Fully Dual-Stable S-system

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Abstract

An S-system M is fully stable if α (N) ⊂ N for each subsystem N and S –homomorphism α of N into M . In this paper we study the dual concept of full stability. Duo property of an S- system being a necessary condition for both full stability and full dual stability, and quasi-projectivity is sufficient condition for duo to be fully dual stable system. Several properties and characterizations of full dual stability are investigated.

Keywords. dual-stable subsystem, fully dual-stable S- system, duo S-system, quasi projective S-system, Hopfian and Co-Hopfian S-system.

الخلاصة

يقال للنظام M , انه تام الاستقرارية اذا α(N) ⊂ N لكل نظام جزئي N ولكل تشاكل α من N إلى M . في هذا البحث ندرس مفهوم الرديف لتمام الاستقرارية . الصفة الثنائية للنظام كونه شرطا ضروريا لكل من تمام الاستقرارية وتمام رديف الاستقرارية , وشبه الاسقاطي هو شرط كافي للثنائي ليكون نظام تام رديف الاستقرارية. لقد تم دراسة عدد من الصفات والتوصيفات تام رديف الاستقرارية.

الكلمات ألمفتاحيه نظام جزئي لرديف الاستقرارية, لنظام تام رديف الاستقرارية , النظام الثنائي, نظام شبه الاسقاطي, هوبفن(Hopfian) ونظام رديف هوبفن(Co–Hopfian).

1.Introduction and Preliminaries

The notion of full stability and full dual stability were studied on modules by (Abbas, 1990; Abbas and Al-Hosainy, 2012).

Most of the modules notions, were reversed to S-system (S-acts), and interesting results were obtained. The notions of full stability on S-system, were studied, recently, by Abbas and Baanoon (Abbas and Baanoon, 2015). In this paper, the notion of full dual-stability on S-system, is investigated.

A subsystem B_S of an S-system A_S is said to be dual stable if $B_S \times B_S \subseteq \ker \alpha$, for each S-homomorphism $\alpha : A_S \to A_S/B_S$. The S-system A_S is said to be fully dual stable (shortly, fully d-stable) if each subsystem of A_S is dual-stable. An S-system A_S is said to be strongly dual stable if $\ker g \subseteq \ker f$ whenever f and g are S-homomorphism of A_S into B_S and g is surjective, where B_S is any S-system.

The above two conditions are equivalent in the case of modules, but in S-system, the second condition implies the first. This difference occurred because of the fact that in modules, the kernel of a homomorphism is a submodule, while the kernel of S-homomorphism does not need to be induced by a subsystem.

In this section, some preliminaries about S-system and related concepts were given. For more information about S-system (s-act) see (Kilp and Mikhlev, 2000).

In section 2, the main results about full d-stability and the related concepts(duo, multiplication, quasi-projective) were given. More results about full d-stability and quasi-projectivity discussed in section 3.

(1.1) Definition(Kilp and Mikhalev, 2000).Let S be a monoid and A is a nonempty set. If we have a mapping **,** $\mu : A \times S \to A$**,** $(a, s) \longmapsto as := \mu(a, s)$ such that:

a) a1= a and

b) a(st) = (as)t for a ∈ A, s,t ∈ S,

we call A a right S –system or right system over S and write it as A_S. More informally we often say that μ defines a right multiplication of element from A by element of S. Analogously, we define a left S-system A and write $_SA$.

(1.2) Definition (Abbas and Dahash, 2014). A subsystem N of S-system M_S is called fully invariant if $\alpha(N) \subseteq N$ for each S-endomorphism α of M_S, M_S is duo S-system if each subsystem of M_S is fully invariant.

The following lemma is a part of lemma (1.3)(Roueentan and Ershad, 2012), for completeness we give a proof for it.

(1.3) Lemma: An S-system M_S is duo iff for each endomorphism f of M and each element m, there exists r ∈ S (depending on m) such that, f (m) = mr.

Proof: (⟹) Assume M is a duo S-system , f ∈ End_S M and m ∈ M. Then mS is a subsystem of M_S and m ∈ mS.

Since M is duo, we have f(mS) ⊆ mS , hence f (m) ∈ mS, that is ∃ r ∈ S such that f (m) = mr.

(⟸) It is clear.

Recall that U is said to be a generating set of A_S if for all a ∈ A, a = us for some u ∈ U and s ∈ S.

(1.4) Definition(Kilp and Mikhalev, 2000)**:** **A** set U of generating elements of a right S-system A_S is said to be a **basis** of A_S if every element a ∈ A_S can be uniquely presented in the form a = us, u ∈ U, s ∈ S, if $a = u_1s_1 = u_2s_2$, then $u_1 = u_2$ and $s_1 = s_2$. If an S-system A_S has a basis U, then it is called a **free S-system** or, more precisely, a |U|-free S-system. In particular, S_S is 1-free with basis{1}. Also, we say that A_S is of rank |U|.

(1.5) Definition (Kilp and Mikhalev, 2000): Let M_S be an S-system. An equivalence relation ρ on M is called an **S-system congruence** or a **congruence** on M_S, if (m,n) ∈ ρ implies (ms,ns) ∈ ρ for m,n ∈ M_S, s ∈ S. If S is a monoid then any right(semigroup) congruence ρ on S is an S- system congruence on S_S.

(1.6)Definition(Kilp and Mikhalev, 2000): Any subsystem $B_S \subseteq A_S$ defines the **Rees congruence** ρ_B on A, by setting $(a, \acute{a}) \in \rho_B$ if $a, \acute{a} \in B$ or $a = \acute{a}$.

(1.7) Definition(Kilp and Mikhalev, 2000): Let A_S be a right S-system. An element θ ∈ A_S is called a **fixed element** of A_S if $\theta s = \theta$ for all s ∈ S. If A_S has a unique fixed element θ, then θ is called **zero element** of A_S, we will denote the zero element of A_S by $\mathcal{O}$.

(1.8)Definition(Kilp and Mikhalev, 2000): We call an S-system A_S decomposable if there exist two subsystems B_S,$C_S \subseteq A_S$ such that $A_S = B_S \cup C_S$ and $B_S \cap C_S = \emptyset$. In this case $A_S = B_S \cup C_S$ is called a decomposition of A_S . Otherwise A_S is called **indecomposable**.

If we consider S-system with zero θ, then we have to replace $\emptyset$ by { θ} to define decomposable and indecomposable S-system with zero.

(1.9)<u>Definition</u> (Kilp and Mikhalev, 2000): An S-system A_S is called **torsion free** if for any x, y ∈ A_S and right cancellable element c ∈ S the equality xc = yc ⟹ x = y.

(1.10) <u>Definition</u>(Kilp and Mikhalev, 2000): An S-system A_S is called quasi-projective if for any epimorphism $\pi : A_S \rightarrow B_S$ and homomorphism $\alpha : A_S \rightarrow B_S$ there exists an endomorphism f of A_S such that $\pi f = \alpha$.
Note it is clear that if M_S is quasi-projective then for all N subsystem M_S and for all $\alpha : M \rightarrow M / N$ there exists f ∈ End M such that $\pi_N \circ f = \alpha$ where π_N is the natural epimorphism of M onto M / N.

(1.11)<u>Definition</u> (Kilp and Mikhalev, 2000): Let A_S and B_S be two S-systems. Consider an S-homomorphism $f : A_S \longrightarrow B_S$. Then f is called a **retraction** if f is right invertible, i.e. there exists g ∈ $Hom_S(B,A)$ with f g = id_B; B is called a retract of A.

2.Fully dual stable S-system

We start by introducing the dual concept of fully stable S-system.

(**2.1**)**<u>Definition:</u>** Let M be an S-system and N is a subsystem of M. N is said to be d-stable subsystem of M if, N× N ⊆ ker α, for all $\alpha : M \longrightarrow M / N$. M is said to be fully d-stable if, any subsystem of M is d-stable.

(2.2)<u>Remark :</u> If S is a monoid, then S is fully d-stable if S_S is fully d-stable.

(**2.3**)**<u>Lemma:</u>** If $f : A_S \longrightarrow B_S$ is homomorphism, $\varphi : B_S \longrightarrow C_S$ is an isomorphism then ker $\varphi \circ f$ = ker f.
Proof: Is clear.

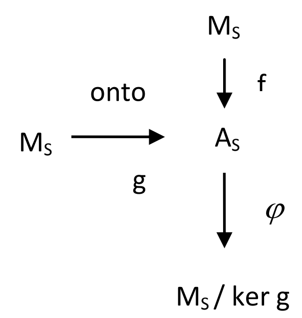
(2.4)<u>Proposition:</u> If M_S is fully d-stable S-system, then M / N is fully d-stable for any subsystem N_S of M_S.
Proof: Let K / N be a subsystem of M / N and $\alpha : M/N \longrightarrow (M/N)/(K/N)$ be a homomorphism, consider the composition

$$M \xrightarrow{\pi_N} M/N \xrightarrow{\alpha} (M/N)/(K/N) \overset{\varphi}{\cong} M/K$$

since M is fully d-stable, it follows, K × K ⊆ ker $\varphi\alpha\pi_N$ = ker $\alpha\pi_N$ (lemma 2.3). Now, if $([k_1]_N, [k_2]_N) \in K/N \times K/N$, then $(k_1, k_2) \in K \times K$, hence $(\alpha\pi_N)(k_1) = (\alpha\pi_N)(k_2)$, that is $\alpha(\pi_N(k_1)) = \alpha(\pi_N(k_2))$. Then $\alpha([k_1]_N) = \alpha([k_2]_N)$, that is $([k_1]_N, [k_2]_N) \in$ ker α. Therefore K / N × K / N ⊆ ker α, and M / N is fully d-stable.

(**2.5**)**<u>Theorem:</u>** Let M be an S-system. The following two statements are equivalent.

1. For each congruence ρ on M and for each S-homomorphism $\alpha : M \longrightarrow M/\rho$, $\rho \subseteq \ker\alpha$ holds.
2. For any S-system A_S, and for each two homomorphisms $f, g : M_S \longrightarrow A_S$, with g onto, ker g ⊆ ker f holds.

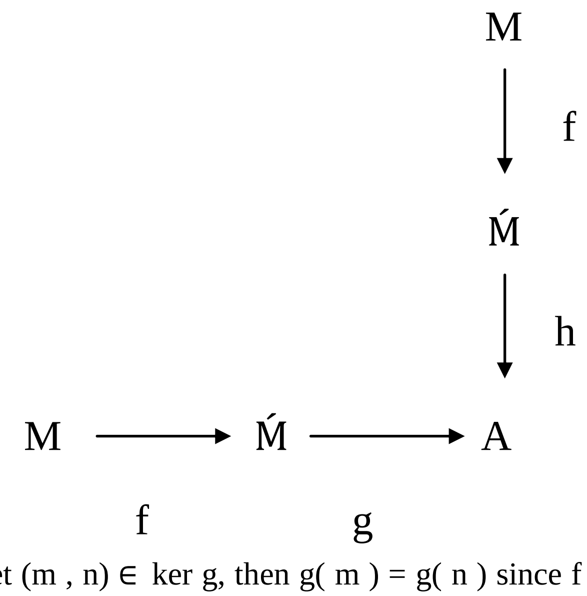


Proof:- (1) $\Longrightarrow$(2)
Since $g : M_S \longrightarrow A_S$ is onto hence $A_S \cong M_S / \ker g$, let $\varphi : A_S \longrightarrow M_S / \ker g$ be an isomorphism therefore $\varphi \circ f : M_S \longrightarrow M_S / \ker g$. By hypothesis,
$\ker g \subseteq \ker \varphi \circ f = \ker f$.(since φ is one to one).
(2) $\Longrightarrow$(1)
ρ congruence on M_S, $\alpha : M_S \longrightarrow M_S /\rho$, let $\pi : M_S \longrightarrow M_S /\rho$ natural epimorphism $\pi(m) = [m]_\rho$, then $\ker \pi = \rho$, ($[m_1]_\rho = [m_2]_\rho \Leftrightarrow (m_1, m_2) \in \rho$). By(2) $\ker \pi \subseteq \ker\alpha$, therefore $\rho \subseteq \ker \alpha$.

(2.6)<u>Definition:</u> An S- system M_S is said to be strongly d-stable S-system if it satisfies any of the two equivalent conditions of Theorem (2.5).

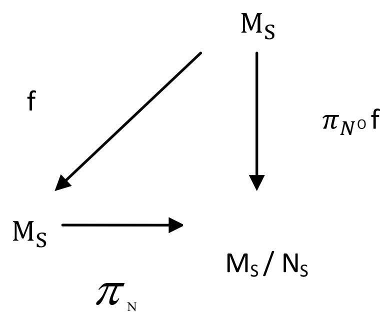
(2.7) <u>Remark:</u> Any strongly d- stable S-system is fully d-stable.
Proof: Let N_S be a subsystem of M_S and α: $M_S \longrightarrow M_S/ N_S$, then ρ_N is a congruence on M and $N\times N \subseteq \rho_N$ (where ρ_N is the Rees congruence on M). Since M_S is strongly fully d-stable then $\rho_N \subseteq \ker \alpha$, and hence $N\times N \subseteq \ker \alpha$. ■

(**2.8) <u>Proposition</u>:** A homomorphic image of a strongly d-stable S-system is strongly d-stable S-system too.
Proof: Assume that $f : M \longrightarrow \acute{M}$ is an epimorphism. Let g , h: $\acute{M} \longrightarrow A$ be two S-homomorphisms with g surjective, then $g \circ f$, $h \circ f : M \longrightarrow A$ are S-homomorphisms , we have $\ker g \circ f \subseteq \ker h \circ f$, (M is strongly d-stable, $g \circ f$ onto).



To prove $\ker g \subseteq \ker h$, let (m , n) $\in$ ker g, then g(m) = g(n) since f is surjective we have m = f(x) , n = f (y) for some x,y $\in$ M , hence g(f(x)) = g(f(y)) ,that is (x,y) $\in \ker g \circ f \subseteq \ker h \circ f$, hence $(x,y) \in \ker (h \circ f) \Longrightarrow h(f(x)) = h(f(y))$,
$h(m) = h(n) \Longrightarrow (m,n) \in \ker h$. Therefore, $\ker g \subseteq \ker h$, which implies $\acute{M}$ is strongly fully d-stable. ■

(**2.9**) **<u>Proposition:</u>** Let M be an S-system, with the property, that either it has no zero element, or a unique zero element which is contained in any subsystem of M.
If M is fully (strongly) d-stable then it is duo.
Proof: By Remark (2.7), it is enough to prove the case (fully d-stable).



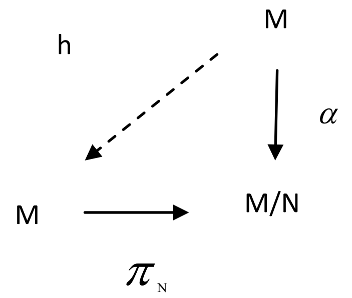
Let M_S be a fully d-stable S-system, f an endomorphism of M_S, and N_S a subsystem of M_S. Then $\pi_N \circ f : M_S \longrightarrow M_S / N_S$ is a homomorphism, which implies $N \times N \subseteq \ker \pi_N \circ f$.(where π_N is the natural epimorphism). Let $x \in N$, we have two cases :

(1) For all $y \in N$, $f(x) = f(y)$, then f(N) is a one –element subsystem, and hence f(x) is a fixed element of M which must be unique and contained in any subsystem of M, (by hypothesis), that is, $f(x) \in N$.

(2) There exists $y \in N$,with $f(x) \neq f(y)$, then $(x, y) \in \ker \pi_N \circ f$ implies $\pi_N(f(x)) = \pi_N(f(y))$, then $f(x), f(y) \in N$, therefore $f(x) \in N$ then, in the two cases $f(x) \in N$ $\forall$ $x \in N$, that is $f(N) \subseteq N$. So, M_S is duo. ■

(**2.10**) **Proposition:** Every quasi- projective duo S-system is fully d-sable.

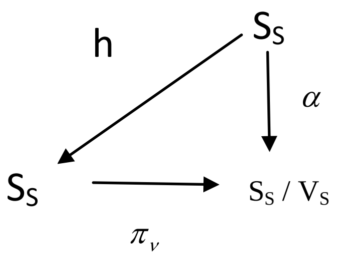
Proof:



Let N be a subsystem of M, $\alpha : M \rightarrow M / N$ is a homomorphism, since M is quasi-projective there exists an endomorphism h of M such that $\pi_N h = \alpha$ where π_N is the natural epimorphism of M onto M / N. Let $x, y \in N$ then $h(x) = xs$, $h(y) = yt$ for some $s,t \in S$(since M is duo) that is $(h(x), h(y))$ in $N \times N$ hence $\alpha(x) = \pi(h(x)) = \pi(xs) = \pi(yt) = \pi(h(y) = \alpha(y)$ therefore $N \times N \subseteq \ker \alpha$. ■

(**2.11**) **Corollary:** Any duo monoid is fully d-stable.

Proof:



Let V_S be a subsystem of S_S and $\alpha : S_S \longrightarrow S_S/V_S$ be a homomorphism, with π_V surjective.
Assume that $\alpha(1) = [t]_V$, for some t ∈ S. Define h by h(s) = ts, ∀ s ∈ S ,
$\pi_V(h(s)) = \pi_V(ts) = \pi_V(t)\, s = [t]_V s = \alpha(1)s = \alpha(s)$ that is $\pi h = \alpha$, therefore S_S is quasi-projective. Since S_S is duo, then by (proposition 2.10), S_S is fully d-stable. ∎

(**2.12**) **<u>Theorem:</u>** Let M be a fully d-stable S-system and N is a subsystem of M such that $\alpha : M \longrightarrow M/N$ is any S-homomorphism. Then for each m ∈ M there exists r ∈ S such that $\alpha(m) = [m]_N r$ (r depends on α and m).

Proof: Let M_S be a fully d-stable S-system, and N a subsystem of M_S, define $\lambda : M/N \longrightarrow M/N$ by $\lambda([m]_N) = \alpha(m)$. For all $m_1, m_2 \in M$, $[m_1] = [m_2] \Longrightarrow m_1 = m_2$ or $m_1, m_2 \in N$(since $N{\times}N \subseteq \ker\alpha$), then $\alpha(m_1) = \alpha(m_2)$, hence λ is well defined. It is clear that λ is an S-homomorphism. Since M/N is fully d-stable, it is duo (proposition 2.9). By(lemma 1.3) . $\lambda([m]_N) = [m]r \Longrightarrow \alpha(m) = [m]r$, for some r ∈ S. ∎

(**2.13**) **<u>Corollary:</u>** An S-system M is fully d-stable if and only if for each subsystem N each S-homomorphism $\alpha : M \longrightarrow M/N$ has the property that for each m ∈ M, there exists r ∈ S such that $\alpha(m) = [m]_N r$ (r depends on α and m).
Proof: (⟹) By (Theorem 2.12).
(⟸) Assume that for each m ∈ M, there exists r ∈ S such that $\alpha(m) = [m]_N r$, let m,n ∈ N, then $\alpha(m) = [m]_N r = [mr]_N = \{N\}$ also $\alpha(n) = [n]_N s = [ns]_N = \{N\}$ for some r,s ∈ S (mr ,ns∈ N), so $\alpha(m) = \alpha(n)$ then (m,n) ∈ ker α that is $N{\times}N \subseteq \ker\alpha$. ∎

(**2.14**) **<u>Proposition:</u>** If S is a monoid , then any free S-system with rank greater than one, is not fully d- stable.
Proof : Let A_S is free S-system with rank more than one.
Let $\{x_1,x_2\}$ be a subset of some basis X of A_S with ($x_1 \neq x_2$).
Define $f : X \longrightarrow S_S$ by

$$f(x) = \begin{cases} x & \text{if } x \notin \{x_1,x_2\} \\ x_2 & \text{if } x = x_1 \\ x_1 & \text{if } x = x_2 \end{cases}$$

since A_S is free f can be extended to an endomorphism $\bar{f}$ of A_S.
Note that $\bar{f}(x_1S) = x_2S \nsubseteq x_1S$, that is, A_S is not duo, so by(proposition 2.9), A_S cannot be fully d-stable. ∎

(**2.15**) **P<u>roposition:</u>** Let S be a commutative monoid all of its elements satisfy left cancellation . If A_S is a duo torsion free and indecomposable S-system then for all f ∈ End A_S, there exists r ∈ S such that f(a) = ar for all a ∈ A. (r depends only on f)
Proof: Assume A_S is duo torsion free and $f : A_S \longrightarrow A_S$, then for all a ∈ A there exists s ∈ S such that f(a) = as. Assume a, b are distinct elements of A_S and (s, r ∈ S), f(a) = as and f(b) = br. To prove s = r, aS ∩ bS ≠ ∅ , ∃ u , v ∈ S such that au = bv, f(au) = asu, f(bv) = brv ⟹ asu = brv ⟹ aus = bvr (S is commutative) ⟹ s = r. (since

A_S is torsion free) therefore for all f : $A_S \longrightarrow A_S$, there exists r(depending on f only) such that f(a) = ar for all a ∈ A_S.

(2.16) Corollary: Let S be a commutative monoid all of its elements satisfy left cancellation. If A_S is a duo torsion free and indecomposable S-system, then End $A_S \cong$ S (as monoids).

Proof: α : End $A_S \longrightarrow$ S, for all f ∈ End A_S, there exists unique r such that f(a) = ar for all a∈ A by Proposition(2.15), define α(f) = r. Now, if f, g ∈ End A_S, α(f) = r, α(g) = s, then (gf)(a) = g(f(a)) = g(ar) = (ar)s = a(rs) ⟹ α(gf) = rs, ⟹ α(gf) = α(g) α(f), hence α is a monoid homomorphism. α is onto, r ∈ S, let f(a) = ar, f ∈ End A_S. If α(f) = α(g) = r ⟹ f(a) = ar ∀ a ∈ A, g(a) = ar ∀ a ∈ A ⟹ f = g that is α is one to one, therefore α is an isomorphism.

(2.17)Proposition: Let S be a commutative monoid all its elements satisfy left cancellation. If M is fully d-stable S-system and N is a subsystem of M such that M / N is torsion free. Then for each homomorphism $\alpha : M \longrightarrow M/N$ there is r ∈ S such that α(m) = $[m]_N$r. for all m ∈ M.

Proof: Recall, to the proof of Theorem (2.12), λ is an S-endomorphism of the torsion-free duo S-system M / N. Then by proposition (2.16) there exists r ∈ S such that λ([m]) = $[m]_N$r for all $[m]_N$ ∈ M / N. Then α(m) = λ([m]) = $[m]_N$r.

The concepts of Hopfian and Co-Hopfian, where discussed for modules, see (Ozcan,ital..,2006). These concepts can be defined analogously in systems.

(2.18) Definition: Let M be an S-system.
M is called Hopfian (Co- Hopfian) if every surjective (injective) endomorphism of M is an isomorphism.

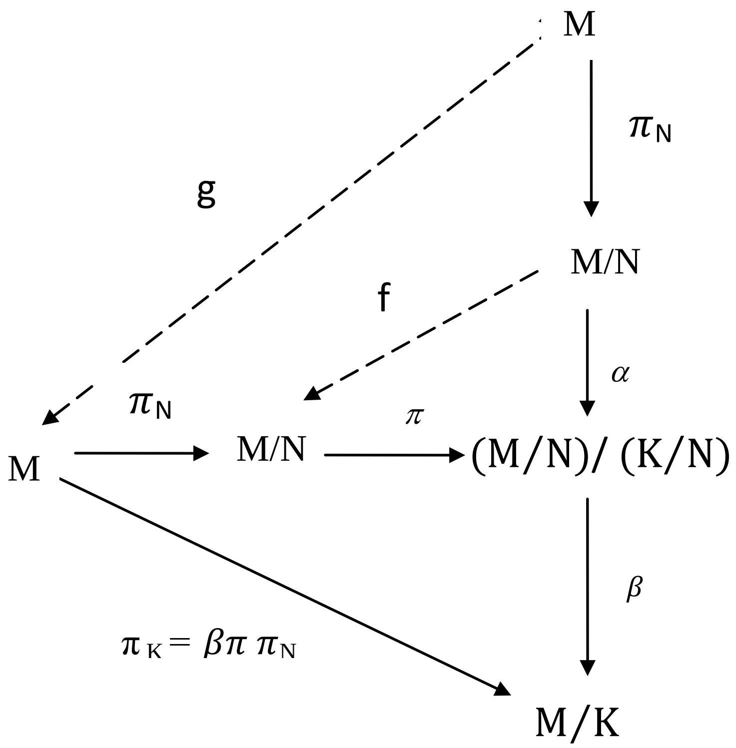
(2.19) Proposition: Every fully d-stable S-system is Hopfian.

Proof: Let M be a fully d-stable S-system. f : $M_S \longrightarrow M_S$ surjective. then $M_S \cong M_S$ / ker f, let α : $M_S \longrightarrow M_S$ / ker f be an isomorphism, hence ker α = Δ_M since M_S is fully d-stable, that is ker f ⊆ kerα = Δ_M that is ker f = Δ_M, (where Δ_M = {(x, x) | x ∈ X }), therefore f is an isomorphism.

(2.20) Example: Let S = (N, .),then N_S is a fully d-stable S- system (Remark 2.2), it is not Co-Hopfian since f : $N_N \rightarrow N_N$, f(n) = 2 n is an injective homomorphism, which is not isomorphism.

3. Dual stability and quasi-projective S-system

(3.1) Proposition: If N is fully invariant subsystem of a quasi-projective S-system M then M / N is likewise quasi-projective.



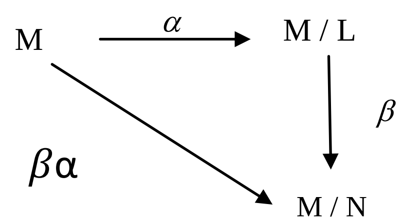
Proof: Let K/N be a subsystem of M/N and α : M/N ⟶ (M/N)/(K/N).
Let β : (M/N)/(K/N) ⟶ M/K be an isomorphism. Then ∃ g : M⟶ M such that
$\pi_K \circ g = \beta \circ \alpha \circ \pi_N \implies (\beta\,\pi\,\pi_N) \circ g = \beta \circ \alpha \circ \pi_N \implies \pi\,\pi_N \circ g = \alpha \circ \pi_N$ ….(1)
Where, (π_K :M ⟶ M/K, π_N :M ⟶ M/N , π : M/N ⟶ (M/N)/(K/N)) .
Define f: M/ N ⟶ M/ N by f($[m]_N$) = $[g(m)]_N$. Note that $[m_1]_N = [m_2]_N$, implies $m_1 = m_2$ or $m_1, m_2 \in$ N, hence g (m_1) =g (m_2) or g (m_1), g (m_2) ∈ N (N fully invariant), and so $[g(m_1)]_N = [g(m_2)]_N$, Therefore, f is well defined, f (π_N (m)) = π_N (g (m)) for From (1) and (2) ⟹ π f ∘ π_N = α∘π_N for each m ∈ M, π (f (π_N(m)) = α ((π_N(m)) ⟹ π(f $[m]_N$) = π ($[m]_N$) for each $[m]_N$ ∈ M/N ⟹ $\pi \circ f = \alpha$. ■

(3.2) Proposition (Kilp and Mikhalev, 2000): An S-system B_S is a retract of an S-system A_S if and only if there exists a subsystem $\acute{A}_S$ of A_S and an epimorphism h : A_S ⟶ $\acute{A}_S$ such that $B_S \cong \acute{A}_S$ and h($\acute{a}$) = $\acute{a}$ for every $\acute{a} \in \acute{A}_S$.

(3.3) Lemma: If N, L are two subsystems of an S-system M and if L is a subsystem of N, then there exists an epimorphism β : M / L → M /N with (N/L×N/L) ∪ $\Delta_{M/L}$ = ker β .
Proof: Let β: M / L→ M / N be defined by $[m]_L \longmapsto [m]_N$, β is well defined, since ($[m_1]_L = [m_2]_L$ hence $m_1, m_2 \in$ L (since L subsystem of N) then $m_1, m_2 \in$ N therefore $[m_1]_N = [m_2]_N$). It is clear that β is a homomorphism.

ker β = {($[m_1]_L$, $[m_2]_L$) | β($[m_1]_L$ = β($[m_2]_L$)}
= {($[m_1]_L$, $[m_2]_L$) | $[m_1]_N = [m_2]_N$)}
= {($[m_1]_L$, $[m_2]_L$) | $m_1 = m_2$ or $m_1, m_2 \in$ N}
= (N / L× N / L) ∪ $\Delta_{M/L}$.

(**3.4**) **Proposition:** Let N be a d-stable retract of an S-system M and L is a subsystem of N, then L is d-stable in N if and only if L is d-stable in M.



Proof: (⟹) Let α **:** $M \longrightarrow M/L$ be homomorphism and $\beta : M/L \longrightarrow M/N$ be as in Lemma(3.3).
Note that $ker\ \beta = (N/L \times N/L) \cup \Delta_{M/L}$. If for all x, y ∈ N, α(x) = α(y), then $L\times L \subseteq N\times N \subseteq \ker\alpha$. Now, let x ∈ N, and there exists y ∈ N such that
α(x) ≠ α(y) since $\beta\alpha : M \longrightarrow M/N$, N is d-stable in M and $(x, y) \in N \times N \subseteq \ker\beta\alpha$, we have $\beta(\alpha(x)) = \beta(\alpha(y))$, so $(\alpha(x), \alpha(y)) \in \ker\beta$, this implies $\alpha(x) \in N/L$ for all x ∈ N, that is $\delta = \alpha|_N : N \longrightarrow N/L$, by d-stability of L in N, $L\times L \subseteq \ker\delta = (\ker\alpha) \cap (N \times N)$, therefore $L\times L \subseteq \ker\alpha$. So, L is d-stable in M.
(⟸) Let L be a d-stable in M and $\alpha : N \longrightarrow N/L$ is an S-homomorphism, let β be a homomorphism of M onto N, such that $\beta|_N = i_N$. (since N is a retract of M), by Definition(1.11). Then $\alpha \circ \beta : M \longrightarrow M/L$, hence $L \times L \subseteq \ker\alpha\beta$

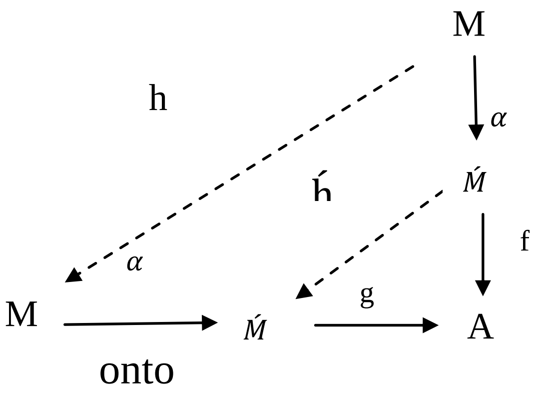
$$= \{(x, y) \in M \times M | (\alpha\beta)(x) = (\alpha\beta)(y)\}$$
$$= \{(x, y) \in M \times M | \alpha(\beta(x)) = \alpha(\beta(y))\}$$

$(x, y) \in L \times L \Longrightarrow x, y \in N \Longrightarrow \beta(x) = x$ and $\beta(y) = y \Longrightarrow \alpha(x) = \alpha(y) \Longrightarrow (x, y) \in \ker\alpha$. ■

Note that the retract property in Proposition (3.2) used only in the sufficient condition. So the following corollary clarifies the transitivity of d-stability.

(**3.5**)**Corollary:** Let A, N and K be subsystems of a system M with $A \subset N \subset K$. If A is d-stable in N and N is d-stable in K, then A is d-stable in K. ■

(3.6)Corollary: A homomorphic image of a strongly d-stable quasi-projective S-system is likewise quasi-projective. .
Proof: Let M be a strongly d-stable quasi-projective S-system, $\alpha : M \rightarrow \acute{M}$ is an S-epimorphism.



First, note that if h ∈ End M, then α and αh are two homomorphisms from M into Ḿ, with α onto, by strong d-stability we have ker α ⊆ ker αh, that is, α(m) = α(ḿ) implies α(h(m)) = α(h(ḿ)**....(*)**

Assume that f , g : Ḿ ⟶ A be homomorphisms, with g onto, then gα, fα are homomorphisms from M into A with gα onto, since M is quasi-projective, there exists h : M ⟶ M such that gαh = f α. Define h́ : Ḿ ⟶ Ḿ by, h́(x) = α(h(m)), where m ∈ M

with x = α(m) (α is onto), h́ is well defined by(*), also gh́α = gαh = f α, but α is onto implies gh́= f. therefore Ḿ is quasi-projective.

References

Abbas M.S. and Baanoon Hiba R.,(2015). On Fully Stable Acts, Italian Journal of Pure and Applied Mathimatics , to appear.

Abbas Mehdi S. and Dahash Angham A., (2014). Semi-Injective Systems, International Journal of Algebre, Vol. 8, no. 10, 459 – 470 .

Abbas M.S., (1990). On Fully Stable Modules, Ph.D. Thesis, University of Baghdad. Iraq.

Abbas M.S. and Al-Hosainy A.M.A., (2012). Fully Dual-Stable Modules, Archives Des Sciences Journal, Vol. 65, No. 12; Dec. 643-651.

Kilp M.U Knauer, and Mikhalev, A.V., (2000). Monoids, Acts and Categories, Walter de Gruyter, Berlin, New York.

Ozcan A.C., Harmanci A. and Smith P.F.(2006). Duo Modules, Glasgow, Math. J. 48.p. 533-545.

Roueentan, Mohammad and Ershad, Majid,(2014).Strongly Duo and Duo Right S-act, Italian Journal of pure and Applied Mathematics-N.32,(134-154).