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FINITE ELEMENT METHOD FOR TWO DIMENSIONAL COUPLED BBM-SYSTEM

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ABSTRACT

In this paper the Matrix Equation for the two-dimensional nonlinear Coupled-BBM system of type Boussinesq is obtained by using Finite element method. In this regard triangular element is used to get the results. Tsunami wave is used to test the efficiency of this method. The wave generation and evolution are described by numerical experiment.

1. Introduction.

Boussinesq systems have been used in the study of a varity of water wave phenomena in ports, channels, coastal areas, and in the open sea. They have been also used in studies of tsunami wave generation and propagation [11]. These systems may be written as :

$$v_t + \nabla u + \nabla(vu) + a\Delta\nabla u - b\Delta v_t = 0$$
$$u_t + \nabla v + \frac{1}{2}\nabla|u|^2 + c\Delta\nabla v - d\Delta u_t = 0. \quad (1)$$

These systems have been derived by [6] to describe irrotational free surface flow of an ideal fluid over a horizontal bottom. The independent variable $X = (x, y)$ represents the position, t is proportional to elapsed time, $v = v(X,t)$ is proportional to the deviation of the free surface from its rest position, while u is proportional to the horizontal velocity of the fluid at some height.

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The wellposedness and regularity of (1) are given in [4,5]. The existence of line solitary waves, line cnoidal waves, symmetric and a symmetric periodic wave patterns are proved in [2,7,9,10].

This paper deals with two-dimensional Coupled BBM-system of type Boussinesq. i.e.

$$v_t + \nabla u + \nabla(vu) - \frac{1}{6}\Delta v_t = 0 \quad (2a)$$

$$u_t + \nabla v + \frac{1}{2}\nabla|u|^2 - \frac{1}{6}\Delta u_t = 0, \quad (2b)$$

where v, u are mappings from $\Omega \times (0,\infty)$ to $\Re$, $\Omega = (0, L) \times (0, L)$ with boundary conditions: $v|_{\partial\Omega} = 0, u|_{\partial\Omega} = 0, t \in [0,\infty)$ and the initial conditions are $v|_{t=0} = v_0, u|_{t=0} = u_0, X \in \Omega$. [8] Theoretical and numerical aspects of these systems in the case of a horizontal bottom, i.e., for the systems (1) and (2), were studied recently in[6,8,12,13].

Bona and Chen in [4] proved that, the initial-boundary value problem for the Coupled BBM-system in one space dimension with nonhomogeneous Dirichlet boundary conditions at the endpoints of a finite interval is locally well-posed. In [1] we proved the convergence analysis of the solution of one-dimensional nonlinear Coupled-BBM system by using implicit finite difference method.. Coupled BBM-system on a smooth plane domain with homogeneous Dirichlet boundary conditions, homogeneous Neumann boundary conditions and to the (normal) reflective boundary conditions established in [13]. Existence of the solitary waves for these one-dimensional systems has been studied by Toland [3].

The content of this paper is as follows: section two is devoted to deriving the matrix equation using finite element method. Triangular element is used in this regard. In section three of this paper. Generation and evolution of a Tsunami wave is used to examine the efficiency of the method. The conclusions of this paper are mentioned in the last section.

2. Derivation of the matrix equation using the Finite Element Method.

Let u^j and v^j are the discretized solution which satisfies the system (2), then the system (2) is rewritten as:

$$v_t^j + \nabla u^j + \nabla(v^j u^j) - \frac{1}{6}\Delta v_t^j = 0 \qquad (3a)$$

$$u_t^j + \nabla v^j + \frac{1}{2}\nabla\left|u^j\right|^2 - \frac{1}{6}\Delta u_t^j = 0 \qquad (3b)$$

Consider the Dirichlet boundary conditions $v|_{\partial\Omega} = 0, u|_{\partial\Omega} = 0, t \in [0,\infty)$, these become the essential boundary conditions for constructing the weak formulation for the system.

Multiply equation (3a) by test function $w \in W_0^{1,\infty}(\Omega)$, where $\Omega = (a,b)\times(a,b)$ and integrate over the finite element Ω_e. Then we get the following expression

$$\int_{\Omega_e} w.(v_t^j + \nabla u^j + \nabla(v^j u^j) - \frac{1}{6}\Delta v_t^j)dxdy = 0$$

For all $j \in N$. Since w satisfies essential boundary conditions, the boundary terms vanish after the integration by parts of the above equation and we get the following equation

$$\int_{\Omega_e}(wv_t^j + wu_x^j + wu_y^j + wv^j u_x^j + wu^j v_x^j + wv^j u_y^j + wu^j v_y^j + \frac{1}{6}w_x v_{xt}^j + \frac{1}{6}w_y v_{yt}^j)dydx = 0 \quad (4)$$

We define

$$v^j(x,y) = \sum_{s=1}^{n_e} v_s^j(t) N_s(x,y)$$
$$u^j(x,y) = \sum_{s=1}^{n_e} u_s^j(t) N_s(x,y) \qquad (5)$$
$$w(x,y) = N_i(x,y), \qquad i = 1,\ldots,n_e$$

Here $u_s^j(t)$ and $v_s^j(t)$, $s = 1,\ldots,n_e$ are undetermined time dependent quantities and $N_s(x,y)$ are the interpolation functions. Then equation (4) Becomes

$$\sum_{s=1}^{n_e}\int_{\Omega_e}\left(N_i(x,y)N_s(x,y)dxdy + \frac{1}{6}(\frac{\partial N_i(x,y)}{\partial x}\frac{\partial N_s(x,y)}{\partial x} + \frac{\partial N_i(x,y)}{\partial y}\frac{\partial N_s(x,y)}{\partial y})dxdy\right)\overset{\bullet}{v}{}_s^j +$$
$$\sum_{s=1}^{n_e}\int_{\Omega e} N_i(x,y)\frac{\partial N_s(x,y)}{\partial x}u_s^j dxdy + \sum_{s=1}^{n_e}\int_{\Omega_e} N_i(x,y)\frac{\partial N_s(x,y)}{\partial y}u_s^j dxdy +$$
$$+\int_{\Omega_e} N_i(x,y)\sum_{s=1}^{n_e} N_s(x,y)u_s^j \sum_{s=1}^{n_e}\frac{\partial N_s(x,y)}{\partial x}v_s^j dydx$$
$$+\int_{\Omega_e} N_i(x,y)\sum_{s=1}^{n_e} N_s(x,y)v_s^j \sum_{s=1}^{n_e}\frac{\partial N_s(x,y)}{\partial x}u_s^j dydx$$
$$+\int_{\Omega_e} N_i(x,y)\sum_{s=1}^{n_e} N_s(x,y)v_s^j \sum_{s=1}^{n_e}\frac{\partial N_s(x,y)}{\partial y}u_s^j dydx = 0 \qquad (6)$$

Where

$$\overset{\bullet}{v}{}_s^j \approx \frac{v_s^{j+1} - v_s^j}{\Delta t}$$

After multiplying both sides of equation(6) by Δt and moving the previous time solutions to the right hand side, we get the same of matrix equation on each finite element Ω_e as in the one dimensional case as follows

$$[A^e + B^e]\{v^{j+1}\} = [A^e + B^e]\{v^j\} - \Delta t F^e(u^j, v^j)$$

Where

$$A^e_{i,s} = \int_{\Omega_e} N_i(x,y) N_s(x,y) dxdy \qquad (7)$$

$$B^e_{i,s} = \frac{1}{6}\int_{\Omega_e} \left(\frac{\partial N_i(x,y)}{\partial x}\frac{\partial N_s(x,y)}{\partial x} + \frac{\partial N_i(x,y)}{\partial y}\frac{\partial N_s(x,y)}{\partial y}\right) dxdy$$

And

$$\{F^e(u^j,v^j)\}_i = \int_{\Omega_e} \sum_{s=1}^{n_e} N_i(x,y)\frac{\partial N_s(x,y)}{\partial x} u_s^j dxdy + \int_{\Omega_e} \sum_{s=1}^{n_e} N_i(x,y)\frac{\partial N_s(x,y)}{\partial y} u_s^j dxdy$$
$$+ \int_{\Omega_e} N_i(x,y)\sum_{s=1}^{n_e} N_s(x,y) u_s^j \sum_{s=1}^{n_e}\frac{\partial N_s(x,y)}{\partial x} v_s^j dxdy$$
$$+ \int_{\Omega_e} N_i(x,y)\sum_{s=1}^{n_e} N_s(x,y) u_s^j \sum_{s=1}^{n_e}\frac{\partial N_s(x,y)}{\partial y} v_s^j dxdy$$
$$+ \int_{\Omega_e} N_i(x,y)\sum_{s=1}^{n_e} N_s(x,y) v_s^j \sum_{s=1}^{n_e}\frac{\partial N_s(x,y)}{\partial x} u_s^j dxdy$$
$$+ \int_{\Omega_e} N_i(x,y)\sum_{s=1}^{n_e} N_s(x,y) v_s^j \sum_{s=1}^{n_e}\frac{\partial N_s(x,y)}{\partial x} u_s^j dxdy$$

By the same approach, equation (3b) after multiplication by a test function w and integration by parts become

$$\int_{\Omega_e} (wu_t^j + wv_x^j + wv_y^j + wu^j u_x^j + wu^j u_y^j + \frac{1}{6} w_x u_{xt}^j + \frac{1}{6} w_y u_{yt}^j) dydx = 0 \quad (8)$$

Using equations (5). equation (8) become

$$\sum_{s=1}^{n_e}\int_{\Omega_e}\left(N_i(x,y)N_s(x,y)dxdy + \frac{1}{6}\left(\frac{\partial N_i(x,y)}{\partial x}\frac{\partial N_s(x,y)}{\partial x} + \frac{\partial N_i(x,y)}{\partial y}\frac{\partial N_s(x,y)}{\partial y}\right)dydx\right)\dot{u}_s^j +$$
$$\sum_{s=1}^{n_e}\int_{\Omega_e} N_i(x,y)\frac{\partial N_s(x,y)}{\partial x} v_s^j dydx + \sum_{s=1}^{n_e}\int_{\Omega_e} N_i(x,y)\frac{\partial N_s(x,y)}{\partial y} v_s^j dydx +$$
$$\int_{\Omega_e} N_i(x,y)\sum_{s=1}^{n_e} N_s(x,y) u_s^j \sum_{s=1}^{n_e}\frac{\partial N_s(x,y)}{\partial x} u_s^j dydx +$$
$$\int_{\Omega_e} N_i(x,y)\sum_{s=1}^{n_e} N_s(x,y) u_s^j \sum_{s=1}^{n_e}\frac{\partial N_s(x,y)}{\partial x} u_s^j dydx = 0 \qquad (9)$$

where $, \dot{u}_s^j \approx \frac{u_s^{j+1} - u_s^j}{\Delta t}$

After multiplying the both sides of equation(9) by Δt and moving the previous time solutions to the right hand side, we get the same of matrix equation on each finite element Ω_e as in the one dimensional case as follows

$$[A^e + B^e]\{u^{j+1}\} = [A^e + B^e]\{u^j\} - \Delta t G^e(u^j, v^j)$$

Where A^e, B^e as in equations (7)

And

$$\{G^e(u^j,v^j)\}_i = \int_{\Omega_e} \sum_{s=1}^{n_e} N_i(x,y)\frac{\partial N_s(x,y)}{\partial x} v_s^j dxdy + \int_{\Omega_e} \sum_{s=1}^{n_e} N_i(x,y)\frac{\partial N_s(x,y)}{\partial y} v_s^j dxdy$$
$$+ \int_{\Omega_e} N_i(x,y)\sum_{s=1}^{n_e} N_s(x,y) u_s^j \sum_{s=1}^{n_e}\frac{\partial N_s(x,y)}{\partial x} u_s^j dxdy$$
$$+ \int_{\Omega_e} N_i(x,y)\sum_{s=1}^{n_e} N_s(x,y) u_s^j \sum_{s=1}^{n_e}\frac{\partial N_s(x,y)}{\partial y} u_s^j dxdy$$

The assembling of the global stiffness matrix from the finite element equation depends on the elements. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be the three components of the triangle. Then the following functions are interpolation functions of the triangular elements

$$N_s^e(x,y) = \frac{1}{2A^e}(a_s + b_s x + c_s y), \quad s = 1,2,3$$

Where

$a_1 = x_2 y_3 - x_3 y_2, a_2 = x_3 y_1 - x_1 y_3,\ a_3 = x_1 y_2 - x_2 y_1$
$b_1 = y_2 - y_3,\ b_2 = y_3 - y_1, b_3 = y_1 - y_2$
$c_1 = x_3 - x_2,\ c_2 = x_1 - x_3, c_1 = x_2 - x_1$

A^e = Area of the triangle

Choose the right triangle on each equilegthed cubic of length h. then we can obtain the local matrices A^e, B^e as follows

$$A^e = \frac{h^2}{24}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad , \quad B^e = \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

3. Numerical experiment

In this section we present the results of a simulation of the propagation of a tsunami wave using Finite Element Method that was analyzed in Section 2. In the case of Finite Element Method we use the standard Galerkin-finite element method with continuous piecewise linear elements on triangles with 1800 elements.

The generated wave by a source which is not necessarily axisymmetric, for example in the 2004 Asian tsunami, the waves were generated by a fault line which is about 1200km long in a north-south orientation. It is observed that the greatest strength of the tsunami waves was in an east-west direction [8]. The initial data in this sequence of tests be based on

$$v(x,y,0) = 5\alpha^2 e^{-\alpha^{2m}(\sigma^m (x-x_0)^{2m} + \sigma^{-m}(y-y_0)^{2m})},$$
$$u(x,y,0) = 0,$$

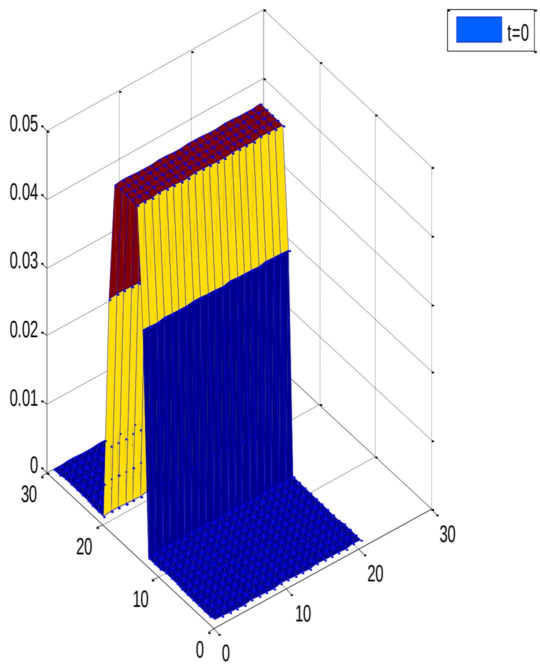
where $\alpha = 0.1, m = 8, \sigma = 10$, [8].

with homogenous Dirichlet boundary condition.

The simulations are executed up to time t=10. u and v values at y=15, t=8 with different values of x presented in table 1. In figure 1, the initial wave profile of v and its surface plot are presented to give a view on the rectangular nature of the initial data. Similar plots are presented with different time in figure 2. The results shows that the waves have the same behavior with the results in [8], where the leading wave in the positive and negative x-directions (east-west directions) are much bigger than that in the north-south directions. The waves in the north-south directions are very small. Also as in [8] the maximum amplitude is 0.05 at t=0, and its decreases with the time. These observations are confirmed by graph in figures 2 and 3.

Table 1: u and v values by finite element method at $y = 15$,t=8.

x	u	v
0	0	0
1	0	0
2	-4.7513E-07	1.6657E-06
3	-8.0045E-06	2.4963E-06
4	-1.2363E-05	3.6175E-05
5	-1.4721E-04	5.5621E-05
6	-2.2523E-04	5.3383E-05
7	-0.0017	8.111E-04
8	-0.0026	0.0047
9	-0.0107	0.0069
10	-0.0151	0.020
11	-0.0284	0.0266
12	-0.0336	0.029
13	-0.0216	0.0266
14	-0.0153	0.0194
15	-0.0221	0.0127
16	-0.0185	0.009
17	3.8615E-07	-0.0019
18	0.0185	0.009
19	0.0221	0.0127
20	0.0153	0.0194
21	0.0216	0.0266
22	0.0336	0.029
23	0.0284	0.0266
24	0.0151	0.020
25	0.0107	0.0069
26	0.0026	0.0047
27	0.0017	8.111E-04
28	2.2476E-04	5.3222E-04
29	1.5522E-04	5.811e-05
30	0	0



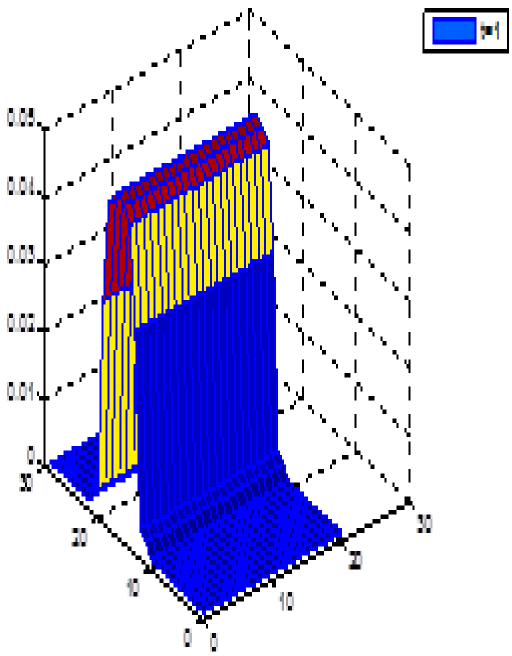
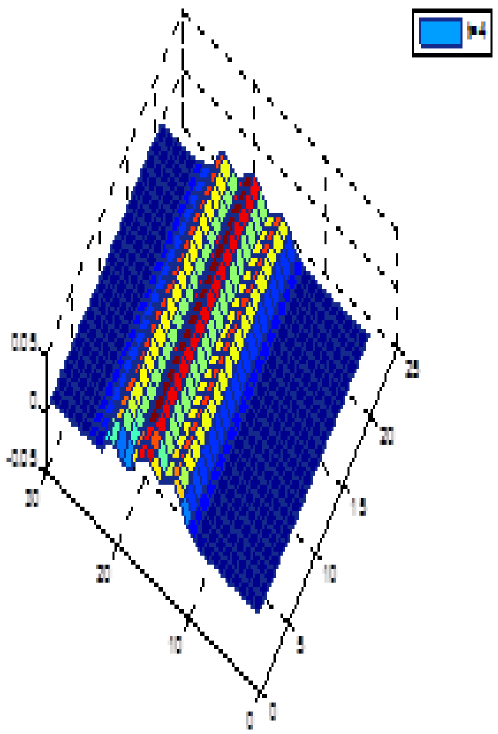
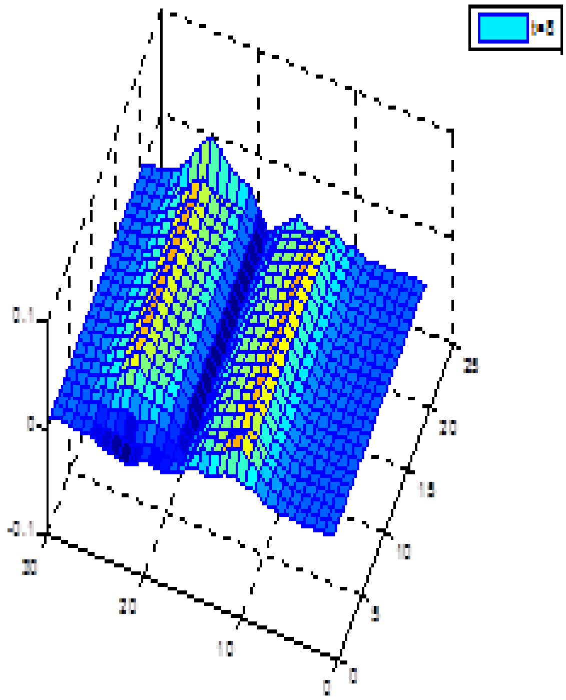


Figure 1. plot of initial data $v(x, y, o)$





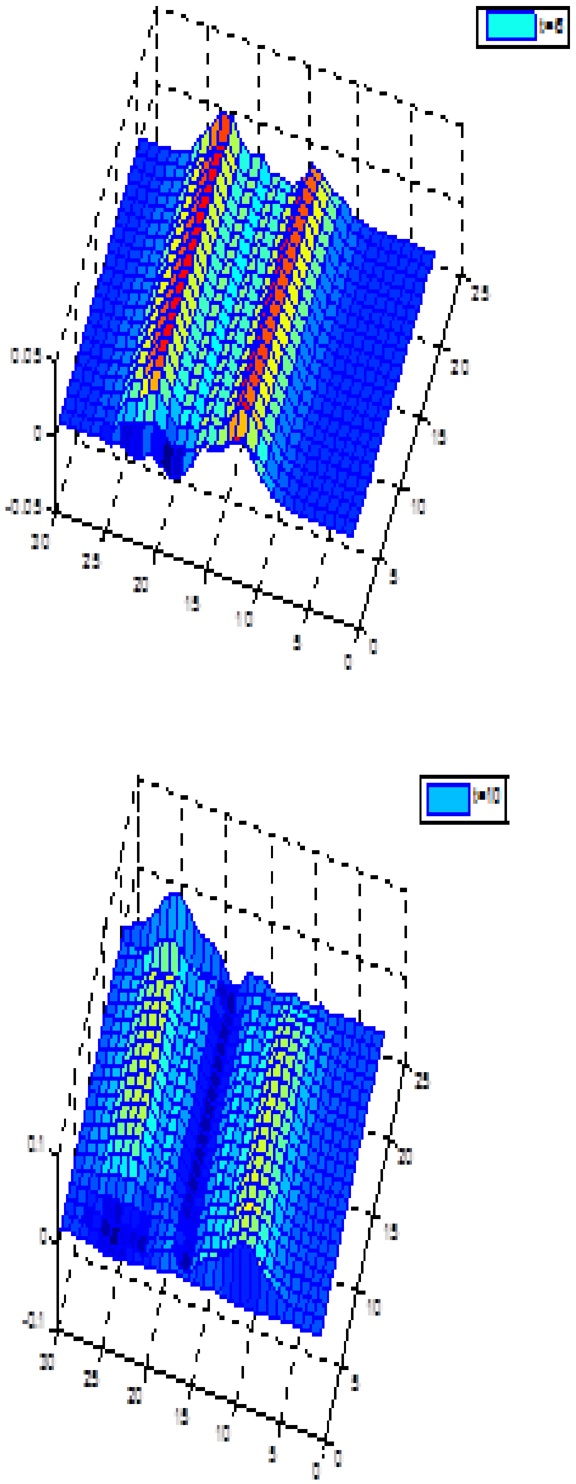


Figure 2. Solution describes the evolution of v at different time.

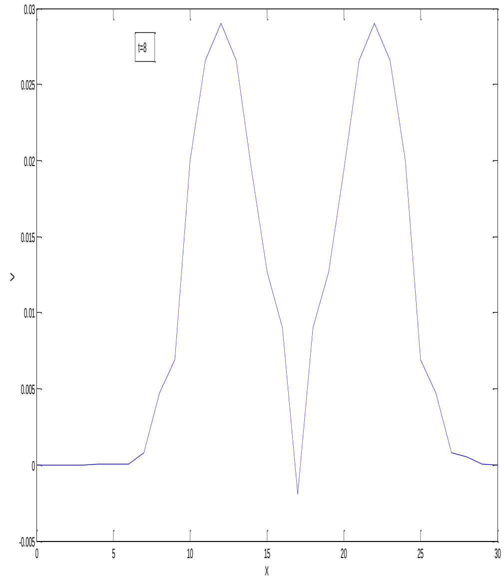


Figure 3. graph of $v(x,15,8)$, $0 \le x \le 30$.

4. Conclusion

In this paper the matrix equation for two dimensional Coupled BBM-system by using finite element method has been proposed. Linear triangular element adapted in this regard. The results show that the method is good candidate to generate and evaluate these types of waves.

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طريقة العنصر المنتهية لنظام بي بي إم ثنائي الأبعاد المزدوج

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الخلاصة

في هذا البحث تم اشتقاق معادلة المصفوفة لنظام BBM غير الخطي المزدوج من نوع Boussinesq في البعد الثنائي وباستخدام طريقة العناصر المنتهية. في هذا السياق تم استخدام العنصر ألمثلثي لإيجاد الحل العددي للنظام. موجة تسونامي تم استخدامها لاختبار كفاءة الطريقة ولوصف تولد وتطور هذه الموجة استخدمت تجربة عددية.