cong Z_6$ is not simple Z-module while Z_6 is 2-regular Z-module

(2) "Every semi-maximal submodule of an R-module M" is 2-maximal ,while the convers is not true , for example , the submodule 25Z of Z-module Z is 2-maximal submodule But notsemi-maximal Since $\frac{Z}{25Z} \cong Z_{25}$ is 2-regular Z-module but not semi-simple Z-module.

(3) Every weak-maximal submodule of an R-module M is 2-maximal submodule of M, while the convers is not true , for example the submodule 4Z of a Z-module Z is 2-maximal submodule of Z But not weak-maximal submodule , Since $\frac{Z}{4Z} \cong Z_4$ is 2_regular module over Z , but not regular Z-module.

"The following theorem is characterization of 2-maximal submodules ".

Theorem (2.3): "Let M be an R-module then the following statements are equivalent ":

(1) Every submodule of M is 2-maximal submodule.

(2) (2) Every cyclic submodule of M is 2-maximal submodul.

(3) (3) Every finitely generated submodule of M is 2-maximal submodule.

Proof: (1) $\Longrightarrow$ (2) Follows directly.

(2) $\Longrightarrow$ (3) Assume every cyclic submodule K of M is 2-maximal submodule of M , and let N be a finitely generated submodule of M. Since K is 2-maximal submodule of M , then $\frac{M}{K}$ is 2-regular R-module , Hence by [3, prop.(1.1.5)] M is 2-regular R-module. Again by [3,prop.(1.1.5)] $\frac{M}{N}$ is 2-regular R-module. Therefor N is 2-maximal submodule of M.

(1) $\Longrightarrow$ (3) Follows directly.

(3) $\Longrightarrow$ (2) Since cyclic submodule is finitely generated and by hypothesis every finitely generated submodule is 2-maximal , Then every cyclic submodule of M is 2-maximal submodule.

The following proposition is another characterization of 2-maximal submodule.

Proposition (2.4) Let M be an R-module. Then the following statements are equivalent :

(1) Every submodule of M is 2-maximal submodule.

(2) Every cyclic submodule of M is 2-pure.

(3) Every finitely generated submodule of M is 2-pure.

Proof:

(1)$\Longrightarrow$ (2) Suppose that every submodule N of M is 2-maximal submodule of M , then $\frac{M}{N}$ is 2-regular R-module.Thus by [3,Prop.(1.1.5)] M is 2-regular R-module. Hence by [3,Theo.(1.1.14)] every cyclic submodule of M is 2-pure.

(2) $\Longrightarrow$ (3) Follows by [3,Theo.(1.1.14)].

(3) $\Longrightarrow$ (1) "Since every finitely generated submodule of" M is 2-pure then by [3,The.(1.1.14)] M is 2-regular R-module ,Thus by[3,Prop.(1.1.15)] $\frac{M}{N}$ is 2-regular for each submodule N of M. Hence N is 2-maximal submodule of M , that is every submodule of M is 2-maximal submodule of M.

"The following propositions are give basic properties of 2-maximal submodule".

Proposition (2.5) The homomorphic image of 2-maximal submodule of an R-module M is 2-maximal submodule.

Proof: Let M and $\acute{M}$ be two R-modules , and $f: M \longrightarrow \acute{M}$ be an R-epimorphism and K is2-maximal submodule of M , then $\frac{M}{K}$ is 2-regular R-module. Now let $g: \frac{M}{K} \longrightarrow \frac{\acute{M}}{f(K)}$

be a function define by $g(m+K) = f(m) + f(K)$, $m \in M$ toprove that g is well-defined

suppose that $m_1 + K = m_2 + K$, $m_1, m_2 \in M$, then $m_1 - m_2 \in K$, then $f(m_1 - m_2) \in f(K)$

Then $f(m_1) - f(m_2) \in f(K)$, hence $f(m_1) + (K) = f(m_2) + (K)$.

that is $g(m_1 + K) = g(m_2 + K)$.To prove that g is an R-homomorphism.Let $m_1 + K, m_2 + K \in \frac{M}{K}$, $m_1, m_2 \in M$ and $r \in R$, then

$$g(m_1 + K) \oplus (m_2 + K)) = g(m_1 + m_2 + K) = f(m_1 + m_2) + f(K) = f(m_1) + f(m_2) + f(K) = (f(m_1) + f(K)) \oplus (f(m_2) + f(K)) = g(m_1 + K) \oplus g(m_2 + K)$$

$$= g\big(r \odot (m_1 + K)\big) = g(rm_1 + K)$$

$$f(rm_1) + f(K) = rf(m_1) + f(K) = r \odot g(m_1 + K)$$

Clearly g is onto. Hence by [3,Cor.(1.1.12)] $\frac{\grave{M}}{f(k)}$ is 2-regular 2-module , Hence f(k) is 2-maximal submodule of $\grave{M}$.

Proposition (2.6) Let M , $\grave{M}$ be R-module and $f: M \longrightarrow \grave{M}$ be an R-epimorphsim with $\grave{M}$ is 2-regular R-module , Then Kerf is 2-maximal submodule of M.

Proof: Let $f: M \longrightarrow \grave{M}$ be an R-epimorphism and $\grave{M}$ is 2-regular R-module. Hence by first isomorphism theorem $\frac{M}{Kerf} \cong \grave{M}$, which implies that $\frac{M}{Kerf}$ is 2-regular R-module Thus Kerf is a 2-maximal submodule of M.

Proposition (2.7) "Let M be an R-module , and N_1 , N_2 are 2-maximal submodules of M Then $N_1 \cap N_2$ is a 2-maximal submodule of M".

Proof: Since N_1 , N_2 are 2-maximal submodulse of M , Then , $\frac{M}{N_1}$, $\frac{M}{N_2}$ are 2-regularR-modules. Hence by [3,coro.(1.1.18)] $\frac{M}{N_1} \oplus \frac{M}{N_2}$ is 2-regular R-module. Then by [4,Prop.(1.1.50)] $\frac{M}{N_1 \cap N_2} \cong \frac{M}{N_1} \oplus \frac{M}{N_2}$ which implies that $\frac{M}{N_1 \cap N_2}$ is 2-regular R-module Hence $N_1 \cap N_2$ is 2-maximal submodule of M.

Proposition (2.8) "Let M be an R-module , and N_1 , N_2 are submodules of M" with $N_1 \subseteq N_2$. Then N_2 Is " a 2-maximal submodule of M if and only If $\frac{N_2}{N_1}$ is a 2-maximal submodule of $\frac{M}{N_1}$".

Proof: Suppose that N_2 is a 2-maximal submodule of M , Then $\frac{M}{N_2}$ is 2-regular R-module. Since $N_1 \subseteq N_2$, Then $\frac{N_2}{N_1}$ is submodule of $\frac{M}{N_1}$. By third isomorphism theorem we get $\frac{M}{N_2} \cong \frac{M/N1}{N2/N1}$ is 2-regular R-module , Hence $\frac{N_2}{N_1}$ is 2-maximal submodule o f $\frac{M}{N_1}$.

For the convers suppose that $\frac{N_2}{N_1}$ is a 2-maximal submodule of $\frac{M}{N_1}$,then $\frac{M/N1}{N2/N1}$ is 2-regularR-module , Again by third isomorphism theorem we have $\frac{M/N1}{N2/N1} \cong \frac{M}{N_2}$ 2-regular R-module Hence N_2 is a 2-maximal submodule of M.

Proposition (2.9) "If M be an R-module , and N_1 , N_2 are submodules of M with $N_1 \subseteq N_2$, if N_1 is 2-maximal submodule of M , Then N_2 is 2-maximal submodule of M".

Proof: Let N_1 be 2-maximal submodule of M , then $\frac{M}{N_1}$ is 2-regular R-module Since $N_1 \subseteq N_2$, then [by Prop. 2.8]$\frac{N_2}{N_1}$ is submodule of $\frac{M}{N_1}$. Hence by [3.Prop.(1.1.15)] $\frac{M/N1}{N2/N1}$ is 2-regularR-module. But by Third isomorphism theorem we get $\frac{M}{N_2} \cong \frac{M/N1}{N2/N1}$ is 2-regular R-module , Hence N_2 is 2-maximal submodule of M.

Definition (2.10) [5] : " Let N be a proper submodule of an R-module M , define $\sqrt{N} = \cap\{K: K \text{ is a prime submodule of } M \text{ containing } N\}$, $\sqrt{N}$ is a submodule of M with $N \subseteq \sqrt{N}$ ".

As a direct application of proposition (2.9) we have the following corollary :

Corollary (2.11) " Let M be an R-module , and N is 2-maximal submodule of M, then $\sqrt{N}$ is 2-maximal submodule of M."

Definition (2.12) [6] " Let N be a submodule of an R-module M , the closure of N denoted by cl(N) define $cl(N) = \{m \in M: [N:(m)] \text{ essential ideal in } R\}$
cl(N) is a submodule of M and $N \subseteq cl(N)$".

Corollary (2.13) " Let M be an R-module , and N is 2-maximal submodule of M , Then cl(N) is 2-maximal submodule of M ".

Definition (2.14) [2] "Let N be a submodule of an R-module M , and I is an ideal of R , define $[N:I] = \{x \in M : xI \subseteq N\}$ is a submodule of M with $N \subseteq [N:I]$ and $[N:R] = N$,
[I:R]=I"

Corollary (2.15) " Let M be an R-module , and N ba a 2-maximal submodule of M , If I is an ideal of R, then [N:I] is 2-maximal submodule of M ".

Proposition (2.16) Let M be an R-module , and N_1, N_2 are two submodules of M with $N_1 \subseteq N_2$, and N_1 is 2-maximal submodule of M Then N_1 is 2-maximal submodule of N_2

Proof: Since N_1 is 2-maximal submodule of M , then $\frac{M}{N_1}$ is 2-regular R-module.Since $N_1 \subseteq N_2$, then $\frac{N2}{N1}$ is a submodule of $\frac{M}{N1}$, then by [3,Prop.(1.1.20)] $\frac{N2}{N1}$ is 2-regularR-submodule of $\frac{M}{N1}$, Hence N_1 is 2-maximal submodule of N_2.

Proposition (2.17) "Let M be an R-module. Then M is 2-regular R-module if and only if every submodule of M is 2-maximal submodule of M ".

Proof: Let M be 2-regular R-module ,and N is a submodule of M , then by [3,Prop.(1.1.15)] $\frac{M}{N}$ is 2-regular R-module , Hence N is 2-maximal submodule for every submodule N of M.

For the convers , Let K be 2-maximal submodule of M , then $\frac{M}{K}$ is 2-regular R-module , Hence by [3, Prop.(1.1.15)] M is 2-regular R-module.

Remark (2.18) Proposition (2.17) is not true if "all proper submodule of an R-module M is 2-maximal submodule of M". For the example : "the module Z_8 as Z-module is not2-regular Z-module" , Since the submodule $<\bar{4}>=\{\bar{0},\bar{4}\}$ of Z_8 is not 2-pure because $2^2 Z_8 \cap <\bar{4}> = <\bar{4}>$ but $2^2 <\bar{4}> = <\bar{0}>$,implis that $2^2 Z_8 \cap <\bar{4}> \neq 2^2 <\bar{4}>$. similarly for the submodule $<\bar{2}>=\{\bar{0},\bar{2},\bar{4},\bar{6}\}$ is not 2-pure in Z_8. but all proper submodule of Z_8 $<\bar{2}>, <\bar{4}>$ are 2-maximal submodules of Z_8 Since $\frac{Z8}{<2>} \cong Z_2$, $\frac{Z8}{<4>} \cong Z_4$ are2-regular Z-module.

Proposition (2.19) Let M be an R-module , and K be 2-maximal submodule of M , Then for each y in M and a in R there exist l in R such that $a^2y = a^2la^2y$.

Proof: Suppose that K is 2-maximal submodule of M , then $\frac{M}{K}$ is 2-regular R-module. Hence by[4,Prop.(1.1.15)] M is 2-regular R-module , let $y \in M$ and a in R,then $a^2y \in a^2M$ and $a^2y \in <a^2y>$,then $a^2y \in a^2M \cap <a^2y>$. Since M is 2-regular R-module then , $a^2M \cap <a^2y> = <a^2y>$. Hence $a^2y \in a^2 <a^2y>$. Hence there exists l in R such that $a^2y = a^2la^2y$. The convers of Proposition (2.19) is true if R is principle ideal ring.

Proposition (2.20) Let M be an R-module over principle ideal ring R. if for each y in M and a in R there exist l in R such that $a^2y = a^2la^2y$, Then every submodule of M is 2-maximal submodule.

Proof: Let K be a submodule of M , and J is an ideal of R, let $y \in a^2M \cap K$, then $y \in a^2M$ and $y \in K$,implies that there exists m in M such that $y = a^2m$. but by hypotheses $\exists$ l in R such that $a^2m = y = a^2la^2m$. Thus $y \in a^2K$. But it is given that R is principle ideal ring , implies that $J^2M \cap K = J^2K$. that is K is 2-pure submodule of M , Hence M is 2-regular R-module , Hence by Proposition (2.4) K is 2-maximal submodule of M. Thus every submodule of M is 2-maximal.

The following is another characterization of 2-maximal submodule.

"Recall that an ideal I of a ring R is 2-maximal in R if and only If $\frac{R}{I}$ is 2-regular ring" [7].

Proposition (2.21) "Let M be an R-module over principle ideal ring R ,and N is a submodule of M. Then N is 2-maximal submodule of M if and only if $ann_R(m)$ is2-maximal ideal of R for each m in M ".

Proof: "Suppose that N is a 2-maximal submodule of M", Then $\frac{M}{N}$ is 2-regular R-module , by [3,Prop.(1.1.15)] , M is 2-regular R-module.Hence by [3,Prop.(1.1.28)] $\frac{R}{ann_R(m)}$ is 2-regular ring for all m in M. Hence $ann_R(m)$ is 2-maximal ideal of R.

For the convers , suppose that $ann_R(m)$ is 2-maximal ideal of R , then $\frac{R}{ann_R(m)}$ is2-regular ring , Thus by [3,Prop.(1.1.28)] , M is 2-regular then by [3,Prop.(1.1.15)] $\frac{M}{N}$ is 2-regular R-module. Hence N is 2-maximal submodule of M.

"From Proposition (2.20) and Proposition (2.21) we get the following corollary" :**Corollary (2.22)** Let M be an R-module over principle ideal ring R ,and N is a submodule of M. Then the following statements are equivalent :

(1) N is 2-maximal submodule of M.

(2) $ann_R(m)$ is 2-maximal ideal of R.

(3) For each m in M and a in R there exist l in R such that $a^2m = a^2la^2m$.

Proof: (1) $\Longrightarrow$ (2) Follows by Proposition (2.21)

(2) $\Longrightarrow$ (3) : Suppose that $ann_R(m)$ is 2-maximal ideal of R for each m in M , then $\frac{R}{ann_R(m)}$ is 2-regular ring, Thus by [4,Prop.(1.1.28)] $a^2m = a^2la^2m$ for each m in M and a in R and for some l in R.

(3) $\Longrightarrow$ (1) Follows by Proposition (2.20)

Before we introduce the next proposition , we need to give the following lemma.

Lemma (2.23) Let M be an R-module , with ann_RM is 2-maximal ideal of R , Then $ann_R(x)$ is 2-maximal ideal of R for each x in M.

Proof: Let ann_RM is 2-maximal ideal of R and x in M , since $<x> \subseteq M$, Then $ann_RM \subseteq ann_R(x)$. but ann_RM is 2-maximal ideal of R , Hence by [7, Prop 12] $ann_R(x)$ is 2-maximal ideal of R.

Proposition (2.24) Let M be an R-module over principle ideal ring R with ann_RM is 2-maximal ideal of R. Then every submodule of M is 2-maximal submodule.

Proof: Let N be a submodule of M , Since ann_RM is 2-maximal ideal of R , then by lemma (2.23) $ann_R(x)$

is 2-maximal ideal of R. Hence $\frac{R}{ann_R(x)}$ is 2-regular ring Thus by [3,Prop.(1.1.28)] , M is 2-regular R-module ,and then by [3,Prop.(1.1.15)] $\frac{M}{N}$ is2-regular R-module. Hence N is 2-maximal submodule of M.

Proposition (2.25) Let M be an R-module over principle ideal ring R and N is a submodule of M. Then $[N:_R M]$ is 2-maximal ideal of R if and only if N is "2-maximal submodule of M".

Proof: Suppose that $[N:_R M]$ is 2-maximal ideal of R then $\frac{R}{[N:_R M]}$ is 2-regular ring. But $[N:_R M] = ann_R\left(\frac{M}{N}\right)$, then $\frac{R}{ann\left(\frac{M}{N}\right)}$ is 2-regular ring. Hence by [3,Coro.(1.1.30)] $\frac{M}{N}$ is 2-regular R-module, Hence N is 2-maximal submodule of M.

For the convers , suppose that N is 2-maximal submodule of M , then $\frac{M}{N}$ is 2-regularR-module. Hence by [3,Prop.(1.1.15)]M is 2-regular R-module then by [3,Prop.(1.1.28)] $\frac{R}{ann_R(x)}$ is 2-regular ring for any x in M.Thus $ann_R(x)$ is 2-maximal ideal of R. But $ann_R(x) = [(0):_R (x)]$ and if $r \in ann_R(x)$ then $rx \in (0)$, then $rx = 0$, then $rx \in N$ for all x in M. Thus $rM \subseteq N$. that is $r\in [N: M]$. Thus , $ann_R(x) \subseteq [N:_R M]$ Therefor by [7,Prop12] $[N:_R M]$ is 2-maximal ideal of R.

Proposition (2.26) If I is 2-maximal ideal of a principle ideal ring R , and M is an R-module , Then IM is 2-maximal submodule of M.

Proof: Since $aM \subseteq IM$ for each a in I , then $a \in [IM:_R M]$ which implies that $I \subseteq [IM:_R M]$. But I is 2-maximal ideal of R , then by [7,Prop.12] $[IM:_R M]$ is 2-maximal ideal in R. Hence by Proposition (2.25) IM is 2-maximal submodule of M.

Proposition (2.27) Let M be an R-module over principle ideal ring R and J(R) is 2-maximal ideal of R, if $J(R)\frac{M}{N} = (0)$ where N is a submodule of M, then N is 2-maximal submodule of M.

Proof: It is given that $J(R)\frac{M}{N} = (0)$ implies that $J(R) \subseteq ann_R\left(\frac{M}{N}\right) = [N:_R M]$ but J(R) is 2-maximal ideal of R then by [7,Prop.12] $[N:_R M]$ is 2-maximal ideal in R. Hence by Proposition (2.25) N is 2-maximal submodule of M.

Proposition (2.28) : "Let N be a submodules of an R-module M, and N is the intersection of finite number of maximal submodule of M. Then N is 2-maximal submodule of M".

Proof : Let $N = N_1 \cap N_2 \cap \dots \cap N_n$, where N_i is maximal submodule of M $\forall i = 1,2,\dots,n$. Hence $\frac{M}{N_1}, \frac{M}{N_2}, \dots, \frac{M}{N_n}$ are simple R-modules. Then $\frac{M}{N} = \frac{M}{N_1 \cap N_2 \cap \dots \cap N_n} \cong \frac{M}{N_1} \oplus \frac{M}{N_2} \oplus \dots \frac{M}{N_n}$ [1]. That is $\frac{M}{N}$ isomorphic to a direct sum of simple R-module. Hence $\frac{M}{N}$ is semisimple R-module , which implies that $\frac{M}{N}$ is regular R-module. Hence $\frac{M}{N}$ is 2-regular R-module. Therefore N is 2-maximal submodule of M.

Proposition (2.29) : Let M_1, M_2 be two R-module over principal ideal ring R, and N_1, N_2 are 2-maximal submodule of M_1 and M_2 respectively. Then $N_1 \oplus N_2$ is 2-maximal submodule of $M_1 \oplus M_2$.

Proof : Since N_1, N_2 are 2-maximal submodule of M_1 and M_2 respectively. Then $\frac{M_1}{N_1}$ and $\frac{M_2}{N_2}$ are 2-regular R-modules , then by [3. Corr. (1.1.15)] $\frac{M_1}{N_1} \oplus \frac{M_2}{N_2}$ is 2-regular R-module. Now , let $f: \frac{M_1}{N_1} \oplus \frac{M_2}{N_2} \longrightarrow \frac{M_1 \oplus M_2}{N_1 \oplus N_2}$ be a map define by $f(a + N_1, b + N_2) = ((a, b) + N_1 \oplus N_2)$ where $a \in M_1, b \in M_2$. To prove that f is well defined. Let $(a + N_1, b + N_2) = (N_1, N_2)$ where $a \in M_1, b \in M_2$, then $(a, b) \in N_1 \oplus N_2$, implies that $(a, b) + N_1 \oplus N_2 = (N_1, N_2)$. That is $f(a + N_1, b + N_2) = (N_1, N_2)$.Therefore f is well defined. It is clear that f is an R-homomorphism. Now consider $Im f \circ \{f(a + N_1, b + N_2): a \in M_1, b \in M_2\} = \{(a, b) + N_1 \oplus N_2): a \in M_1, b \in M_2\} = \frac{M_1 \oplus M_2}{N_1 \oplus N_2}$, hence f is an epimorphism . Therefore by [3, Corr. (1.1.19)] $\frac{M_1 \oplus M_2}{N_1 \oplus N_2}$ is 2-regular. Hence$N_1 \oplus N_2$ is 2-maximal submodule of $M_1 \oplus M_2$

3. 2-Maximal submodule in certain type of modules

"In this section we study the behavior of 2-maximal submodule in some classes of modules as projective" , semi simple modules and modules with pure sum property , endo 2-regular modules.

"We start this section by recall the following detentions" :-

"Recall that an R-module F is called free if it is isomorphic to infinite direct sum of copies of R as R-module and write $F \cong \oplus_\Lambda R$ where Λ is index set ".[8]

"Recall that an R-module M is projective if and only if M is (isomorphic to) direct summand of a free R-module ". [9]

Proposition (3.1) Let R be any ring with $\oplus_\Lambda R$ is 2-regular R-module for any index set Λ. Then every submodule of projective R-module is 2-maximal submodule.

Proof: Let M be projective R-module , and N is a submodule of M , then there exists a free R-module F and an R-epimorphism $f: F \longrightarrow M$ and $F \cong \bigoplus_{\Lambda} R$ where Λ is index set. then $0 \longrightarrow \mathrm{Ker}f \xrightarrow{i} \bigoplus_{\Lambda} R \xrightarrow{f} M \longrightarrow 0$, where i is the inclusion mapping. Since M is projective then $\bigoplus_{\Lambda} R =$ Kerf⊕M.But $\bigoplus_{\Lambda} R$ is 2-regular R-module , Then by[3, coro.(1.1.18)] M is 2-regular R-module. Therefore $\frac{M}{N}$ is 2-regular , Hence N is 2-maximal submodule of M.

Proposition (3.2) Every submodule of semi-simple R-module is 2-maximal.

Proof: Let M be a semi-simple R-module , and N is a submodule of M , then $\frac{M}{N}$ isa semi-simple R-module.But semi-simple R-module is 2-regular R-module ,Hence by proposition (2.17) Nis a 2-maximal submodule.

Definition (3.3) [4] "Let M be an R-module , and K(M) is the submodule of M containing every 2-regular submodule of M , K(M) is called maximal 2-regular submodule of M. if M=R then K(M) is an ideal of R , and M a 2-regular if and only if K(M)=M ".

Proposition (3.4) Let M be an R-module , Then K(M)=M if and only if every submodule of M is 2-maximal.

Proof: Suppose K(M)=M , then M is 2-regular R-module , Hence by proposition (2.17) every submodule of M is 2-maximal submodule of M.

For the convers every submodule of M is 2-maximal submodule of M ,then again by proposition (2.17) M is 2-regular R-module , thus K(M)=M.

Recall that a submodule N of an R-module M is said to dense in M , if N generates M, that is $M = \sum_{f \in \mathrm{Hom}_R(N,M)} f(N)$.[8]

Proposition (3.5) Let M be an R-module , and K(M) be a dense submodule in M ,Then every submodule of M is 2-maximal submodule of M.

Proof: Let N be a submodule of M. since K(M) is dense in M , Then , $M = \sum_{f \in \mathrm{Hom}_R(K(M),M)} f(K(M))$. But by [3,Prop.(1.3.15)] K(M) is stable submodule of M Then $f(K(M)) \subseteq K(M)$, Hence $M = \sum_{f \in \mathrm{Hom}_R(K(M),M)} f(K(M)) \subseteq K(M)$, Then M=K(M). Thus by proposition (3.4) every submodule of M is 2-maximal submodule of M

"Recall that the Jacobson Radical of an R-module M denoted by J(M) is define to be the sum of all small submodule of M" , where a submodule N of an R-module M is called small submodule of M if for any submodule L of M such that M=N+L , implies that L=M. [9]

It is well-known that if M isfinitely generated then J(M) is small submodule of M. [1] **Proposition (3.6)**Let M be a finitely generated R-module and K(M)+J(M)=M. Then every submodule of M is 2-maximal submodule of M.

Proof: Since M is finitely generated R-module , then J(M) is small submodule of M. And since K(M)+J(M)=M and K(M) is a submodule of M , then K(M)=M. Hence by proposition (3.4) every submodule of M is 2-maximal submodule of M.

"Recall that an R-module is said to have the 2-pure sum property if the sum of any two2-pure submodule of M is 2-pure". [3]

Proposition (3.7) Let M be an R-module with Ry⊕M have 2-pure sum property for every non-zero y in M , Then every submodule of M is 2-maximal submodule of M.

Proof: Let y be a non-zero element in M , then Ry is a submodule of M, and there exist the inclusion map $i: Ry \longrightarrow M$.

But Ry⊕M has 2-pure sum property , Then $IMi = Ry$ is 2-pure in M. that is every cyclic submodule of M is 2-pure. Hence by proposition (2.4) every submodule of M is2-maximal submodule of M.

Proposition (3.8) Let M be an R-module with R⊕M has 2-pure sum property.Then every submodule of M is 2-maximal submodule of M.

Proof: Let m be a non-zero element in M, then there exists an epimorphism $f: R \longrightarrow Rm$ define by $f(r) = rm$ for each $r \in R$. Now let $i: Rm \longrightarrow M$ be the inclusion map , and consider $i \circ f: R \longrightarrow M$ is an R-homomorphism. Since R⊕M has2-pure sum property , Then $\mathrm{Im}((i \circ f)(R) = \mathrm{Im}(i(f(R))) = i(f(R)) = i(Rm) = Rm$is 2-pure in M. Thus every cyclic submodule of M is 2-pure. Hence by proposition (2.4) every submodule of M is 2-maximal submodule of M.

Recall that an R-module M is called endo 2-regular module if $\mathrm{End}_R(M)$ is 2-regular ring [3].

Proposition (3.9) Let M be a cyclic R-module with $\mathrm{ann}_R M$ is 2-maximal ideal of R , Then M is endo 2-regular ring.

Proof: Let M be a cyclic R-module. Since $\mathrm{ann}_R M$ is 2-maximal ideal of R , then $\frac{R}{\mathrm{ann}_R M}$ is 2-regular ring.

But by [10] $End_R(M) \cong \frac{R}{ann_R M}$ Then $End_R(M)$ is 2-regular ring. Hence M is endo 2-regular ring.

Proposition (3.10) Let M be an R-module with $ann_R(x)$ is 2-maximal for each x in M. Then Rx is 2-regular submodule of M.

Proof: Since $ann_R(x)$ is 2-maximal ideal of R , then $\frac{R}{ann_R(x)}$ is 2-regular R-module. But Rx $\cong \frac{R}{ann_R(x)}$, then Rx is 2-regular submodule of M.

4. Sufficient conditions for 2-maximal submodule to be semi (weak)- maximal submodules.

We notes that : semi-maximal submodules $\Longrightarrow$ weak-maximal submodules $\Longrightarrow$ 2_maximal submodule. But neither the convers of implication is hold In this section we give sufficient condition on this implications to hold.

"In the following propositions we show that in the class of semi-prime R-module over local ring" , the class of 2-maximal submodules is equivalent with class of semi-maximal (weak-maximal) submodules.

Proposition (4.1) Let M be a semi-prime R-module over local ring R , and N be a submodule of M , Then N is 2-maximal submodule of M if and only if N is asemi-maximal submodule of M.

Proof: ($\Longrightarrow$) Suppose that N is 2-maximal submodule of M , then $\frac{M}{N}$is 2-regularR-module , Hence by [3,Prop.(1.1.15)] M is 2-regular R-module. Therefore by [3,Coro.(1.2.5)] , M is semi-simple R-module , Thus $\frac{M}{N}$ is semi-simple R-module. Hence N is semi-maximal submodule of M.

($\Longleftarrow$)For the convers, is a straight fort.

Proposition (4.2) Let M be a semi-prime R-module over local ring R , and N isa submodule of M , Then N is 2-maximal submodule of M if and only if N is a weak-maximal submodule of M.

Proof: ($\Longrightarrow$) Suppose that N is 2-maximal submodule of M , then $\frac{M}{N}$ is 2-regular R-module , Hence by [3,Prop.(1.1.15)] M is 2-regular R-module. Therefore by [3,Coro.(1.2.5)] , M is regular R-module , Thus $\frac{M}{N}$ is regular R-module. Hence N is weak-maximal submodule of M.

($\Longleftarrow$) For the convers direct.

From proposition (4.1) and proposition (4.2) we get the following result.

Corollary (4.3) Let M be a semi-prime R-module over local ring R , and N is a submodule of M , Then the following statements are equivalent :

(1) N is 2-maximal submodule of M.

(2) N is a semi-maximal submodule of M.

(3) N is a weak-maximal submodule of M.

Proof : (1) $\Longrightarrow$ (2) Follows by proposition (4.1).

(3)$\Longrightarrow$ (1) Let N be a weak-maximal submodule of M, Then $\frac{M}{N}$ is regular module , Hence by [4,Rem.and Ex.(1.1.3)] $\frac{M}{N}$ is 2-regular module , Thus N is 2-maximal submodule of M

It is well-known that a prime R-module is a semi-prime R-module. [11]. So we get the following result.

Corollary (4.4) Let M be a prime R-module over local ring R , and N is a submodule of M , Then the following statements are equivalent :

(1) N is 2-maximal submodule of M.

(2) N is a semi-maximal submodule of M.

(3) N is a weak-maximal submodule of M.

"Recall that an R-module M is said to be I-multiplication module , if each submodule N of M is of the form IM for some idempotent ideal I of R". [3]

Proposition (4.5) If M be is I-multiplication R-module , and N is a submodule of M , Then N is 2-maximal submodule of M if and only if N is a weak-maximal submodule of M

Proof: ($\Longrightarrow$) Suppose that N is 2-maximal submodule of M , then M/N is 2-regular R-module. To prove that $\frac{M}{N}$ is I-multiplication R-module : Let $\overline{K}$ be a submodule of $\frac{M}{N}$, Then there exist a submodule K of M with N $\subseteq$ K such that $\overline{K} = \frac{K}{N}$. Since M isI-multiplication , Then K=JM for some idempotent ideal J of R. that is $\overline{K} = \frac{K}{N} = \frac{JM}{N} = J\left(\frac{M}{N}\right)$. Hence $\frac{M}{N}$ is I-multiplication R-module. Hence by [4,Prop.(1.2.11)] $\frac{M}{N}$ is regular R-module , Thus N is a weak-maximal submodule of M.

($\Longleftarrow$) For the convers direct.

It is well-known that a regular ring are characterized as those rings all its ideal are idempotent [9], so we get the following result.

Corollary (4.6) Let M be a multiplication R-module over regular ring R , and N be a submodule of M Then N is 2-maximal submodule of M if and only if N is a weak-maximal submodule of M.

References

1. Khalaf , H. Y. ((Semi-maximal Submodules)) , ph.D thesis , Univ. of Baghdad , 2007.

2. Shwkaea M. R. ((Weak Maximal Ideals and Weak Maximal Submodules)) ,M.Sc. thesis , Univ, of Tikrit , 2012.

3. Ghaleb , A. H. ((Generalization of Regular Modules and Pure Submodules)) , ph.D , thesis , Univ. of Baghdad , 2015.
4. Desale , G. And Nicholson , W. ((Endoprimitive Rings)) J. Algebra , 70, 545-560 ,(1981).
5. El-Bast , Z. A. And Smith , P.P. ((Multiplication Modules)) , Comm. In Algebra , 16 , 755-770 ,(1988).
6. Goldie , K. R. ((Torsion Free Modules and Rings)) , J. Algebra , 1,268-287 , (1964).
7. Mohammad ali H.K , Mohammod A.S. and Sallman M.D. ((2-Maximal Ideals and 2-Maximal R-Modules)) ,To a paper.
8. Lam , T. Y. ((Lecture on Modules and Rings)) , Berkely California , 1998
9. Kasch , F. ((Modules And Rings)) Acad , Press., London , 1982.
10. Al-Aubaidy W. K. ((The Ring of Endomorphism of Multiplication Modules)) M.Sc. thesis , Univ. of Baghdad ,1993.
11.Sharide F.A((S-compactly packed submodules and Semi prime Modules)) , M.Sc. thesis Univ. of Tikrit , 2008.

الخلاصة:

خلال هذا البحث R تمثل حلقة إبداليه بمحايد و M مقاسا احاديا ايسر. الغرض من هذا البحث هو دراسة مفهوم جديد, يسمى المقاس الجزئي الاعظمي من النمط -2 والذي هو اعمام للمقاسات الجزئية الاعظمية , حيث ان المقاس الجزئي N من المقاس M يدعى بالمقاس الجزئي الاعظمي من النمط -2 اذا وفقط اذا المقاس $\frac{M}{N}$ منتظما منالنمط -2.تم اعطاء العديد من التشخيصات والصفات لمفهوم المقاس الجزئي الاعظمي من النمط -2. بالإضافة الى ذلك درسنا سلوك المقاسات الجزئية الاعظمية في بعض المقاسات. واخيرا درسنا الشروط الكافية التي نضعها على المقاسات الجزئية الاعظمية من النمط- 2 لتصبح مقاسات جزئية شبه اعظمية , مقاسات جزئية اعظمية ضعيفة.