An Approach for Constructing Liapounv Functions of Differential – Algebraic Equations

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1 - Abstract

In this paper, we present a new approach for constructing Liapunov functions[Rao, 1980] for differential – algebraic equations depend on Krasovaskii's theorem, in this approach and modification of finding Liapunov functions theories with illustrative examples

Keywords: Implicit Function theorm , Krasovaskii's method , differential algebraic equation **.**

الخلاصة

في هذا البحث قدمنا طريقة جديدة لإيجاد دوال ليبانوف للمعادلات التفاضلية الجبرية وهذه الطريقة معتمدة على مبرهنة كروسوفسكي. في هذه الطريقة والتعديل لايجاد دوال ليبانوف قدمنا اساس نظري مع امثلة .

الكلمات المفتاحية : نظرية الدالة الضمنية ، طريقة كرسوفسكي ، المعادلة التفاضلية الجبرية .

2 - Introduction

The field of differential equation goes back to the time of Newton and Leibniz , who invented calculus because they realties that many of the laws of nature are governed by what we now call differential equation [Robert *et al.*, 2011].

The dynamic behavior of physical processes is usually modeled via differential equation. But if the states of physical system are in some ways constrained, like for example by conservation laws such as Kirchhoff's laws in electrical networks or by position constraints such as movement of mass points on surface then the mathematical model also contains algebraic equations to describe these constrains.

Such systems, consisting of both differential and algebraic equation called differential algebraic equation [Xuefeng Song, 2003].

3–Construction of Liapunov Functions for Differential Algebraic Equations

In this section , we will introduce a new modification for Krasovaskii's method for constructing Liapuniv functions of differential algebraic equations .

3–1 : Krasovaskii's Method [Ogata, 1967]:

Consider the non – linear system $x^{\circ} = f\ (x)$, where x is an n–dimensional vector (called the state vector) , and f(x) is a vector whose elements are non-linear functions of X_1 , X_2, ..., Xn

The Jacobian matrix for this system is given by:

$$J = \left[\frac{\partial\,(f_1, f_2, \dots, f_n)}{\partial\,(x_1, x_2, \dots\dots, x_n)}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \dots\dots & \frac{\partial f_n}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \dots\dots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} \dots\dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

In this non-linear system, there may be more than one equilibrium state space, but however, it is possible to transfer the equilibrium state under consideration to the origin of the state space by an appropriate transformation of coordinates . We shall , therefore , consider the equilibrium state vector consideration at the origin , we shall now presume Krasovskii's theorem .

Theorem (3. 1) [Ogata, 1967]:

Consider the system x^0= f(x), and define $J^{\wedge} = J^T + J$, where J is the Jacobean matrix of f , and J^T is the transport of J .

If J^ is negative definite then the zero solution is asymptotically stable.

A Liapunov function for this system is

$V(x) = f^T(x) f(x) = f_1^2 + f_2^2 + \cdots + f_n^2 = \sum_{i=1}^{n} f_1^2$

Example (3.1)

In order to study the stability of the zero solution of the system

$\overset{\circ}{x_1} = -x_1 - x_2 - x_1^3 = f_1$

$\overset{\circ}{x_2} = -x_1 - x_2 - x_1^3 = f_2$

Using the Krasovskii's method, define

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1-3\,x_1^2 & -1 \\ 1 & -1-3\,x_2^2 \end{bmatrix}$$

And therefore:

$$J^T = \begin{bmatrix} -1-3\,x_1^2 & 1 \\ -1 & -1-3\,x_2^2 \end{bmatrix}$$

And since $J^{\wedge} = J^T + J$, hence we have:

$$J^{\wedge} = \begin{bmatrix} -2-6\,x_1^2 & 0 \\ 0 & -2-6\,x_2^2 \end{bmatrix}$$

Considering the principal minors of the above last matrix of J^ one obtain that :

$\Delta_1 = -2 - 6\,x_1^2 < 0$ and

$$\Delta_2 = \begin{bmatrix} -2-6\,x_1^2 & 0 \\ 0 & -2-6\,x_2^2 \end{bmatrix} = 4 + 12\,x_2^2 + 12\,x_1^2 + 36\,x_1^2 . x_2^2 > 0$$

Therefore J^ is negative definite , and by applying Krasovskii's Liapunov function is :

$V(x) = (-X1 - X2 - X1^3)^2 + (X1 - X2 - X2^3)^2$

Theorem (3.2) , [Ibrahim Hussein , 2013] (Implicit function theorem)

Let f be given in

$f\,(x, \mu) = 0$ (vector form)

Suppose that
$f\ (x_0, \mu_0) = 0$ And
Det $J\ (x_0, \mu_0) \neq 0$
Then there exist a neighborhoods
U of x_0 in R^n and V of μ_0 in R^k and a function X : V → U
Such that for every $\mu \in V$, $f\ (x, \mu) = 0$
Has the unique solution $X = X\ (\mu),\ in\ V$
Moreover if $f \in C^s\ (R^n)$ then $X \in C^s\ (R^k)$ in symbols $F\ (x(\mu)\mu = 0$
$x(\mu_0) = x_0$

Approach (3.1) :
Now, we introduce the new approach for construction of Liapunov functions for differential algebraic equation.
Since we have differential algebraic equation system.
$x^{\circ} = f(\mathrm{x}, \mathrm{y})$ ... ... (1)
$0 = g(\mathrm{x}, \mathrm{y})$ ... ... (2)
Now we use implicit function theorem, from (2) we obtain:
$g_x\, x^{\circ} + g_y\, y^{\circ} = 0$
$y^{\circ} = g_y^{-1} g_x\, f$
Since g_y^{-1}exist therefore det . $g_y \neq 0$
So we have :
$y^{\circ} = g\ (x, y)$
Then :

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \neq 0$$

$$J^T = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} \end{bmatrix} \neq 0$$

$$J^{\wedge} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} + \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$$J^{\wedge} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial y} \end{bmatrix}$$

If $|J^{\wedge}| < 0$ then J^ is negative definite . By applying Krasovskii's theorem implies that zero solution is asymptotically stable .
So , Liapunov nor function is

$V(x) = f^T \ f$

Example (3.2):

$x^{\cdot} = -2\,x + y$

$0 = \ x + y + x^2$

$x = -y - x^2 \rightarrow -x - \ x^2 = y$

$x^{\cdot} = -3\,x - \ x^2$

By using Krasovskii's method :

$J = [-\,3 - 2\,x]\,,J^T = [-\,3 - 2\,x],$

And :

$J^{\wedge} = J + J^T = -6 - 4\,x \ < 0\,,$

Therefore J^ is negative definite

Karosovskii's theorem implies that solution is asymptotically stable

And Liapunov function

$v(x) = (-3\,x - x^2)^2$

Example (3.3)

$x^{\cdot} = -\,x - 3\,y - \ x^3$

$0 = -\,y + x$

$x^{\cdot} = -\,x - 3\,y - \ x^3$

$y^{\cdot} = x^{\cdot} \quad , \quad y^{\cdot} = -\,x - 3\,y - \ x^3$

$J = \begin{bmatrix} -\,1 - 3x^2 & -\,3 \\ -\,1 - 3\,x^2 & -\,3 \end{bmatrix}$

$J^T = \begin{bmatrix} -\,1 - 3\,x^2 & -\,1 - 3\,x^2 \\ -\,3 & -\,3 \end{bmatrix}$

$J^{\wedge} = \begin{bmatrix} -\,2 - 6\,x^2 & -\,4 - 3x^2 \\ -\,4 - 3x^2 & -\,6 \end{bmatrix}$

Therefore J^ is negative definite

Applying Krasovskii's theorem , we get this system which is asymptotically stable

and Liapunov function is

$v(x) = (-\,x - 3\,y - \ x^3)^2 + (-\,x - 3\,y - \ x^3)^2$

$v(x) = 2(-\,x - 3\,y - \ x^3)^2$

4 - References

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