Chaotic Properties of Modified the Kaplan York Map

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Abstract

We studied this work investigate the fixed points of modified Kaplan York map k_1 and we focus on found contracting and expanding area of this map ,Moreover we study the dynamical system of modified Kaplan York map,is aslo studied the chaotic properties of k_1 proved the topological entropy of k_1 is positive , k_1 is sensitive dependence into initial condition , k_1 is transitive finally the Lyapunov exponent is positive .we use mat lab program to show the sensitivity and transitivity of Kaplan York map

Key words: The Kaplan York map , Sensitive Depends on Initial Condition , Transitivity Kaplan York Map, Lyapunov exponents of the Kaplan York Map.

الخلاصة

درسنا دالة كابلان– يورك المطوره ووجدنا الخواص العامة لها وحددنا مناطق تقلصها وتمددها وكذلك درسنا خواصها الفوضويه حيث برهنا أنها تمتلك تبولوجي انتروبي موجبا وتملك حساسية عند الشروط الابتدائية وانها متعدية واخيرا اثبتنا انها تمتلك توسيع ليبانوف موجبا, واخيرا استخدمنا برنامج الماتلاب لبيان حساسية وتعدي الداله

الكلمات المفتاحية: دالة كابلان يورك, الحساسية المعتمدة على الشروط الابتدائية , التعدي لدالة كابلان يورك , ثابت ليبانوف لدالة كابلان يورك

1. Introduction

The Kaplan York map has chaotic behavior. It is one of the famous map on discrete dynamical system which has many natural applications

We define the chaotic map as:-

$$K\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2x\ mod1\\ \propto y+cos4\pi x\end{pmatrix}$$

In our work, we modify the Kaplan York map into

$$K_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\propto x\ mod1\\ \beta y+x^2\end{pmatrix}$$

We will simplify the Kaplan York map by replacing $(cos4\pi x)$

To x^2 and when we add the new parameter we get the properties of dynamic behavior which are different from the Kaplan York map , also there are some similar properties.

2. The General Properties of The Modified Kaplan York Map

We study the dynamical system of modified Kaplan York map , We find the fixed point and the Jacobain of K_1 and we study the contracting area and exponding area of K_1

in this section many fundamental concepts which are needed in this work we introduced:

Let G:R²→R² such that $G\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}f(x,y)\\g(x,y)\end{pmatrix}$ be a map .Any pair $\begin{pmatrix}p\\q\end{pmatrix}$ for where f $\begin{pmatrix}p\\q\end{pmatrix}$ =p , g $\begin{pmatrix}p\\q\end{pmatrix}=q$ is called a fixed point of the two dimensional dynamical system G is C^1 ,if all of its first partial derivatives exist and are continuous. G is C^∞ , if its mixed K th partial derivatives exist and are continuous for all K ∈ Z . G is called a diffeomorphism provided that G is one -to-one , G is onto , G is C^∞, its inverse $G^{-1}:R^2 \to R^2$ is C^∞ too. Let V be a subset of R^2, and v_0 be any element in R^2.

Consider $G:V \to R^2$ be a map .Furthermore assume that the first partial of the coordinate map f and g exist at v_0 . The differential of G at v_0 is the linear map $DG(v_0)$ defined on R^2 by

$$DG(v_0) = \begin{bmatrix} \frac{\partial f(v_0)}{\partial x} & \frac{\partial f(v_0)}{\partial y} \\ \frac{\partial g(v_0)}{\partial x} & \frac{\partial g(v_0)}{\partial y} \end{bmatrix}$$, for all v in R^2 . The determinant

$DG(v_0)$ is called the Jacobian of F at v_0 and is denoted ,
By J=det $DG(v_0)$.And if $|\det DG(v_0)| > 0$ then G is called area-expanding at v_0 ,A point x ∈ X is a periodic point of period n> 0 if $f^n(x) \neq x$ for all $r < n$.

If p is period –n point of f such that $|f^{(n)}(p)| < 1$ then f cannot has sensitive dependence on initial conditions at p

Definition (2.1) (Gulick, 1992):

Let $G:R^2 \to R^2$ be any map and let p be any fixed point of G. If λ_1, λ_2 are the eigenvalues of DG(P) then

1.If $|\lambda_i| < 1, \forall i = 1,2$ then p is an attracting fixed point
2.If $|\lambda_i| > 1, \forall i = 1,2$ then p is an repelling fixed point
3.If there exist i∈ {1,2} then $|\lambda_i| > 1$and $|\lambda_j| < 1$ i≠ j then p is a saddle fixed point

Definition (2.2) (Kurka, 1997):

The f:X→X is said to be sensitive dependence on initial conditions if there exists $\varepsilon >$ 0such that for any $x_0 \in X$ and any open set $U \subset X$ containing x_0 there exists $y_0 \in U$ and n $\in Z^+$such that $d(f^n(x_0), f^n(y_0)) > \varepsilon$ that is $\exists\, \varepsilon > 0$,Vx,V$\delta$ >0, ∃y∈ B δ (x) , ∃:$d(f^n(x_0), f^n(y_0)) \geq \varepsilon$

Definition (2.3) (Fotion,2005):

Let f:X→X be a continuous map and X be a metric space . Then the map f is said to be chaotic according to wiggins or W – chaotic if:

1.f is topologically transitive.
2.f is sensitive dependent on initial condition

Definition (2.4) (Sturman, 2006):

The map f: $R^n \to R^n$ will have n Lyapunov exponent , say L_1 (x, v) , L_2 (x ,v) ,……, L_n (x, v) for a system of n variable . then the Lyapunov exponent is the maximum n Lyapunov exponent that is L_1 (x, v) = max{ L_1 (x, v) , L_2 (x, v) ,……, L_n (x, v)} . Where $v=(v_1,v_2,\ldots,v_n)$.

Proposition (2.5):

IF $\propto \neq \frac{1}{2}$ and $\beta \neq 1$ then K_1 has a fixed point.
proof

Since $K_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\propto x\ mod1\\ \beta y+x^2\end{pmatrix}$ so
x=2 $\propto x\ mod1$ and $\beta y+x^2$ =y
this implies x-2 ∝ x=0
(1-2 ∝)x=0
By hypothesis $\propto \neq \frac{1}{2}$ then
X=0 (mod1), that is
X=k ; $\forall\ k\in Z$
Thus $-y=k^2$; $\forall\ k\in Z$
$(\beta-1)y=-k^2$
Since $\beta\neq 1$then y= $\frac{-k^2}{\beta-1}$
Therefore $\begin{pmatrix}k\\ \frac{-k^2}{\beta-1}\end{pmatrix}$ is a fixed point of k_1 ; $\forall\ k\in Z$

Remark (2.6)
If $\propto=\frac{1}{2}$ $and\ \beta=1$ thus y+k^2=y , so k^2=0 then k_1 has unique fixed point which is $\begin{pmatrix}0\\0\end{pmatrix}$

Remark (2.7)
If $\propto=\frac{1}{2}$ and $\beta\neq 1$ then k_1 has infinite points
$\begin{pmatrix}x\ mod1\\ \beta y+x^2\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$
If $|\propto|=\frac{1}{2}$ and $|\beta|=1$
$\begin{pmatrix}2\propto\ x\ mod1\\ y+x^2\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$
then k_1 has $\begin{pmatrix}0\ \ mod1\\0\end{pmatrix}$ is the fixed points

If $|\beta|=1$ and $|\propto|\neq\frac{1}{2}$
$\begin{pmatrix}2\propto x & mod1\\ y+x^2 & \end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$
Then k_1 has $\begin{pmatrix}0 & mod1\\0 & \end{pmatrix}$ as the fixed points

Proposition(2.8):
the Jacobain of the modified Kaplan York map is $(2\propto\beta)$, $\forall\propto,\beta\ \in R$
Proof
The differential matrix of $K_1$is $Dk_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{\partial f1}{\partial x} & \frac{\partial f1}{\partial y}\\ \frac{\partial f2}{\partial x} & \frac{\partial f2}{\partial y}\end{pmatrix}=\begin{pmatrix}2\ \propto & 0\\ 2x & \beta\end{pmatrix}$
J= det $Dk_1\begin{pmatrix}x\\y\end{pmatrix}=2\propto\beta$; $\forall\propto,\beta\ \in R$

Proposition(2.9):

1* If either $|\propto| > \frac{1}{2}$ or $|\beta| > \frac{1}{2}$ Then k_1 is area expanding map.

2*$|\propto| < \frac{1}{2\,|\beta|}$ or $|\beta| < \frac{1}{2|\propto|}$, $\propto \neq 0$, k_1 is area contracting map.

Proof

If $|J| = \left| \det Dk_1 \binom{x}{y} \right|$

$= |2 \propto \beta| > 1$, thus $|\propto \beta| > \frac{1}{2}$

$|2 \propto \beta| > \frac{1}{2}$ either $|\propto| > \frac{1}{2}$ or $|\beta| > 1$

Proposition(2.10):

the modified Kaplan York map is C^{∞}

proof

Note that

$\frac{\partial k_1}{\partial x} = 2 \propto$, $\frac{\partial k_1}{\partial y} = 0$, $\frac{\partial k_2}{\partial x} = 2x$ and $\frac{\partial k_2}{\partial y} = \beta$

$\frac{\partial^n k_1}{\partial x^n} = 0$; $\forall n \in N$ and $\frac{\partial^n k_2}{\partial y^n} = 0$; $\forall n \in N$

And these partial derivatives are exist and continuous then k_1 is C^{∞}

Remark (2.11):-

k_1 is not one - to – one so k_1 is not differentiable map.

Remark (2.12):-

K_1 is not onto if

1* $\propto = 0$, $\beta = 0$

2*$\propto = 0$, $\beta \neq 0$ or

3* $\propto \neq 0$, $\beta = 0$

Remark (2.13):-

The eigen values of Dk_1 at the fixed point are $\lambda_1 = 2 \propto$ and $\lambda_2 = 2k$; $\forall k \in Z$

Proof

Since $Dk_1 \binom{x}{y} = \begin{pmatrix} 2\propto & 0 \\ \beta & 2x \end{pmatrix}$

So that $Dk_1(p) = \begin{pmatrix} 2\propto & 0 \\ \beta & 2k \end{pmatrix}$; $\forall k \in Z$

The eigen values of Dk_1 is

Det $\begin{pmatrix} 2\propto - \lambda & 0 \\ \beta & 2k - \lambda \end{pmatrix} = 0$, this imply

$(2 \propto - \lambda)(2k - \lambda) = 0$,so $\lambda_1 = 2 \propto$ and $\lambda_2 = 2k$; $\forall k \in Z$

To find the type of the fixed points this proposition should be proved .

Proposition(2.14):

1* If $|\propto| < \frac{1}{2}$ and $|k| < \frac{1}{2}$ $\forall k \in Z$ then the fixed point attractor

2* either $|\propto| > \frac{1}{2}$ or $|k| > \frac{1}{2}$ $\forall k \in Z$ then the fixed point of k_1 is saddle

3* If $|\propto| > \frac{1}{2}$ and $|k| > \frac{1}{2}$ $\forall k \in Z$ then the fixed point of k_1 is repelling

Proof

Since the eigen values of Dk_1 are $ƛ_1 = 2 \propto$ and $ƛ_2 = 2k$; $\forall k \in Z$ and by the proposition holds

Proposition(2.15):

K_1 has a positive topological entropy $\forall \propto, \beta \in R$

1* If $|\propto| > \frac{|\beta|}{2}$

H $_{top}$ 1 H top 1 k_1 (v) $\geq \log|2 \propto| > 0$

2* If $\frac{|\beta|}{2} \geq |\propto|$

H $_{top}$ $(k_1|\propto| \geq \log \beta$

3- Chaotic Properties of Modified the Kaplan York Map

There are many chaotic properties , we start with topological entropy we will prove the modified Kaplan York has positive topological entropy as its shown below :-

We recall the theorem(3.5) on [6] by theorem (4.1)

Let f:$R^n \rightarrow R^n$ be a continuous map then h $_{top}$(k_1) $\geq$ log| ƛ|

Where ƛ is the largest eigen value of Dk_1 (v) , where r $\in R^n$

So we can estamite the topological entropy of k_1 as:-

Proposition (3.1)

either $|\propto| > \frac{1}{2}$ or $|\beta| > 1$ then k_1 has sensitive dependence into initial Cond icons

Proof

$$K_1 \binom{x}{y} = \binom{2 \propto x \, mod 1}{\beta y + x^2}$$

$$k_1^2 \binom{x}{y} = \binom{4 \propto^2 x}{\beta(\beta\, y + x^2) + x^2} \propto \binom{4 \propto^2 x}{\beta^2 y + x^2}$$

By induction

$$k_1^n \binom{x}{y} \propto \binom{(2 \propto)^n x}{\beta^n y + x^2}$$

If $\left|\frac{\propto}{2}\right|$ >1 then $k_1^n \rightarrow \infty$ as n→∞

If $|\beta|$ >1 then $k_1^n \rightarrow \infty$ as n→∞

$d\,(k^1(x_1), k_1^n(x_2) = \sqrt{(2 \propto)^n (x_1 - x_2)^2} + \beta^n (y_1 - y_2)^2$

If $|\propto| > \frac{1}{2}$ then d($k_1^n(x_1)$, $k_2^n(x_2)$) →∞ as n→∞

And $|\beta| >$ 1 then d($k_1^n(x_1)$, $k_2^n(x_2)$) →∞ as n→∞

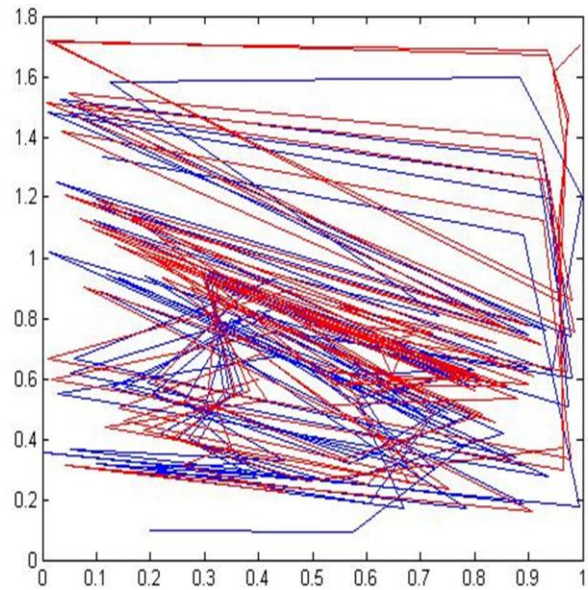


Fig (1-1) ∝=-1.06 , β=0.5 with initial points (0.2,0.1) , (0.4,0.6)

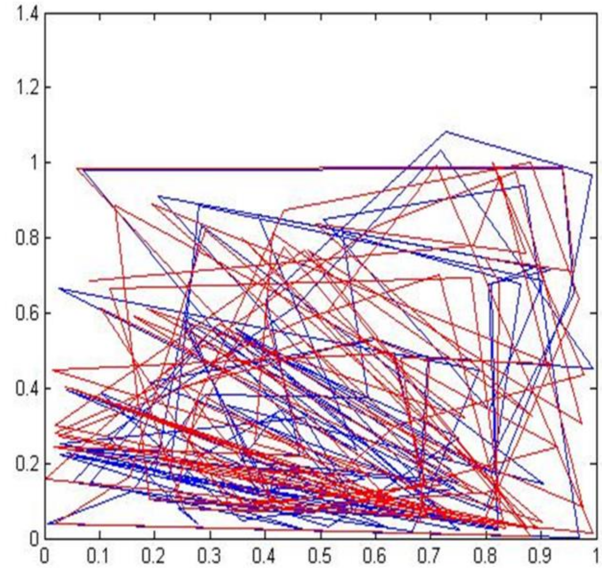


Fig (1-2) ∝=-3.16 , β=0.1 with initial points (0.2,0.1) , (0.4,0.6)

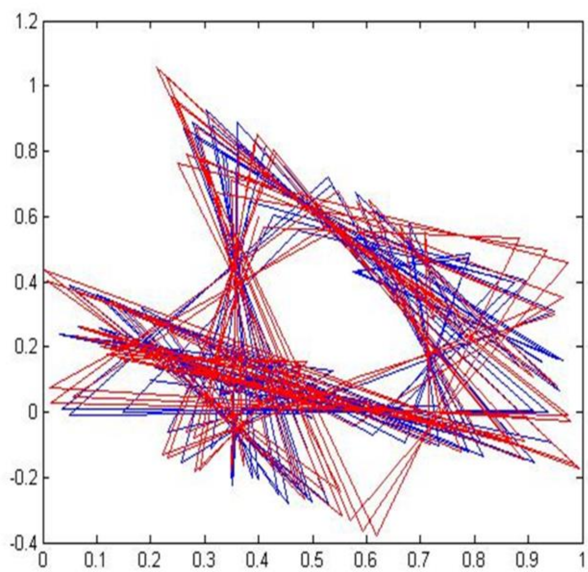


Fig (1-3) ∝=-0.9, β=-0.4 with initial points (0.2,0.1) , (0.4,0.6)

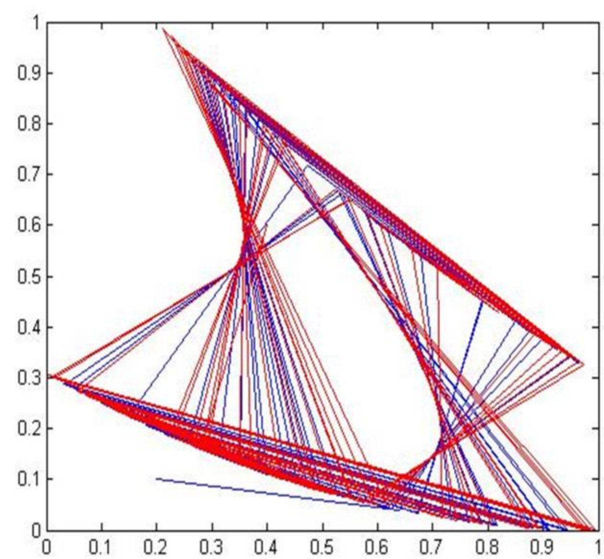


Fig (1-4)∝ =-0.9 , β=0 with initial points (0.2,0.1) , (0.4,0.6)

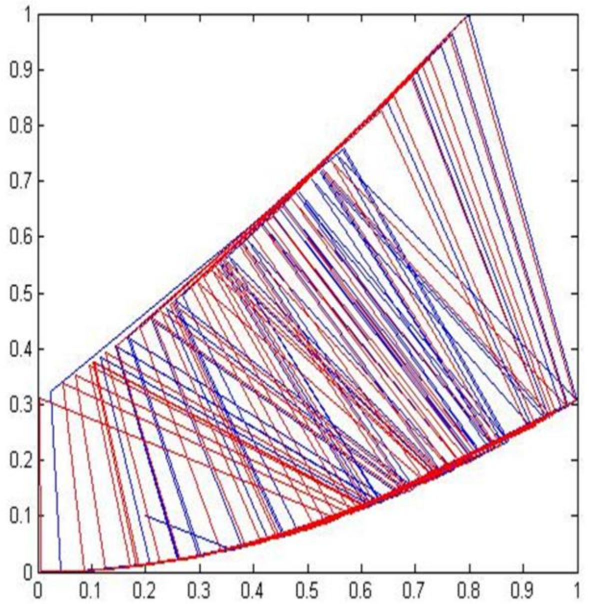


Fig (1-5) ∝=0.9 , β=0 with initial points (0.2,0.1) , (0.4,0.6)

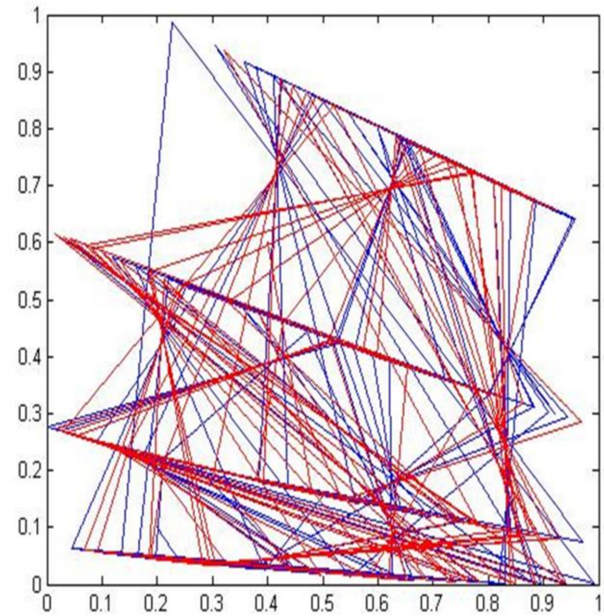


Fig (1-6) ∝=-1.9 , β=0 with initial points (0.2,0.1) , (0.4,0.6)

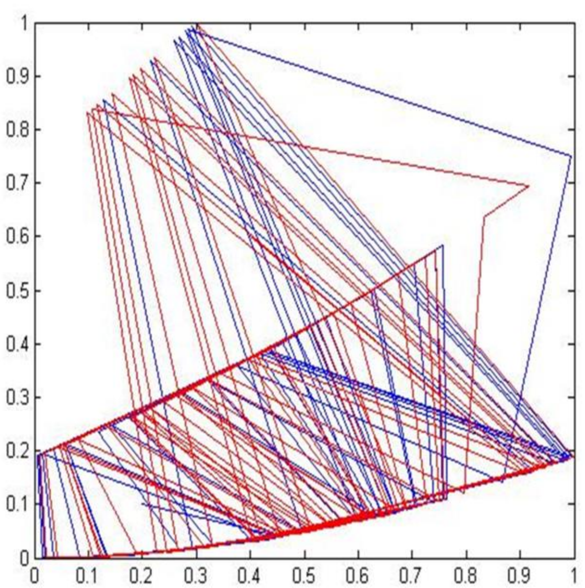


Fig (1-7) ∝=1.15 , β=0 with initial points (0.2,0.1) , (0.4,0.6)

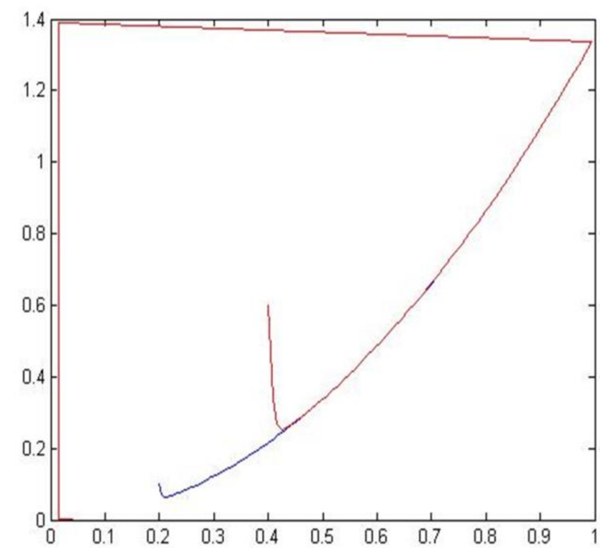


Fig (1-8) ∝=0.51 , β=0.3 with initial points (0.2,0.1) , (0.4,0.6)

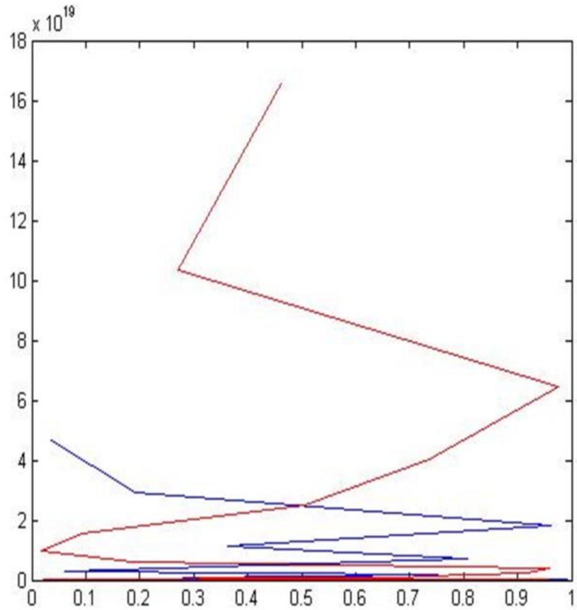


Fig (1-9) $\propto$=2.7 ,β =1.6 with initial points (0.2,0.1) , (0.4,0.6)

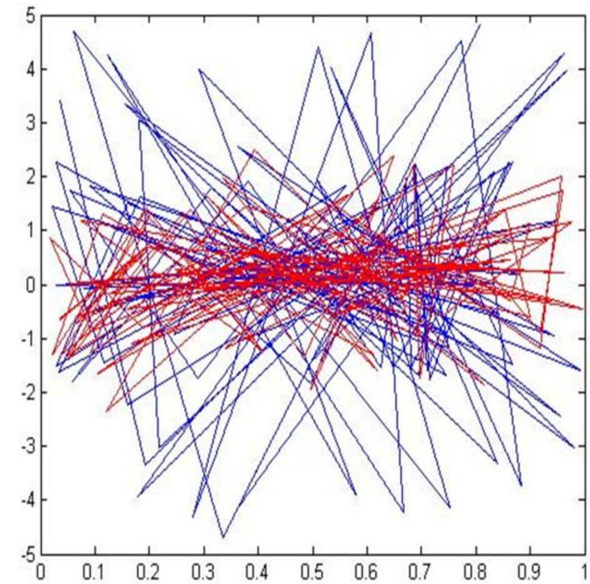
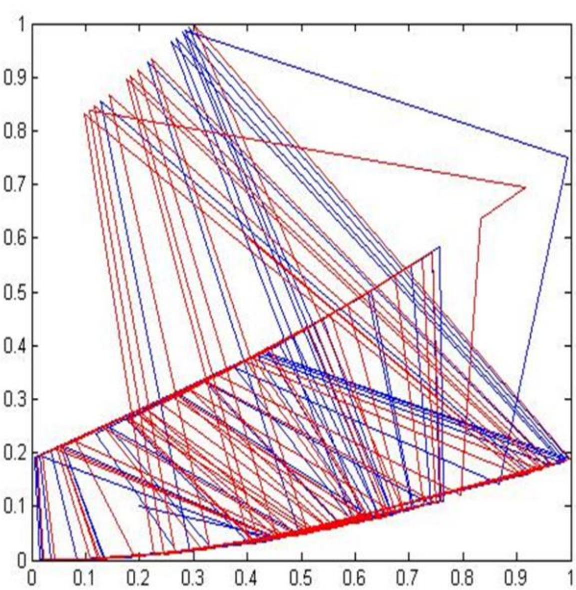
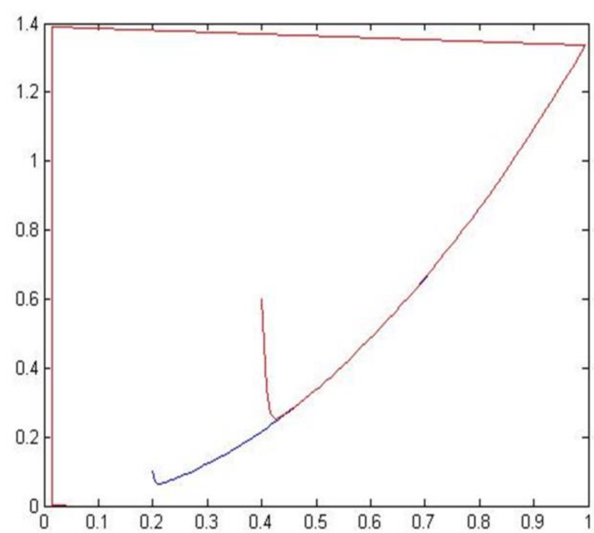


Fig (1-10)$\propto$ =2.7 , β=-1 with initial points (0.2,0.1) , (0.4,0.6)





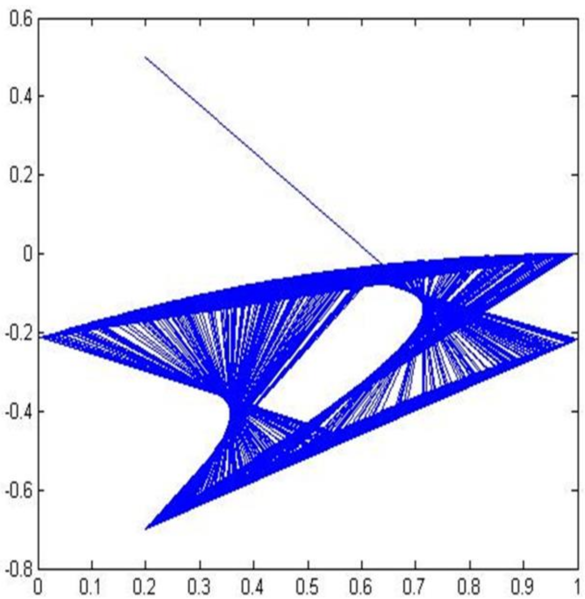


Fig (1-11)∝ =-0.9 , β=-0.7 with initial points (0.1,0.1)

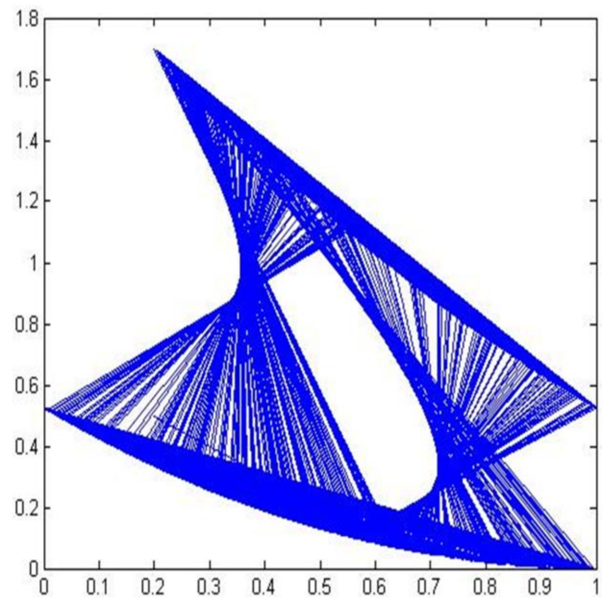


Fig (1-12) ∝=-0.7 ,β =1.7 with initial points (0.1,0.1)

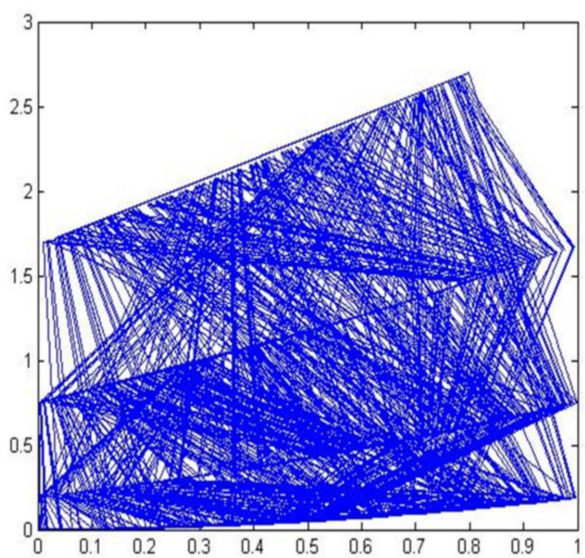


Fig (1-13)∝ =1.9 ,β =2.7 with initial points (0.1,0.1)

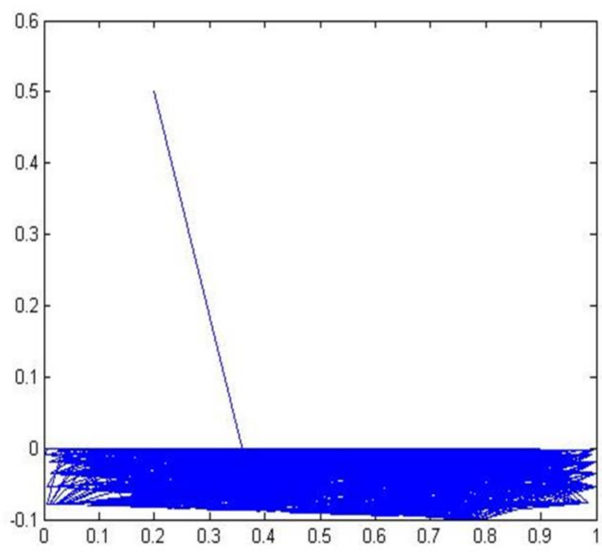


Fig (1-14)∝ =3.4 , β=-0.1 with initial points (0.1,0.1)

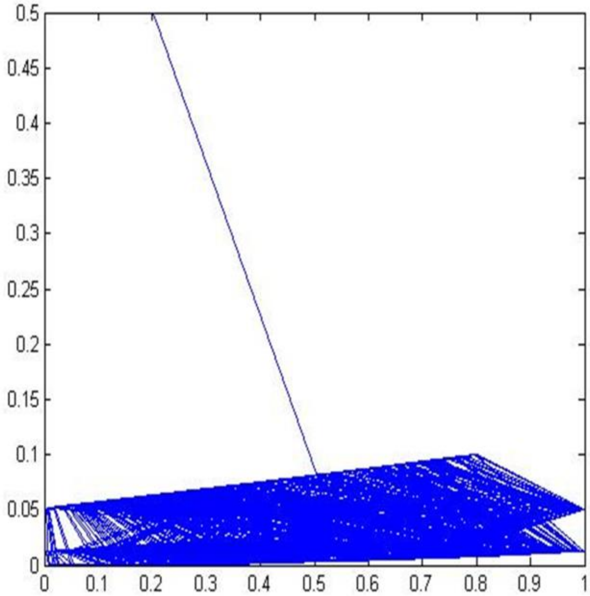


Fig (1-15)$\propto$ =-0.7 , β=0.1 with initial points (0.1,0.1)

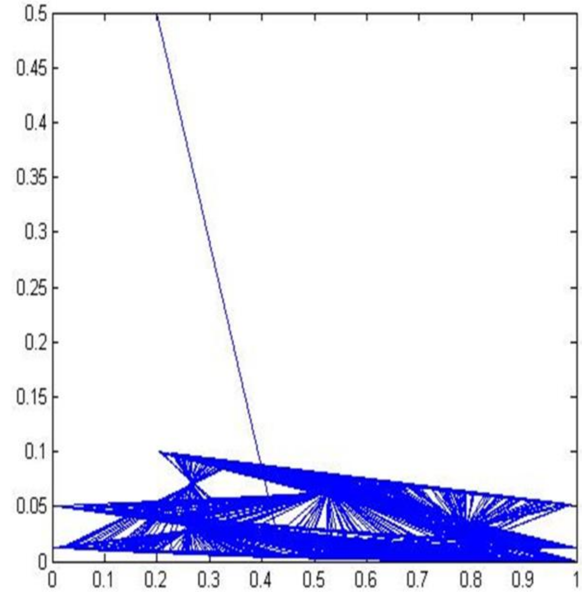


Fig (1-16) $\propto$=-1.4 , β=0.1 with initial points (0.1,0.1)

The final chaotic properties is Lypunov exponent

Proposition (3.2)

k_1 has a positive Lypunov exponent

Proof

Since $\lambda_1 = 2 \propto$ and $\lambda_2 = 2\,k$ and L (k_1) = log $|2 \propto|$ or L (k_1) = log $|k|$; $k \in Z$

If $|\propto| > \frac{1}{2}$ then L (n)> 0 and if k≠0 then L (n_1) >0

The second property is sensitivity

Proposition (3.3)

If $|\propto| > \frac{1}{2}$ or $|\beta| > \frac{1}{2}$ then k_1 has sensitive dependence on the initial conditions.

Proof

$$K_1 \binom{x}{y} = \binom{2 \propto x \, mod 1}{\beta y + x^2}$$

$$k_1^2 \binom{x}{y} = \binom{4 \propto^2 x}{\beta(\beta\, y + x^2) + x^2} \propto \binom{4 \propto^2 x}{\beta^2 y + x^2}$$

By induction

$$k_1^n \binom{x}{y} \propto \binom{(2 \propto)^n x}{\beta^n y + x^2}$$

If $\left|\frac{\propto}{2}\right|$ >1 then $k_1^n \to \infty$ as $n \to \infty$

If $|\beta|>1$ then $k_1^n \to \infty$ as $n\to\infty$

We use the matlab to calculate the Lypunov exponent of k_1

$$J_2=\begin{pmatrix} 2\propto mod\ 1 & 0 \\ 2x & \beta \end{pmatrix} \quad x=0.1,\ y=0.1$$

A	b	L1	L2
-1	0.1	- ∞	NaN
- 0.8	±0.1	- ∞	NaN
-0.4	0.1	- ∞	NaN
0	±0.1	- ∞	NaN
1.2	0.1	- ∞	NaN
0.9	0.1	-0.2217568112	- ∞
0.5	±0.1	- ∞	NaN
0.8	0.1	-0.5104248463	- ∞
0.7	±0.1	- ∞	NaN
0.8	0.7	-0.3530910674	-0.5144095003
-1	±0.7	-0.3550215868	- ∞
-2	±0.7	-0.3555969509	- ∞
0	±0.7	- ∞	NaN
0.4	±0.4	-0.223143513	-0.9162907319
0.23	1	-0.0042964340	-0.7722323555
0.23	-1	-0.0051558170	-0.7713729725
-0.23	-1	0.0009468829	-0.6171330223
-0.23	1	0.0024528523	-0.6186389917

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