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Generalization of Counter Example of B. AlSPACH

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Abstract

In this work, I generalize the counter – example given by B – Alspach. And I find that there exists an infinity number of tournaments which is point – symmetric, but not rotation. **[1]**

Keywords: Counter Example, B. AlSPACH, Tournament.

1. Introduction

Brain Alspach is found a counter example **[1]** for a tournament of cardinal 21 elements which is point symmetric but not rotation. I generalize this counter example to show that if the number of the cardinal of tournament in prime this tournament is with grand arbitrary cardinal point symmetric but it is not rotation.

2. Definitions

- The tournament T is a binary – relation which is antireflexive and

$\forall\, x.v \in E.(\, E\ in\ the\ elements\ of\ T.)$ [3]: $T(x,y) \neq T(y,x) \qquad x \neq y$

- The tournament T is called rotation. Where one of it's circular - permutations base E in an automorphism
- The tournament in point – symmetric. [3] where $\forall\, x, y \in E$ .There exists an automorphism
$f\ of\ T$ such that
$f(x) = y$.

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3. Theorem

For all numbers p . prime congruent to 1 mod 3, there exists a tournament of cardinal $3P$ which is
point – symmetric but not rotation.

Proof :
The tournament T is described and It's automorphism group, then I prove that T is point symmetric then I prove that T is not rotation .
Description of Tournament and it's automorphism group.
$Let\ E = \{0,1,2, \dots, 3p-1\}$ put $E = E_1 \cup E_2 \cup E_3$ such that

$$E_1 = \{i\ ,\ o \le i \le p-1\}.$$
$$E_2 = \{i + p\ , o \le i \le p-1\}.$$
$$E_3 = \{i + 2p\ , o \le i \le p-1\}.$$

Let b and be c two numbers modulo p. on E, I define the three circular permututions as the following t_1 , t_2 , t_3 with the numbers i , $i-1, i+b$ and $i + c$ mod p .
For all $i \in E_1$:
$$t_1(i) = i - 1$$
$$t_2(i) = i + b$$
$$t_3(i) = i + c$$

Let t be the permutation of E defined by .. For all $i \in E_1$:
$$\left.\begin{aligned} t\,(i) &= t_1\,(i) \\ t\,(i+p) &= t_2\,(i) + p \\ t\,(i+2p) &= t_3\,(i) + 2p \end{aligned}\right\} \text{mod p}$$

We have t= $(o\ , p-1\ , p-2\ , \dots, 1)\ (p\ , p+b\ , \dots)\ (2p, 2p+c\ , \dots)$
Let s be the permutation of E detined by $\forall\ i \in E_1$
$$s\,(i) = i + p$$
$$s\,(i+p) = i + 2p$$
$$s\,(i+2p) = i$$
Then s transforms Ei in $Ei + 1$ $(i = 1,2,3)$
By replacing the order of Ei . I define the Tournament . T as Follows : s and t are the automorphism of
T, T is the tournament of base E such that for all i= 1,2,3, the restriction of T to Ei, T/Ei, in isomorphic
to T_h with $h = \frac{p-1}{2}$, $\forall\ i$ $T_h\ (i, i+k) = +$ for $1 \le k \le h$. We verify that t, is an automorphism of T_h
Let M be the matrix of T_h. The permutation s will become automorphism. The matrix M will be classified into blocks A,B,C such that

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$$M = \left(\begin{array}{c|c|c} A & B & C \\ \hline & & \\ \hline C & A & B \\ \hline B & C & A \end{array}\right)$$

We search for b and c , B and C such that t be an automorphism of T, I prove first :

Lemma 1:

There exist two cubic roots mod P not trivial other than unity . **[5]**

Proof:

The integer p is prime number according to Fermat we have for all integer x , $x^{p-1} \equiv 1$ (modulo p)

But p≡1 (mod 3) by hypothesis, then p-1 is a multiple of 3. Let K be such that p – 1=3k . We have then for all x , $x^{3k} \equiv 1$ (mod p). X^k is the cubic roots other than the unity. We know that if P is prime , then the multiplicative group G = $((z/p_z)^*$,o).* is a cyclic group, then there exist a ∈ G which generate G. We have also : 1< $k < p - 1$, then $a^k \neq 1$ there exist then two cubic roots mod P. Not trivial other than + 1 , then there exit also two cubic roots mod p which are not trivial other than – 1

Lemma2:

If t is an automorphism at T therefore b and c are the cubic roots **[2]** not trivial mod p other than -1.

Proof:

Note that B (i, j) is the element of B which lies on the line i and in the colomn j. By considering the three positions of B in the matrix M, we put the following conditions.

For all $i, j \in E_1$

$$B(i,j) = B(t_1(i), t_2(j)) = B(i-1, j+b)$$
$$B(i,j) = B(t_2(i), t_3(j)) = B(i+b, j+c)$$
$$B(i,j) = B(t_3(i), t_1(j)) = B(i+c, j-1)$$

These conditions lead to the existence of two integers h and k such that, which yields to

$-1 = kb = hc$ modulo p.

$$b = kc = -h$$

Then we get

$$k = c \quad , \quad hk = 1 \quad , \quad h^3 = 1 \quad , \quad k^3 = 1$$

and $b^3 = -1 = c^3$

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Therefore b and c are the cubic roots of -1
Other than -1 (This existence by lemma1)
We have :
$t^{-1} = (0,1,2,\dots,p-1)\ (p,pth,p+2h,\dots)(2p,2p+1),\dots)$
We want to prove that
$\forall\, i\,,j : (i-1\,,j+c) = (i+kb\,,j-k) = (i+hc,j+hb)$
Then $-1 = kb = hc$
$c = -k = hb$ These relations are verified before.
We detine B and C as follows.
$B\ (i,j)$ indicates to the element of B lie on the line i, calumn j)
$\forall\ i,j$ mod p .
We have $B(o,o) = 1$ and $B(o,j) = o$ for $j \neq o$
B $(i,j) = B\ (i-1,j+b)$.
and $C\ (i,j) = 1 - B\ (i,j)$.
and t and s are two automorphismes of T we shall prove that T in point-symmetric tournament

let $i \in E$,

o transform in $i \in E_1$ by t^{-i}
o transform in $i+p \in E_2$ by $s\,t^{-1}$
o transform in $i+2p \in E_3$ by $s^2\,t^{-1}$

The Tournament then point-symmetric

2. The Tournament T in not rotation
let $< s,t >$ be the group generate by s and t and Aut (T) is the automorphisme group of T .
We can prove:
A) *$< s,t > = Aut\ (T)$.*
B) *The number of elements of $< s\,,t\ > = 3p$.*
C) *The group $< s\,,t\ >$ is not cyclic*

Suppose that there exists a permutation of E which in an automorphisme of T . let a be regarded as automorphisme . Then a is of order $3P$ because T is a point symmetric tournament according to A

we have $a \in < s\,,t > \ni\ < a > \subset < s\,,t\ >$

but $< a >$ has $3P$ elements by (B)

we have $< s\,,t > = < a >$ then.

$< s\,,t >$ is cyclic Contradiction by (c)

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we prove first:

Lemma3
The group $< s\,, t\ >$ is not cyclic [4] .

Proof:
we have $st(o) = s(p-1) = 2p-1$
$ts(o) = t\ (p) = p + b$.

if $st\ (o) = ts\ (o)\,.Then$.
$$2p - 1 = p + b\ and\ b = -1\ \ mod\ p.$$
Which contradicts to definition of b
Therefore $st(0) \neq ts(0)\ then\ < s, t)$ is not commutative so $< st >$ is not cyclic
Lemma 4
The group $< s, t >$ verify the relation
$$sts^{-1} = t^{-c}$$

Proof:
$b\ c$ are the cubic roots not trivial $\neq -1$
We have $1 - b + b^2 = 0$ and $1 - c + c^2 = 0$
$$b^2 = -c\quad , and\quad c^2 = -b$$
and $1 = b + c$

We shall verify the relation.

1) $sts^{-1}\ (x) = t^{-c}(x)$. Is true for $x = 0$

Calculate $sts^{-1}\ (0)$
$$s^{-1}\ (0) = 2p$$
$$t(2p) = 2p + c$$
$$s(2p + c) = c$$
On the other Side $n = t^n(0) = -n$ (mod p)
and $t^{-c}\ (0) = c$.
and (1) is true for $X = 0$
We prove if (1) is true for x then
(1) Is true for $s(x)$.
We have .
$$Sts^{-1}\ (s\ (x)) = st(x)$$
$st(x) = t^{-c}\ (s(x))$ by hypothesis

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Then $sts^{-1}\big(s(x)\big) = t^{-c}(s(x))$
We verify the relation (1) $\forall\, x\ of\ the\ from\quad t^{i}\ (0)$.
Let i be element culculate

$$sts^{-1}\left(t^{i}(0)\right)$$

$$t^{i}(0) = -i$$

$$s^{-1}(-i) = 2p - i$$

$$t(2p - i) = 2p - i + c$$

$$s(2p - i + c) = c - i$$

On the other hand :

$$t^{-c}\left(t^{i}(0)\right) = t^{i-c}(0) = c - i$$

Then $$sts^{-1} = t^{-c}$$

Corollary :
The cardinal of $< s, t >$ is equal to $3P$.

Proof:
Each element of $< s, t >$ is product of the power of s and t.
By lemma 4 we can put each product $St\ \ by\ \ t^{-c}s$
By repeating these steps we can put each element of $< s, t >$ in the form $t^{i}\, s^{j}$

$$i.e\quad t^{p} = identity = s^{3}$$

Then the cardinal of $< s, t > = 3p$

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Conflict of Interests.
There are non-conflicts of interest .

4. Reference

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الخلاصة

في هذا البحث ، قمت بتصميم مثال مضاد وجه B.Alspach لعلاقة دورية ذات 21 عنصر . وتم التعميم لعناصر العلاقة الجديدة التي يكون عدد عناصرها 3P حيث P عدد اولي والتي تكون متماثلة الرؤوس ولكن ليست دائرية.