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Some Probability Characteristics of the Solution of Stochastic Fredholm Integral Equation Contain Brownian Motion.

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ABSTRACT

In this paper, some probability characteristics (probability density, characteristic, covariance, and spectral density) functions are derived depending on the smallest variance of the solution of a supposing stochastic Fredholm integral equation contains the Brownian motion by adomian decomposition method.

1. Introduction:

In the beginning of 1980, a method for solving linear and non-linear integral (differential) equations for various kind has been proposed by G. Adomian, the so called Adomian decomposition method (A.D.M). In this method, the solution of stochastic Fredholm integral equation is considered as an infinite series (geometrical series) which converging to the solution [1].The convergence of this method applied to the one dimensional integral equations is discussed in [2].In this paper our aim is derive the probability characteristics functions of the solution of supposing stochastic Fredholm integral equation, this solution is found analytically by the Adomian decomposition method.

Suppose the following stochastic Fredholm integral equation having the formula

$$\mathrm{Y}(t,w)=\mathrm{X}(t,w)+\int_a^b k(t,s,w)\mathrm{Y}(t,s)\,\mathrm{d}s$$

…(1)

Where

- $w \in \Omega$, Ω is a sample space supporting of the probability measure space $(\Omega,\mathcal{F},\mathbb{P})$.
- $\mathrm{Y}(t,w)$ is the unknown stochastic process for the time $t > 0$.
- $\mathrm{X}(t,w)$ is known stochastic process, with the one dimension Brownian motion[3]

i- $\mathrm{X}(t,w) \sim \mathrm{N}(0,t)$ for all $t \geq 0$, $-\infty < w < \infty$, (i.e.) $\mathrm{X}(t,w)=\frac{1}{\sqrt{2\pi t}}\mathrm{e}^{\frac{-w^2}{2t}}$ …(2)

ii- $\mathrm{X}(t,w)$ has stationary independent increments with the distribution $\mathrm{X}(t,w)-\mathrm{X}(s,w)\sim \mathrm{N}(0,t-s)$ for $0 \leq s < t$.

- $K(t,s,w)$ is known kernel defined by $t > 0$ and $s \in$ S, where S is a compact space.
- $\mathrm{Y}(t,s)$ is a scalar function defined for the time $t > 0$, $s \in$ S.

2. Preliminaries

By considering the kernel $k(t,s,w)=stw$ and by substituting formula (2) into equation (1), it will be

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$$Y(t,w)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{w^2}{2t}}+\int_0^1(stw)Y(t,s)ds$$

…(3)

To find the stochastic solution of (3), Adomian decomposition method [4, 5] will be used which briefly depend on the following steps,

$$Y_0(t,w)=X(t,w)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{w^2}{2t}}$$

…(3a)

$$Y_0(t,s)=X(t,s)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{s^2}{2t}}$$

…(3b)

$$Y_{m+1}(t,w)=\int_0^1 k(t,s,w)\,Y_m(t,s)\,ds\ ,\ m=0,\ 1,\ 2,\ldots \qquad \ldots(3c)$$

That is to obtain the following solution

$$Y(t,w)=Y_0(t,w)+\sum_{m=1}^{\infty}Y_m(t,w)$$

…(4)

By substituting respectively m=0,1,2,… in (3c) we get:

For m = 0

$$Y_1(t,w)=\int_0^1 swt\,Y_0(t,s)\,ds$$

$$=\int_0^1 swt\,\frac{1}{\sqrt{2\pi t}}e^{-\frac{s^2}{2t}}\,ds$$

$$=\frac{wt}{\sqrt{2\pi t}}\int_0^1 s\,e^{-\frac{s^2}{2t}}\,ds$$

$$Y_1(t,w)=\frac{wt^2}{\sqrt{2\pi t}}\left[1-e^{-\frac{1}{2t}}\right],\ -\infty<w<\infty,\ t>0.$$

For m = 1

$$Y_2(t,w)=\int_0^1 swt\,Y_1(t,s)\,ds$$

$$=\int_0^1 swt\,\frac{st^2}{\sqrt{2\pi t}}\left[1-e^{-\frac{1}{2t}}\right]ds$$

$$=\frac{wt^3}{\sqrt{2\pi t}}\left[1-e^{-\frac{1}{2t}}\right]\int_0^1 s^2\,ds$$

$$Y_2(t,w)=\frac{wt^3}{3\sqrt{2\pi t}}\left[1-e^{-\frac{1}{2t}}\right],\ -\infty<w<\infty,\ t>0.$$

For m = 2

$$Y_3(t,w)=\int_0^1 swt\,Y_2(t,s)\,ds$$

$$=\int_0^1 swt\,\frac{st^3}{3\sqrt{2\pi t}}\left[1-e^{-\frac{1}{2t}}\right]ds$$

$$=\frac{wt}{3\sqrt{2\pi t}}\left[1-e^{-\frac{1}{2t}}\right]\int_0^1 s\,ds$$

$$Y_3(t,w)=\frac{wt^4}{9\sqrt{2\pi t}}[1-e^{\frac{-1}{2t}}],\ -\infty<w<\infty,\ t>0.$$

and by repeating for m = 3,4,5,…, one can get

$$Y_k(t,w)=\frac{w}{\sqrt{2\pi t}}\left[\frac{t^{k+1}}{3^{k-1}}\right][1-e^{-\frac{1}{2t}}],\ k=4,5,\ldots$$

Therefore, (4) can be written as

$$Y(t,w)=Y_0(t,w)+\left[\frac{wt^2}{\sqrt{2\pi t}}(1-e^{-\frac{1}{2t}})+\frac{wt^3}{3\sqrt{2\pi t}}(1-e^{-\frac{1}{2t}})+\ldots\right]$$

$$=\frac{1}{\sqrt{2\pi t}}e^{-\frac{w^2}{2t}}+\frac{1}{\sqrt{2\pi t}}(1-e^{-\frac{1}{2t}})w\left[t^2+\frac{t^3}{3}+\frac{t^4}{3^2}\ldots\right]$$

Therefore, we get

$$Y(t,w)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{w^2}{2t}}+\frac{1}{\sqrt{2\pi t}}(1-e^{-\frac{1}{2t}})w\sum_{n=1}^{\infty}\frac{t^{n+1}}{3^{n-1}}$$

where $\sum_{n=1}^{\infty}\frac{t^{n+1}}{3^{n-1}}$ is a geometrical series which converges at 0 < t <3

Hence

$$Y(t,w)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{w^2}{2t}}+\frac{w}{\sqrt{2\pi t}}\left(1-e^{-\frac{1}{2t}}\right)\left(\frac{3t^2}{3-t}\right),-$$

$\infty<w<\infty, 0<t<3$ …(5)

or

$$Y(t,w)=\alpha(t)e^{-\frac{w^2}{2t}}+\beta(t)\,w$$

…(6)

where

$$\alpha(t)=\frac{1}{\sqrt{2\pi t}}$$

$$\beta(t)=\frac{1}{\sqrt{2\pi t}}\left(1-e^{-\frac{1}{2t}}\right)\left(\frac{3t^2}{3-t}\right)$$

Furthermore the function Y(t,w) can also be considered as a solution over the interval $-1 < w < 1$ which permit to derive the probability characteristics of the solution over this interval and over any division of its ten equal subintervals each of length 0.1 starting from (–1 to +1). and choosing many suitable different values for $t \in$ T such that T={0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1,1.1,1.2,1.3,1.4,1.5,1.6,1.7,1.8, 1.9,2}.

3. Moments, Variances:

In order to find the variance of the solution (6) over the interval $-1 < w < 1$, $0 < t < 3$, it must be that this solution is a p.d.f. of w. So, we multiply this solution by a constant A and equate it by one and find the value of A which makes the solution is a p.d.f. of w, (i.e.) we write

$$\int_{-1}^{1} A\,Y(t,w)\,dw=1$$

$$\int_{-1}^{1} A\,Y(t,w)\,dw=A\int_{-1}^{1}\left[\alpha(t)e^{-\frac{w^2}{2t}}+\beta(t)w\right]dw=1$$

$$A\alpha(t)\int_{-1}^{1} e^{-\frac{w^2}{2t}}dw+A\beta(t)\int_{-1}^{1} w\,dw=1$$

$$A[2N\left(\frac{1}{\sqrt{t}}\right)-1]+0=1$$

Hence,

$$A=\frac{1}{2N\left(\frac{1}{\sqrt{t}}\right)-1}$$

…(7)

where $N\left(\frac{1}{\sqrt{t}}\right)$ is a standard normal distributed and let

$\varphi(t,w)$ = A Y(t,w)

and the solution of (3) over the interval $-1 < w < 1$ is a p.d.f. when

$$\varphi(t,w)=\frac{1}{2N\left(\frac{1}{\sqrt{t}}\right)-1}\frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t}}+\frac{1}{2N\left(\frac{1}{\sqrt{t}}\right)-1}\frac{w}{\sqrt{2\pi t}}(1-e^{-\frac{1}{2t}})\left(\frac{3t^2}{3-t}\right),$$

$0<t<3$ …(8)

$$\varphi(t,w)=\alpha_1(t)e^{-\frac{w^2}{2t}}+\beta_1(t)\,w$$

…(9)

where

$$\alpha_1(t)=\frac{1}{\sqrt{2\pi t}\left[2N\left(\frac{1}{\sqrt{t}}\right)-1\right]}$$ …(9a)

$$\beta_1(t)=\frac{1}{\sqrt{2\pi t}\left[2N\left(\frac{1}{\sqrt{t}}\right)-1\right]}\left(1-e^{-\frac{1}{2t}}\right)\left(\frac{3t^2}{3-t}\right)$$ …(9b)

Calculation $\varphi(t,w)$ is tabulated in table (1).

1st Moment

$$E_{\varphi(t,w)}[.,w]=\int_{-1}^{1} w\,\varphi(t,w)\,dw$$

$$=\alpha_1(t)\int_{-1}^{1} we^{\frac{-w^2}{2t}}dw+\beta_1(t)\int_{-1}^{1} w^2dw$$

$$E_{\varphi(t,w)}[.,w]=\frac{2}{\sqrt{2\pi t}\left[2N\left(\frac{1}{\sqrt{t}}\right)-1\right]}\left(1-e^{-\frac{1}{2t}}\right)\left(\frac{t^2}{3-t}\right)$$

…(10)

2nd Moment

$$E_{\varphi(t,w)}[.,w^2]=\int_{-1}^{1} w^2\,\varphi(t,w)\,dw$$

$$= \alpha_1(t)\int_{-1}^{1} w^2 e^{-\frac{w^2}{2t}}\,\mathrm{d}w + \beta_1(t)\int_{-1}^{1} w^3\,\mathrm{d}w$$

$$\mathrm{E}_{\varphi(t,w)}[.,w^2] = \frac{-2t}{\sqrt{2\pi t}\left[2\mathrm{N}\left(\frac{1}{\sqrt{t}}\right)-1\right]}e^{-\frac{1}{2t}} + t$$

…(11)

While, the variance of the solution can be obtained from the first and second moments. Table (2) shows that, the smallest variance of (9) is when (t = 1.6). Hence, the solution of the supposing Freddholm integral equation (9) over the interval $-1 < w < 1$ that will be adopted takes the following form

$$\varphi(1.6,\mathrm{w}) = \alpha_1(1.6)e^{-\frac{w^2}{2(1.6)}} + \beta_1(1.6)\,w$$

…(12)

$$\varphi(1.6,\mathrm{w}) = 0.552539e^{-\frac{w^2}{3.2}} + 0.46435\,w$$

…(13)

Figure (1) represents the curve of φ(1.6,w).

4. Characteristic Function of φ (1.6,w):

The characteristic function [6] of φ (1.6,w) can be found as following,

$f\,(u,w,t) = \mathrm{E}[e^{iuw}]$

$$= \int_{-1}^{1} e^{iuw}\varphi(t,w)\,\mathrm{d}w$$

$$= \int_{-1}^{1} e^{iuw}\left[\alpha_1(t)e^{-\frac{w^2}{2t}} + \beta_1(t)w\right]\mathrm{d}w$$

$$= \alpha_1(t)\int_{-1}^{1} e^{iuw-\frac{w^2}{2t}}\,\mathrm{d}w + \beta_1(t)\int_{-1}^{1} w\,e^{iuw}\,\mathrm{d}w$$

$$= \alpha_1(t)e^{-\frac{tu^2}{2}}\int_{-1}^{1} e^{\frac{-(w-itu)^2}{2t}}\,\mathrm{d}w + \beta_1(t)\int_{-1}^{1} we^{iuw}\,\mathrm{d}w$$

$$= \alpha_1(t)e^{-\frac{tu^2}{2}}\int_{\frac{-1-itu}{\sqrt{2t}}}^{\frac{1-itu}{\sqrt{2t}}} e^{-z^2}\sqrt{2t}\,\mathrm{d}z + \beta_1(t)\int_{-1}^{1} we^{iuw}\,\mathrm{d}w$$

$$= \alpha_1(t)e^{-\frac{tu^2}{2}}\sqrt{2t}\int_{\frac{-1-itu}{\sqrt{2t}}}^{\frac{1-itu}{\sqrt{2t}}}\left[1-\frac{z^2}{1!}+\frac{(z^2)^2}{2!}-\frac{(z^2)^3}{3!}+\ldots\right]\mathrm{d}z + \beta_1(t)\int_{-1}^{1} we^{iuw}\,\mathrm{d}w$$

$$= \alpha_1(t)e^{-\frac{tu^2}{2}}\sqrt{2t}\int_{\frac{-1-itu}{\sqrt{2t}}}^{\frac{1-itu}{\sqrt{2t}}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{n!}\,\mathrm{d}z + \beta_1(t)\int_{-1}^{1} we^{iuw}\,\mathrm{d}w$$

$$= \frac{\mathrm{A}}{\sqrt{\pi}}e^{-\frac{tu^2}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{n!(2n+1)}\Bigg|_{\frac{-1-itu}{\sqrt{2t}}}^{\frac{1-itu}{\sqrt{2t}}} + \beta_1(t)\left[\frac{1}{iu}\left(we^{iuw}\Big|_{-1}^{1}\right)-\frac{1}{iu}\int_{-1}^{1} e^{iuw}\,dw\right]$$

$$f\,(u,w,t) = \frac{1}{\sqrt{\pi}[2N(\frac{1}{\sqrt{t}})-1]}e^{-\frac{tu^2}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n[(1-itu)^{2n+1}-(-1-itu)^{2n+1}]}{(n!)(2n+1)(2t)^{\frac{2n+1}{2}}} + \quad \ldots(14)$$

$$\frac{1}{\sqrt{2\pi t[2N(\frac{1}{\sqrt{t}})-1]}}\left[1-e^{-\frac{1}{2t}}\right]\left[\frac{3t^2}{3-t}\right]\left[\frac{1}{iu}(e^{iu}+e^{-iu})+\frac{1}{u^2}(e^{iu}-e^{-iu})\right]$$

$$f\,(u,w,1.6) = (0.988412)e^{-\frac{1.6u^2}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n[(1-i(1.6)u)^{2n+1}-(-1-i(1.6)u)^{2n+1}]}{(n!)(2n+1)(3.2)^{\frac{2n+1}{2}}} + \quad \ldots(15)$$

$$(0.813491)\left[\frac{1}{iu}(e^{iu}+e^{-iu})+\frac{1}{u^2}(e^{iu}-e^{-iu})\right]$$

5. Covariance Function of φ (1.6, w):

For any $s > t = 1.6$, $\tau = s - 1.6 > 0$, the covariance function [6] of φ (1.6,w), with the function φ (1.6 + τ,w) depends only on the difference $|\tau| = |-\tau| = |s - 1.6|$ and can be found as following,

B(τ) = B(s – 1.6)

= E{[φ (1.6,w) – E(φ (1.6,w))][φ (s,w)−E(φ (s,w))]}

B(τ)=0.012292 …(16)

6- Spectral Density Function of φ (1.6,w):

The spectral density function of φ (1.6, w) for known B(τ) = 0.012292 can be found by khinchine's formula [7] as follows

$$f_\varphi(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\tau)e^{-i\lambda\tau}\,\mathrm{d}\tau$$

$$= \frac{0.012292}{2\pi}\int_{-\infty}^{\infty}(\cos\lambda\tau - i\sin\lambda\tau)\,\mathrm{d}\tau \quad , |\lambda| \le 2\pi$$

and for τ = s – 1.6 > 0.

$$f_\varphi(\lambda) = \frac{0.012292}{\pi}\int_{0}^{s-1.6}\cos\lambda\tau\,\mathrm{d}\tau$$

$$f_{\varphi}(\lambda)=0.003913\frac{\sin(\lambda(s-1.6))}{\lambda}\ ,\ |\lambda|\le 2\pi, s>1.6$$

…(17)

Figure (2) represent the curve of $f_{\varphi}(\lambda)$ for $0<\lambda\le 2\pi$ when s = (4.6,10.6).

Conclusion:

The Adomian decomposition method had been successful used to obtain the solution of stochastic Fredholm integral equation. The analytical solution of stochastic Fredholm integral equation by the Adomian decomposition method give a continuous function on closed and bounded interval which permit to predict this solution over any interval, that is by using some statistical methods.

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Table (1) Calculation of $\varphi(t,w)=\alpha_1(t)e^{-\frac{w^2}{2t}}+\beta_1(t)\,w$

W \ t	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
-1	0.039124	0.110177	0.115514	0.078535	0.009540	-0.090424	-0.226299	-0.409246	-0.659859	-1.016511
-0.8	0.155984	0.244832	0.245626	0.209464	0.148422	0.063674	-0.048892	-0.198413	-0.401523	-0.689041
-0.6	0.350514	0.397619	0.377098	0.336234	0.281167	0.2101983	0.121615	0.005652	-0.149721	-0.367837
-0.4	0.599116	0.545284	0.495866	0.449757	0.401481	0.346899	0.281720	0.200195	0.093324	-0.0540720
-0.2	0.820962	0.658353	0.587505	0.541568	0.503817	0.467513	0.428522	0.382977	0.326063	0.248857
0	0.915262	0.711821	0.641150	0.605652	0.584368	0.571216	0.560104	0.552538	0.546663	0.541969
0.2	0.835365	0.695856	0,652756	0.639828	0.641776	0.654043	0.675797	0.708375	0.755411	0.824296
0.4	0.627920	0.620295	0.626368	0.646275	0.677398	0.719958	0.776269	0.850989	0.952473	1.096158

1	0.8	0.6
0.111135	0.213593	0.393721
0.297702	0.394851	0.510135
0.441771	0.506631	0.572852
0.569828	0.602497	0.631010
0.699334	0.700255	0.695043
0.842226	0.809793	0.770573
1.010077	0.940208	0.863441
1.217739	1.103174	0.981843
1.488016	1.316777	1.139004
1.860682	1.612713	1.358481

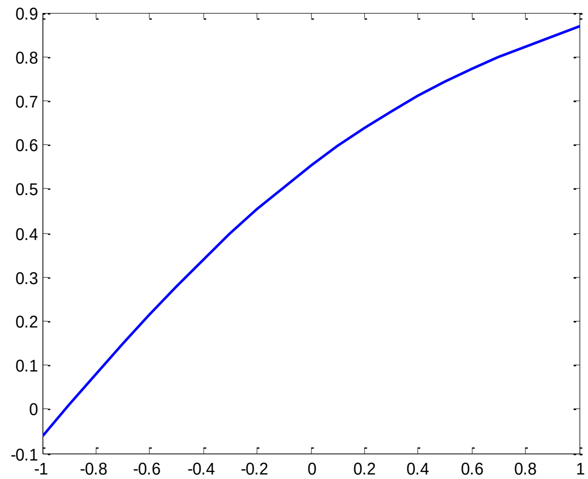

Figure (1)The curve of

$$\varphi(1.6,w) = 0.552539e^{-\frac{w^2}{3.2}} + 0.46435\,w$$

Table (2) The Smallest Variance of

$$\varphi(t,w) = \alpha_1(t)e^{-\frac{w^2}{2t}} + \beta_1(t)\,w$$

t	$E_{\varphi(t,w)}[.,w^2]$	$[E_{\varphi(t,w)}[.,w]]^2$	$var_{\varphi}(t)$
0.1	0.098297	0.000075	0.098222
0.2	0.169948	0.000576	0.169372
0.3	0.211446	0.001785	0.209661
0.4	0.236848	0.003907	0.232941
0.5	0.253704	0.007164	0.246540
0.6	0.265629	0.011827	0.253802
0.7	0.277835	0.018235	0.259600
0.8	0.281308	0.026819	0.254489
0.9	0.286725	0.038126	0.248599
1	0.291126	0.052868	0.238258
1.1	0.294770	0.071969	0.222801
1.2	0.297838	0.096648	0.201190
1.3	0.300453	0.128538	0.171915
1.4	0.302711	0.169848	0.132863
1.5	0.304682	0.223618	0.081064
1.6	0.306412	0.294120	0.012292
1.7	0.307950	0.387459	No Variance
1.8	0.309319	0.512596	No Variance
1.9	0.310546	0.683035	No Variance
2	0.311658	0.919805	No Variance

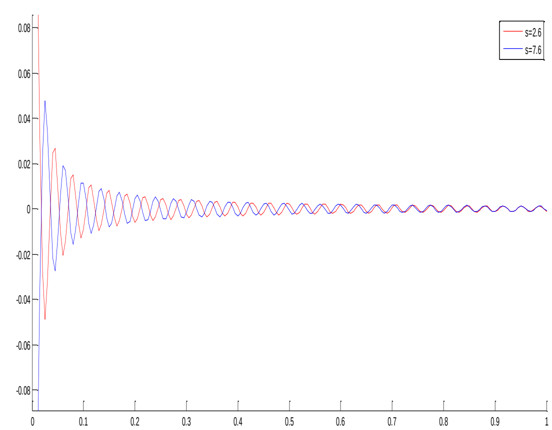

Figure (2)The curve of $f_{\varphi}(\lambda)$ **for** $0 < \lambda \le 2$

بعض مزايا الاحتمالية لحل معادلة فريدهولم التكاملية العشوائية الحاوية على الحركة البراونية

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الخلاصة:

في هذا البحث بعض دوال مزايا الاحتمالية (كثافة الاحتمالية، المميزة، الارتباط والكثافة الطيفية) تم اشتقاقها استناداً لأصغر تباين لحل معادلة افتراضية لفريدهوم التكاملية العشوائية الحاوية على الحركة البراونية بطريقة ادوميان التحليلية.